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METHOD OF CURVATURE IN CIRCULAR COMPLEX ARITHMETIC

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Abstract

Given a disk Z, an analytic function f, and an estimate for the range $f(Z)$ in the form of a disk $\{c; R\}$, the following question arises: does $\{c; R\}$ completely contain the complex-valued range $f(Z)$? In this paper we present an approach to checking the above enclosure by using of the method of curvature, which can be suitably applied in some cases. This method is based on Blaschke's result concerning the intersection of a given simple closed smooth boundary of a circle. The method is used for determining the best including approximations for the ranges $\log Z$ and $Z^{1/k}$.

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1. Introduction

Let $Z = \{z \,|\, |z - c| \le r\}$ be a disk in the complex plane with the center $c = \operatorname{mid} Z$ and radius $r = \operatorname{rad} Z$, denoted shorter by $Z = \{c; r\}$. The boundary of the disk Z will be denoted by Γ and the set of all disks by $K(C)$. Let f be a complex function, defined on the union of all disks from $H \subseteq K(C)$, such that the range

$$f(Z) = \{f(z) \,|\, z \in Z\} = \bigcup_{z \in Z} f(z)$$

is closed for each $Z \in H$. The range $f(Z)$ is not a disk in general. For the sake of circular arithmetic evaluations we introduce a circular outer approximation of the range set $f(Z) = \{f(z) | z \in Z\}$, denoted by $I(f(Z))$, which entirely includes the range $f(Z)$ for each $Z \in H$, that is, $I(f(Z)) \supseteq f(Z)$. The disk $I(f(Z))$ is called a *circular complex extension* of $f(z)$ over Z or a *circular including approximation*, shorter *I-approximation.*

Regarding the best I-approximation there appears the problem of existence of the smallest disk containing the closed range $f(Z)$. If we define

$$R(\zeta) = \max_{z \in Z} |f(z) - \zeta|$$

for an arbitrary complex number ζ, then

$$|f(z) - \zeta| \leq R(\zeta), \quad z \in Z,$$

which means that the region $f(Z)$ is completely contained in the disk $V = \{\zeta; R(\zeta)\}$.

Let $S(\zeta) = \text{area}\,(V) = \pi R(\zeta)^2$. If $S(\zeta_0) = \inf S(\zeta)$, then the disk $V_0 = \{\zeta_0; R(\zeta_0)\}$ is the best I-approximation of the exact range $f(Z)$. In the best case the diameter $2R(\zeta_0)$ of the smallest disk V_0 can be equal to the diameter

$$D = \text{diam}\ \ f(Z) = \max_{z_1, z_2 \in Z} |f(z_1) - f(z_2)|$$

of the closed set $f(Z)$. Then $V_0 = \{\zeta_0; D/2\}$ is called the *diametrical including approximation* or *D-form* for $f(Z)$, denoted by $I_d(f(Z))$. Complex circular functions in the D-form can be of great importance. This topic was the subject of investigation in the papers [3], [5]–[8] for various analytic functions.

2. Method of curvature

For a given disk Z and a function f let us assume that we have found (using some useful technique or an assumption based on geometrical construction) an approximation of the range$f(Z)$ in the form of a disk $\{c; R\}$. Then the following important question arises:

Does disk $\{c; R\}$ completely contain the complex-valued range $f(Z)$, that is, is the inclusion

(1) $$f(Z) \subseteq \{c; R\}$$

valid?

The checking of the inclusion (1) is usually very difficult. In some cases, this problem can besolved by appropriate approaches or techniques. This paper is devoted to the use of the **method of curvature.** The method is based on the following Blaschke's result [1]:

Theorem 1. *If the curvature of the simple closed smooth boundary $w(t)$ of a region G is strictly positive and has exactly 2λ extreme points, then the contour $w(t)$ has at most 2λ intersections with any circle. Tangential intersections are counted as double intersections.*

This theorem enables in some cases to check (1) in an elegant and simple way proving that the curvature of the curve $w(t)$ is greater than the curvature of a possible inclusion disk.

For this purpose we prove first the following assertion.

Lemma 1. *For $k \geq 2$ and $p \in (0,1)$ the inequality*

$$(k-p)(1+p)^{1/k} > (k+p)(1-p)^{1/k} \tag{2}$$

holds.

Proof. Starting from the inequality $2/(1-t^2) > 2$ for $t \in (0,1)$, we obtain

$$\int_0^p \frac{2}{1-t^2}dt = \int_0^p \Big(\frac{1}{1+t} + \frac{1}{1-t}\Big)dt > \int_0^p 2dt,$$

wherefrom

$$\ln \frac{1+p}{1-p} > 2p. \tag{3}$$

This leads to

$$\frac{1+p}{1-p} > e^{2p} > \Big(1 + \frac{2p}{k-p}\Big)^k = \Big(\frac{k+p}{k-p}\Big)^k,$$

which reduces to (2). □

In the next sections we apply the method of curvature to the cases of diametrical disks for $\log Z$ and $Z^{1/k}$.

3. D–form for the logarithmic function

Let $Z = \{1; p\}$ $(0 < p < 1)$ be a disk in z-plane, and let G be the image of Z under $w = \log z$. The boundary Γ_G of G is given by $w(t) = \log(1 + pe^{it})$, $t \in [0, 2\pi)$. Let S denote the disk $|w - c| \leq R$ with

$$c = \frac{1}{2}\log(1 - p^2), \quad R = \frac{1}{2}\log\frac{1+p}{1-p}.$$

The mapping $w = \log z$ sends the points $z = 1 \pm p$ to the points $w = \log(1 \pm p) = c \pm R$, so that G can not have diameter less than $2R$. We shall prove that the disk S is a diametrical form of $\log Z$.

Γ_G is tangential to the circle S at the points $c + R$. To prove that Γ_G lies inside S we compute its curvature.

The curvature k of the curve $w(t)$ in the complex plane is given by

$$k = \frac{\operatorname{Im}(\bar{\dot{w}}\ddot{w})}{|\dot{w}|^3},$$

where dots denote differentiation with respect to t. For $w(t) = \log z(t)$ with $z(t) = 1 + pe^{it}$, we compute

$$\dot{w}(t) = \frac{ipe^{it}}{z(t)} \quad \text{and} \quad \ddot{w}(t) = -\frac{pe^{it}}{z(t)^2} \quad \Rightarrow \quad k(t) = \frac{1 + p\cos t}{p|z(t)|},$$

wherefrom we see that the curvature is strictly positive; therefore the domain G is strictly convex. Further, we compute

$$\dot{k}(t) = \frac{-p\sin t(p + \cos t)}{|z(t)|^3}.$$

We see that $\dot{k}(t)$ has precisely four simple zeros in $[0, 2\pi)$, at $t = 0, \pi$ and $\pm\arccos(-p)$. Since the circle S is tangential to $w(t)$ at two points there are no more points of intersection. Hence $w(t)$ lies either completely inside, or completely outside S.

It remains to show that at the point $c + R$ the curvature of $w(t)$ is greater than the curvature $1/R$ of S, that is $k(0) > \dfrac{1}{R}$. From the expression for $k(t)$ we find $k(0)$ so that the last inequality reduces to

$$\frac{1}{p} > \frac{1}{\frac{1}{2}\ln\dfrac{1+p}{1-p}},$$

which is the inequality (3) proved in Lemma 1.

Regarding the domain $Z = \{\zeta; r\}$ with $p = r/|\zeta|$ it is easy to construct the diametrical disk for the range $\log Z = \log \{\zeta; r\}$.

4. D–form of the k-th root

Now we consider the mapping of a disk Z $(0 \notin Z)$ by the transformation $z \mapsto z^{1/k}$. Denote the boundary of Z by Γ and the interior of Z by $\operatorname{int}\Gamma$. If the point $z \in Z$ makes one closed curve in $\operatorname{int}\Gamma$, then all roots $w_0, w_1, \ldots, w_{k-1}$ of the equation $w^k = z$ vary continuously. As z makes a complete circuit along Γ, the roots $w_0, w_1, \ldots, w_{k-1}$ follow the separated closed curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_{k-1}$ which enclose the ranges, say, $W_0, W_1, \ldots,$ W_{k-1}, respectively.

By $Z^{1/k}$ we understand the union of the closed regions, obtained by mapping the domain Z under the transformation $z \mapsto z^{1/k}$, i.e. $Z^{1/k} = \{z \mid z^k \in Z, 0 \notin Z\}$.

Let $c = \rho \exp(i\theta)$, $\beta_m = (\theta + 2m\pi)/k$ and let $w_m = \rho^{1/k} \exp(i\beta_m)$ $(m = 0, 1, \ldots, k-1)$ be the roots of the equation $w^k = c$. If the domain $Z = \{c; r\}$ does not contain the origin, then $r/\rho =: p < 1$. The aim is to enclose the separated ranges $W_0, W_1, \ldots, W_{k-1}$ by covering disks.

We will now give a representation of the k-th root of a disk (which does not contain the origin) where the inclusion of the exact range $Z^{1/k}$ is performed by disks whose diameters are equal to the diameter of these ranges. This problem was also investigated in [6].

Let U be the boundary of the unit circle, i.e. $U = \{\zeta \mid |\zeta| = 1\}$. The transformation $z \mapsto z^{1/k}$ maps any point $c(1 + p\zeta) \in \Gamma$ $(\zeta \in U)$ to the points $s_1(\zeta), \ldots, s_{k-1}(\zeta)$ belonging to the boundaries $\Gamma_1, \ldots, \Gamma_{k-1}$ respectively, where

$$\begin{aligned} s_m(\zeta) &= w_m(1 + p\zeta)^{1/k} \\ &= \rho^{1/k} \exp(i\beta_m)(1 + p\zeta)^{1/k} \in \Gamma_m \quad (m = 0, 1, \ldots, k-1). \end{aligned}$$

This means that the problem of finding diametrical disks for the ranges $W_0, W_1, \ldots, W_{k-1}$ can be reduced to the determination of the diametrical disk for the range $T_p = \{1; p\}^{1/k}$ $(0 < p < 1)$. Then the obtained disk has to be multiplied by the complex numbers $w_0, w_1, \ldots, w_{k-1}$.

The solution of the above problem will be presented in what follows.

For $0 < p < 1$ and the integer $k \geq 2$, let C_p denote the disk $\{c_p; r_p\} = \{w| \, |z - c_p| < r_p\}$ with the center

$$c_p = \frac{(1+p)^{1/k} + (1-p)^{1/k}}{2}$$

and the radius

$$r_p = \frac{(1+p)^{1/k} - (1-p)^{1/k}}{2}.$$

The boundary of the disk C_p will be denoted by ∂C_p.

Theorem 2. *For $0 < p < 1$ and every integer $k \geq 2$, $T_p \subset C_p$. The boundary of T_p meets ∂C_p only at two points $(1 \pm p)^{1/k}$.*

Proof. The boundary of T_p is a simple closed curve with the parameterization

$$w(t) = (1 + pe^{it})^{1/k}, \quad 0 \leq t < 2\pi.$$

This curve is tangential to the boundary ∂C_p at the points $c_p \pm r_p$. To prove that the curve $w(t)$ lies inside ∂C_p (except for the points $c_p \pm r_p$), we will compute its curvature.

The curvature τ of a curve $w(t)$ in the complex plane is given by

$$\tau = \frac{\operatorname{Im} \, \bar{\dot{w}}\ddot{w}}{|\dot{w}|^3} \tag{4}$$

where the dots denote differentiation with respect to t. Let $z(t) = 1 + pe^{it}$ be the boundary of the disk $Z = \{1; p\}$. For $w(t) = z(t)^{1/k}$ we calculate

$$\dot{w}(t) = \frac{pie^{it}}{k} z(t)^{(1/k)-1}.$$

Using logarithmic differentiation we find

$$\ddot{w}(t) = \dot{w}(t) \left[\frac{((1/k) - 1)pie^{it}}{z(t)} + i \right]$$

so that

$$\begin{aligned} \operatorname{Im} \bar{\dot{w}}(t)\ddot{w}(t) &= |\dot{w}(t)|^2 \operatorname{Re} \left[\frac{((1/k) - 1)pie^{it}}{z(t)} + i \right] \\ &= |\dot{w}(t)|^2 \left[\frac{((1/k) - 1)r(\cos t + p)}{|z(t)|^2} + i \right]. \end{aligned}$$

According to the last three expressions, from (4) we obtain

$$\tau(t) = \frac{((1/k)-1)p(\cos t + p) + |z(t)|^2}{|\dot{w}(t)||z(t)|^2} = \frac{k + p^2 + (k+1)p\cos t}{p|z(t)|^{1+(1/k)}}.$$

Since $k + p^2 + (k+1)p\cos t \geq k + p^2 - (k+1)p = (k-p)(1-p) > 0$, the curvature is strictly positive, and so the domain T_p is strictly convex. Its first derivative

$$\dot{\tau}(t) = \frac{-(k^2-1)(p+\cos t)\sin t}{k|z(t)|^{3+1/k}}$$

has precisely four simple zeros in $[0, 2\pi)$ at $t = 0, \pi$ and $\pm\arccos(-p)$. By Theorem 1 there follows that the curve $w(t)$ has at most four points of intersection with any circle. The circle ∂C_p is tangent to $w(t)$ at the points $c_p \pm r_p$, so there are no more points of intersection. As noted in [2] these intersecting points are counted as double intersections since they are the tangential ones. This "double counting" interpretation is indeed valid, for otherwise we could always produce a circle close to ∂C_p which meets $w(t)$ in at least five points. In regard to the above facts we conclude that $w(t)$ lies either entirely inside or entirely outside ∂C_p.

The proof will be completed if we can show that at the point $c_p + r_p$ the curvature of $w(t)$ is greater than the curvature of ∂C_p. Thus, we should show that $\tau(0) > 1/r_p$, or

$$\frac{p(1+p)^{1/k}}{k+p} < \frac{(1+p)^{1/k} - (1-p)^{1/k}}{2}.$$

Rewriting this, we obtain the inequality (2) proved in Lemma 1, and the proof of Theorem 2 is finished. □

The mapping $w = z^{1/k}$ sends the points $z = 1 \pm p \in U$ to the points $w = (1 \pm p)^{1/k} = c_p \pm r_p \in T_p$. Therefore, the region T_p cannot have the diameter less than $2r_p$. In Theorem 2 it was proved that $T_p \subset C_p$, which means that T_p cannot have the diameter greater than $2r_p$. Therefore, C_p is the diametrical disk for the range T_p.

According to the previous, the diameter of the exact range W_m ($m = 0, 1, \ldots, k-1$) is equal to the distance between the points represented by the complex numbers

$$s_m(1) = (1+p)^{1/k} w_m \quad \text{and} \quad s_m(-1) = (1-p)^{1/k} w_m$$

and it is given by

$$D = 2r_p|w_m| = \rho^{1/k}((1+p)^{1/k} - (1-p)^{1/k}) = 2\rho^{1/k}r_p.$$

The complex numbers

$$c_p w_m = \frac{s_m(1) + s_m(-1)}{2} = w_m \frac{(1+p)^{1/k} + (1-p)^{1/k}}{2}$$

have to be taken for the centers of diametrical disks $I_d(W_m)$ enclosing the sets W_m $(m = 0, 1, \ldots, k-1)$. According to the above, the k-th root in circular arithmetic with the diametrical form, denoted by $I_d(Z^{1/k})$, can be represented by

$$I_d(Z^{1/k}) = \bigcup_{m=0}^{k-1} \{c_p w_m; \rho^{1/k} r_p\} \quad (p \in (0,1)).$$

Note that the points of contact of the boundary Γ_m and the circumference of the corresponding inclusive disk are $s_m(-1)$ and $s_m(1)$.

References

[1] Blaschke, W., Kreis und Kugel 2, Auflage, De Gruyter, Berlin 1956.

[2] McCoy, T.L., Kuijilaars, A. B. J., Answer to a query concerning the mapping $w = z^{1/m}$, Indagat. Math., N.S. 4 (1993), 479–481.

[3] Petković, L.D., On optimal including circular approximation for the range of a complex exponential function, Z. Angew. Math. Mech. 73 (1993), 109–116.

[4] Petković, L.D., Petković, M.S., The representation of complex circular functions using Taylor's series, Z. Angew. Math. Mech. 61 (1981), 661–662.

[5] Petković, L.D., Petković, M.S., On some applications of circular complex functions, Computing 41 (1989), 141–148.

[6] Petković, L.D., Trajković, M., On some optimal inclusion approximations by disks, Interval Computation 1 (1993), 34–50.

[7] Petković, L.D., Trajković, M., Verification methods for inclusion disks, Reliable Computing 1 (1995), 403–410.

[8] Trajković, M., Petković, L.D., Petković, M.S., Diametrical inclusion disk for Z^n, Z. Angew. Math. Mech. 75 (1995), 775–782.