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ON THE EXISTENCE OF A MAXIMAL ELEMENT OF MULTIVALUED MAPPINGS IN H- SPACES

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Abstract

A result on the existence of a maximal element of multivalued mappings in H-spaces is proved.

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1. Introduction

In the last twenty years a lot of have been published on about the existence of a maximal element of multivalued mappings. This problem belongs to mathematical economy, more precisely to exchange economy [10]. An **exchange economy** is a model of a very simple type of **agents, consumers** and **economy without production.** Each consumer owns **resources** of various commodities, which may be **exchanged** with other consumers.

In the exchange economy **preferences** of consumers are an essential part. Let E be the **commodity space** and H the set of consumers. Each consumer $h \in H$ has a **consumption set** $X^h \subset E$ from which he has to obtain a commodity bundle. Each consumer $h \in H$ has **preferences** amongst the commodity bundles in X^h, and these are expressed by a correspondence

$P^h : X^h \to \mathcal{P}(X^h)$. The set $P^h(x)$ contains all commodity bundles that h **strictly prefers** to $x \in X^h$. Hence

$$y \in P^h(x)$$

means that the consumer h considers the bundle y to be **better** than the bundle x. In the literature on preferences a number of assumptions on P^h are introduced. One of the most important assumptions is the **irreflexivity:**

$$(\forall x \in X^h)\ (x \notin P^h(x)),$$

which is a very natural assumption. Namely, P^h is **irreflexive** if x is not better than X. In the language of the fixed point theory, the property of the irreflexivity of P^h means that P^h has no fixed point.

Given an irreflexive preference $P^h : X^h \to \mathcal{P}(X^h)$ and a set $B \subset X^h$, we call $x \in B$ a **maximal element** of B with respect to P^h if and only if

$$(\forall y \in B)\ (y \notin P^h(x)).$$

Hence, x is a maximal element of B with respect to P^h if and only if B does not contain an element better than x. Obviously, x is a maximal element of B with respect to P^h if and only if

$$P^h(x) \cap B = \emptyset.$$

A well known result on the existence of a maximal element of a correspondence follows from the Browder fixed point theorem proved in [3].

Theorem A. *Let K be a nonempty, compact and convex subset of a Hausdorff topological vector space E, $T : K \to \mathcal{P}(E)$ and the following conditions be satisfied:*

i) For each $x \in K$, $T(x)$ is a nonempty convex subset of K.

ii) For each $x \in K$, $T^{-1}(x) = \{y;\ y \in K,\ x \in Ty\}$ is open in K.

Then there exists a point $x_0 \in K$ such that $x_0 \in Tx_0$.

From Theorem A the following theorem can be easily proved.

Theorem A'. *Let K be a nonempty, compact and convex subset of a Hausdorff topological vector space E, $T : K \to \mathcal{P}(E)$ an irreflexive correspondence and let the following conditions be satisfied:*

(a) For each $x \in K$, $T(x)$ is a convex subset of K.

(b) For each $x \in K$, $T^{-1}(x)$ is open in K.

Then there exists $x_0 \in K$ such that $Tx_0 = \emptyset$.

Proof. If we suppose that $Tx \neq \emptyset$, for every $x \in K$, from Theorem A it follows that for some $\bar{x} \in K$, $\bar{x} \in T\bar{x}$. This contradicts to the assumption that T is irreflexive. Hence

$$\{x;\ x \in K,\ Tx = \emptyset\} \neq \emptyset.$$

In this paper the condition that $T(x)$ is a H-convex subset of K, for each $x \in K$, is replaced by the weaker condition that $\bigcap_{u \in U} Tu$ is H-convex, for every open set $U \subset K$ [1] and E will be an H-space (E, Γ) [2].

Example. [1] For every $x \in (0,1)$ let

$$Ax = [0,1] \cup (\mathbf{Q} \cap [x,1]), Bx = [0,x] \cup ((\mathbf{R} \setminus \mathbf{Q}) \cap [x,1]),$$

where $\mathbf{Q}$ is the set of all rational numbers. Let $T : [0,1] \to 2^{[0,1]}$ be defined as follows:

$$T(x) = \begin{cases} Ax, & \text{if } x \in (0,1) \cap \mathbf{Q} \\ Bx, & \text{if } x \in (0,1) \cap (\mathbf{R} \setminus \mathbf{Q}), \end{cases}$$

$$T(0) = \mathbf{Q} \cap (0,1),\ T(1) = [0,1].$$

Then $\bigcap_{x \in U} Tx$ is convex for any open set $U \subset [0,1]$.

Let $\mathcal{F}(D)$ be the family of finite subsets of D. An H-space [2] is a triple $(X, D; \Gamma)$ where X is a topological space, D a nonempty subset of X and $\Gamma = \{\Gamma_A\}_{A \in \mathcal{F}(D)}$ a family of contractible subset of X so that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B$ $(A, B \in \mathcal{F}(D))$. If $X = D$ we shall denote $(X, X; \Gamma)$ by (X, Γ). Any convex space X is an H-space (X, Γ) by putting for $A \subset \mathcal{F}(X), \Gamma_A = coA$, where coA is the convex hull of A and every $n-$ simplex Δ_n is an H-space $(\Delta_n, D; \Gamma)$, where D is the set of vertices and $\Gamma_A = coA$ for $A \in \mathcal{F}(D)$. Let $(X, D; \Gamma)$ be an H-space and C a nonempty subset of X. If for each $A \in \mathcal{F}(D)$ such that $A \subset C$ we have that $\Gamma_A \subset C$ then C is an $H-$convex set.

2. Existence of a maximal element

Let $N \in \mathbf{N}$, $\langle N \rangle$ be the set of all nonempty subsets of $\{0, 1, 2, ..., N\}$, $\Delta_N = co\{e_0, e_1, ..., e_N\}$ be the standard simplex of dimension N, where $\{e_0, e_1, ..., e_N\}$

is the canonical basis of $\mathbf{R}^{N+1}$ and for $J \in \langle N \rangle$ let $\Delta_J = co\{e_j;\ j \in J\}$.

In [4], the following Lemma is proved.

Lemma. *Let X be a topological space and $F : \langle N \rangle \to X$. Suppose that for each $J \in \langle N \rangle$, $F(J)$ is a nonempty, contractible subset of X and that*

$$(\forall J, J' \in \langle N \rangle)(J \subseteq J' \Rightarrow F(J) \subseteq F(J')).$$

Then, there exists a continuous function $g : \Delta_N \to X$ such that

$$g(\Delta_J) \subseteq F(J), \quad \textit{for all } J \in \langle N \rangle.$$

This Lemma will be used in the proof of the next Theorem.

Theorem. *Let (E, Γ) be an H-space, K a compact and H-convex subset of E and $S, T : K \to \mathcal{P}(K)$ such that the following conditions are satisfied:*

1) T is irreflexive.

2) For every open subset $U \subset K$, $\bigcap_{u \in U} Tu$ is an H-convex set.

3) $Sx \subseteq Tx$, for every $x \in K$.

4) $S^{-1}(x)$ is open, for every $x \in K$.

Then there exists at least one maximal element of S.

Proof. Suppose that $Sx \neq \emptyset$, for every $x \in K$. Then from 3) it follows that $Tx \neq \emptyset$, for every $x \in K$. We shall prove that in this case there exists $x_0 \in K$ such that $x_0 \in Tx_0$, which contradicts to the assumption that T is irreflexive. Since $S^{-1}(x)$ is open for every $x \in K$, the family $\{S^{-1}(x)\}_{x \in K}$ is an open covering of K. From the compactness of K it follows that there exists $\{x_0, x_1, x_2, ..., x_n\} \subseteq K$ such that

$$K = \bigcup_{i=0}^{n} S^{-1}(x_i).$$

Let $h_0, h_1, h_2, ..., h_n : K \to [0,1]$ be continuous mappings such that $\sum_{i=0}^{n} h_i(x) = 1$, for every $x \in K$ and for every $i \in \{0, 1, 2, ..., n\}$

$$h_i(x) \neq 0 \Longleftrightarrow x \in S^{-1}(x_i). \tag{2}$$

For every $x \in K$, $I(x) \subseteq \{0,1,2,...,n\}$ is defined in the following way:

$$i \in I(x) \Longleftrightarrow h_i(x) \neq \emptyset.$$

Hence

$$i \in I(x) \Longleftrightarrow x \in S^{-1}(x_i).$$

For every $I \subseteq \{0,1,2,...,n\}$ let

$$F(I) = \bigcap_{i \in I} S^{-1}(x_i).$$

We shall prove that $x \in F(I(x))$ for every $x \in K$ which implies that

(3) $$\bigcap_{u \in F(I(x))} Tu \subseteq T(x), \quad \text{for every } \ x \in K.$$

Since $x \in S^{-1}(x_i)$, for every $i \in I(x)$, it follows that

$$x \in \bigcap_{i \in I(x)} S^{-1}(x_i) = F(I(x)).$$

We shall prove that for every $x \in K$ and $i \in I(x)$

(4) $$x_i \in \bigcap_{u \in F(I(x))} Tu.$$

Relation (4) follows from

(5) $$x_i \in \bigcap_{u \in F(I(x))} Su, \quad \text{for every } \ i \in I(x).$$

Indeed, if $u \in F(I(x))$ then $u \in S^{-1}(x_i)$, for every $i \in I(x)$, i.e. $x_i \in Su$, for every $i \in I(x)$. Hence, (5) holds and condition (3) implies (4). Since $\bigcap_{u \in F(I(x))} Tu$ is H-convex (4) implies that

(6) $$\Gamma_{co\{x_i; i \in I(x)\}} \subseteq \bigcap_{u \in F(I(x))} Tu \subseteq T(x).$$

Let

$$G(I) = \Gamma_{co\{x_i; i \in I\}}, \ \ I \subseteq \{0,1,2,...,n\}.$$

We can apply Horvath's result on the existence of a continuous mapping $g : co\{e_0, e_1, e_2, ..., e_n\} \to E$ (since $G(I)$ is contractible) such that

(7) $$g(\Delta(I)) \subseteq G(I), \quad \text{for every} \quad I \subset \{0, 1, 2, ..., n\}.$$

Here

$$\Delta(I) = co\{e_{i_1}, e_{i_2}, ..., e_{i_k}\},$$

and

$$I = \{i_1, i_2, ..., i_k\} \subset \{0, 1, 2, ..., n\}.$$

Relations (6) and (7) imply that

(8) $$g(\Delta(I(x))) \subseteq G(I(x)) \subseteq T(x), \ x \in K.$$

Since $h \circ g : \Delta_n \to \Delta_n$, where

$$h(x) = (h_0(x), h_1(x), h_2(x), ..., h_n(x)), \quad \text{for every} \quad x \in K$$

there exists $x_0 \in \Delta_n$ such that

$$(h \circ g)(x_0) = x_0.$$

Then

$$(g \circ h)(g(x_0)) = g(x_0).$$

On the other hand

$$(h \circ g)(x) \subseteq \Delta(I(g(x))) \ \text{since} \ h(x) \in \Delta(I(x)), \ \text{for every} \ x \in K.$$

Hence (8) implies that $g(x_0) \in T(g(x_0))$, which contradicts to the irreflexivity of T.

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