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BISEMILATTICE-VALUED FUZZY SETS

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Abstract

Bisemilattice-valued fuzzy set (B-fuzzy set) is defined to be a mapping from a nonempty set to a bisemilattice (an algebraic structure with two binary operations, both commutative, associative and idempotent).

To each B-fuzzy set corresponds two collections of subsets, levels (cuts) of that fuzzy set. These two families are determined by two ordering relations on a bisemilattice. Theorems of decomposition of B-fuzzy sets into levels are formulated and proved in the paper. Starting from two families of subsets of a set X, we give conditions for synthesis of a B-fuzzy set. Some other properties of B-fuzzy sets are also formulated.

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1. Preliminaries

Bisemilattice

A bisemilattice $\mathcal{B} = (B, \wedge, \vee)$ is an algebra of the type $(2, 2)$, where $(B, \wedge)$ and $(B, \vee)$ are semilattices. Therefore, the operations $\wedge$ (called "meet") and $\vee$ ("join") satisfy commutative, associative and idempotency laws. Since a lattice is a bisemilattice satisfying the absorption laws, bisemilattices are a generalization of lattices.

A bisemilattice was introduced by J. Plonka in [6] under the name of quasi-lattice, and Padmanabhan in [5] called it bisemilattice.

There are two orderings corresponding to a bisemilattice $(B, \wedge, \vee)$ [2]:

$$x \leq_\vee y \text{ if and only if } x \vee y = y$$

and

$$x \leq_\wedge y \text{ if and only if } x \wedge y = x.$$

$x \geq_\vee y$ is a denotation for $y \leq_\vee x$, and $x \geq_\wedge y$ for $y \leq_\vee x$.

Since a bisemilattice $(B, \wedge, \vee)$ is uniquely represented by two orderings, it can also be considered as a relational system, as follows. $(B, \leq_\vee, \leq_\wedge)$ is a **bisemilattice**, if B is a nonempty set and $\leq_\vee$, $\leq_\wedge$ are ordering relations on B, such that $(B, \leq_\vee)$ is a join-semilattice ($\vee$-semilattice, i.e., the poset in which every two-element subset has the least upper bound), and $(B, \leq_\wedge)$ is a meet-semilattice ($\wedge$-semilattice: each two-element subset has the greatest lower bound).

Remark. Naturally, a diagram of a bisemilattice consists of two Hasse diagrams, one for each ordering. We use the following convention: if $x \leq_\vee y$ and $z \leq_\wedge t$, then x is below y, and z below t in the corresponding diagrams.

Fuzzy set

As is known, a fuzzy set (by the original definition of Zadeh) is a mapping from a nonempty set to the real interval $[0, 1]$. This notion has been generalized in several steps. A codomain was firstly taken to be a Boolean algebra instead of a real interval, later on it was taken to be a lattice, then a partially ordered set and finally a relational system. By taking a special kind of a relational system, one gets the most general concept in this direction ([9]) (the starting point is the lattice as a relational system). In another direction, when the lattice is considered to be an algebra of the type (2,2), a more general approach could be obtained by taking a weaker structure of the same type, instead of a lattice. This is a subject of the present article.

We advance some common elementary notions and properties for any of the above mentioned (and already defined) fuzzy sets, concerning the ordering relation included in the codomain.

Let P be a foregoing ordered structure (real interval, lattice, poset, etc.) with the order $\leq$, and $\overline{A} : A \rightarrow P$ a P-fuzzy subset of a nonempty set A (or: a P-fuzzy set on A). Then, for $p \in P$, a **level set (level, p-cut,)** of $\overline{A}$ is a (crisp) subset A_p of A, such that $x \in A_p$ if and only if $\overline{A}(x) \geq p$. The characteristic function of A_p is denoted by $\overline{A}_p$ and is called the **level function** of $\overline{A}$. Its codomain is the set $\{0,1\}$. It is well known that for $p, q \in P$, $p \leq q$ implies $A_q \subseteq A_p$.

2. Bisemilattice-Valued Fuzzy Sets

In the sequel we investigate bisemilattice-valued fuzzy sets, and we give some of their properties, specially connected with the decomposition and synthesis of a fuzzy set.

A **bisemilattice valued fuzzy set** (B-fuzzy set) is a mapping $\overline{A} : X \longrightarrow B$ from a nonempty set X to a bisemilattice $\mathcal{B} = (B, \wedge, \vee)$.

For each $p \in B$, there are two level subsets defined as follows:

$$A_p^{\vee} = \{x \in X \mid \overline{A}(x) \geq_{\vee} p\}$$

and

$$A_p^{\wedge} = \{x \in X \mid \overline{A}(x) \geq_{\wedge} p\}.$$

The corresponding level functions are:

$$\overline{A}_p^{\vee}(x) = 1 \text{ if and only if } \overline{A}(x) \geq_{\vee} p$$

and

$$\overline{A}_p^{\wedge}(x) = 1 \text{ if and only if } \overline{A}(x) \geq_{\wedge} p.$$

Thus, for a B-fuzzy set $\overline{A} : X \longrightarrow B$, there are two families of level subsets:

$$A_B^{\vee} = \{A_p^{\vee} \mid p \in B\}, \quad \text{and} \quad A_B^{\wedge} = \{A_p^{\wedge} \mid p \in B\}.$$

Let $\overline{A} : X \longrightarrow B$ be a B-fuzzy set. Relation $\sim$ on X, given by: $x \sim y$ if and only if $\overline{A}(x) = \overline{A}(y)$ is an equivalence relation. The corresponding partition of the set X is said to be the **partition induced** by the B-fuzzy set $\overline{A}$. Similarly to the notions given above, this one (partition) is also independent of the fact that $\overline{A}$ is a B-fuzzy set; it is defined in the same way for other types of fuzzy sets.

Recall that the smallest element in an ordered set, if such an element exists, is denoted by $\mathbf{0}$, and the greatest, if it exists, by $\mathbf{1}$.

Theorem 1. *Let $\overline{A} : X \longrightarrow B$ be a B-fuzzy set. Then, the following is satisfied.*

1. *If there is a bottom element* $\mathbf{0}$ *in* $(B, \wedge)$, *then*

$$A^{\wedge}_{\mathbf{0}} = X.$$

2. *If $p \leq_{\vee} q$, then $A^{\vee}_q \subseteq A^{\vee}_p$, and if $p \leq_{\wedge} q$, then $A^{\wedge}_q \subseteq A^{\wedge}_p$.*

3.

$$\overline{A}(x) = \bigvee_{\wedge}\{p \in B \mid \overline{A}^{\wedge}_p(x) = 1\};$$

$$\overline{A}(x) = \bigvee_{\vee}\{p \in B \mid \overline{A}^{\vee}_p(x) = 1\}$$

(i.e., the supremum at the right side of each equality exists for every x and is equal to $\overline{A}(x)$).

Proof.

1. Indeed,

$$A^{\wedge}_{\mathbf{0}} = \{x \in X \mid \overline{A}(x) \geq_{\wedge} \mathbf{0}\} = X.$$

2. Let $p \leq_{\vee} q$. Then $x \in A^{\vee}_q$ if and only if $\overline{A}(x) \geq_{\vee} q \geq_{\vee} p$, i.e., $x \in A^{\vee}_p$. The proof goes analogously for $p \leq_{\wedge} q$.

3. Let $\overline{A}(x) = r \in B$. Then $\overline{A}^{\vee}_r(x) = 1$. If for $p \in B$, $\overline{A}^{\vee}_p(x) = 1$, then $r = \overline{A}(x) \geq_{\vee} p$. Hence, r is the required supremum, and

$$\overline{A}(x) = \bigvee_{\vee}\{p \in B \mid \overline{A}^{\vee}_p(x) = 1\}.$$

The proof is the same for the second equality. □

Theorem 2. *Let $\overline{A} : X \longrightarrow B$ be a B-fuzzy set. Then, the following holds.*

1. *If $B_1 \subseteq B$, then*

$$\bigcap(A^{\vee}_p \mid p \in B_1) = A^{\vee}_{\bigvee_{\vee}\{p \mid p \in B_1\}}.$$

If for $B_1 \subseteq B$ there exists the supremum $\bigvee_{\wedge} B_1$, then,

$$\bigcap(A^{\wedge}_p \mid p \in B_1) = A^{\wedge}_{\bigvee_{\wedge}\{p \mid p \in B_1\}};$$

2.

$$\bigcup(A^{\wedge}_p \mid p \in B) = X \quad \textit{and} \quad \bigcup(A^{\vee}_p \mid p \in B) = X;$$

3. *For all $x \in X$,*

$$\bigcap(A^{\wedge}_p \mid x \in A^{\wedge}_p) \in A^{\wedge}_B \quad \textit{and} \quad \bigcap(A^{\vee}_p \mid x \in A^{\vee}_p) \in A^{\vee}_B.$$

Proof.

1. Since the semilattice $(B, \vee)$ is complete, the supremum of $B_1 \subseteq B$ exists, and for $x \in X$,

$x \in \bigcap(A_p^\vee \mid p \in B_1)$, if and only if $x \in A_p^\vee$ for all $p \in B_1$, if and only if $\overline{A}(x) \geq_\vee p$ for all $p \in B_1$, if and only if $\overline{A}(x) \geq_\vee \bigvee_\vee\{p \mid p \in B_1\}$, if and only if $\overline{A}^\vee_{\bigvee_\vee\{p|p\in B_1\}}(x) = 1$, if and only if $x \in A^\vee_{\bigvee_\vee\{p|p\in B_1\}}$.

The proof of the second equality is similar.

2. If $x \in X$, then $\overline{A}(x) = p \in B$. Hence, $x \in A_p^\vee$ and $x \in A_p^\wedge$. Thus, $x \in \bigcup(A_p^\vee \mid p \in B)$ and $x \in \bigcup(A_p^\wedge \mid p \in B)$, and the required equalities are satisfied.

3. Let $x \in X$. Since $x \in A_p^\vee$ if and only if $\overline{A}(x) \geq_\vee p$ if and only if $\overline{A}_p^\vee(x) = 1$, by Theorem 1 (3) and by the first part of this theorem,

$$\bigcap(A_p^\vee \mid x \in A_p^\vee) = \bigcap(A_p^\vee \mid \overline{A}_p^\vee(x) = 1) =$$

$$A^\vee_{\bigvee_\vee\{p|\overline{A}_p^\vee(x)=1\}} \in A_B^\vee.$$

The proof is the same for the second equality. □

An important difference between B-fuzzy sets and the L-valued ones is that the collection of level sets of an L-fuzzy set is always a lattice (under inclusion), but the corresponding collection for a B-fuzzy set is not a bisemilattice. The reason is that the collections of level subsets $A_B^\vee$ and $A_B^\wedge$ of a B-fuzzy set $\overline{A}$ do not coincide; in general, they even do not have the same cardinality. The following example illustrates this case.

Example 1.

Let $(B, \wedge, \vee)$ be a bisemilattice in Fig. 1, let $X = \{x, y, z\}$, and a B-fuzzy set be given by:

$$\overline{A} = \begin{pmatrix} x & y & z \\ a & c & d \end{pmatrix}.$$

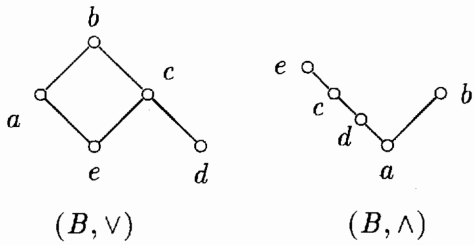

Figure 1

The corresponding families of level sets are: $A_a^{\vee} = \{x\}; A_b^{\vee} = \emptyset; A_c^{\vee} = \{y\}; A_d^{\vee} = \{y, z\}; A_e^{\vee} = \{x, y\}$.

$A_a^{\wedge} = \{x, y, z\}; A_b^{\wedge} = \emptyset; A_c^{\wedge} = \{y\}; A_d^{\wedge} = \{y, z\}; A_e^{\wedge} = \emptyset$.

Hence,

$A_B^{\vee} = \{\emptyset, \{x\}, \{y\}, \{y, z\}, \{x, y\}\}$ and

$A_B^{\wedge} = \{\emptyset, \{x, y, z\}, \{y\}, \{y, z\}\}$.

□

The following theorem gives conditions under which two collection of subsets of a set X are families of the level subsets of a B-fuzzy set.

Let $\mathcal{B}$ be a family of subsets of a nonempty set X union of which is X, such that for every $x \in X$,

$$\bigcap(p \in \mathcal{B} \mid x \in p) \in \mathcal{B}.$$

For every $x \in X$, let

$$\mathcal{B}(x) := \bigcap(p \in \mathcal{B} \mid x \in p).$$

Further, let $\pi_{\mathcal{B}}(x) := \{y \in X \mid \mathcal{B}(y) = \mathcal{B}(x)\}$. The collection $\{\pi_{\mathcal{B}}(x) \mid x \in X\}$ is, by the construction, partition of X, and we shall call it the **partition induced by** $\mathcal{B}$. Every partition of X is induced by some family of the above type: the simplest one is the partition itself, together with the empty set. Obviously, different families can induce the same partition.

Theorem 3. (Theorem of synthesis) *Let X be a nonempty set, and* Π *a partition of X. Further, let $\mathcal{B}_1 \subseteq \mathcal{P}(X)$, $\mathcal{B}_2 \subseteq \mathcal{P}(X)$ be two families of the subsets of X, satisfying the following.*

(i) $|\mathcal{B}_1| = |\mathcal{B}_2|$.

(ii) The poset $(\mathcal{B}_1, \subseteq)$ *is a* $\wedge$*-semilattice and* $(\mathcal{B}_2, \subseteq)$ *is a* $\vee$*-semilattice.*

(iii) $\bigcup \mathcal{B}_1 = \bigcup \mathcal{B}_2 = X$.

(iv) For every $x \in X$

$$\bigcap(p \in \mathcal{B}_1 \mid x \in p) \in \mathcal{B}_1 \quad \textit{and} \quad \bigcap(p \in \mathcal{B}_2 \mid x \in p) \in \mathcal{B}_2.$$

(v) Both $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *induce the partition* Π.

Let $\overline{A}^{\vee} : X \longrightarrow \mathcal{B}_1$ *be a join-fuzzy set, defined by:*

$$\overline{A}^{\vee}(x) = \bigcap(p \in \mathcal{B}_1 \mid x \in p),$$

as a mapping to a semilattice $(\mathcal{B}_1, \leq_\vee)$, *where* $p \leq_\vee q$ *if and only if* $q \subseteq p$.

Similarly, let $\overline{A}^{\wedge} : X \longrightarrow \mathcal{B}_2$ *be a meet-fuzzy set, defined by:*

$$\overline{A}^{\wedge}(x) = \bigcap(p \in \mathcal{B}_2 \mid x \in p),$$

as a mapping to a semilattice $(\mathcal{B}_2, \leq_\wedge)$, *where* $p \leq_\wedge q$ *if and only if* $q \subseteq p$.

Then, there is a bisemilattice $\mathcal{B} = (B, \leq_\vee, \leq_\wedge)$, *and a* B*-fuzzy set* $\overline{A} : X \longrightarrow B$, *such that* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are collections of level sets of* $\overline{A}$.

Proof.

Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are of the same cardinality, and partitions on X induced by $\overline{A}^{\wedge}$ and by $\overline{A}^{\vee}$ are the same (straightforward by *(v)*), we have that also the sets $C_1 = \mathcal{B}_1 \setminus \overline{A}^{\vee}(X)$ and $C_2 = \mathcal{B}_2 \setminus \overline{A}^{\wedge}(X)$ have the same cardinality. Let ϕ be an arbitrary bijection from C_1 to C_2.

Now, we define a bijection $\varphi : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ in the following way:

If $p \in \mathcal{B}_1$ is the image of an $x \in X$ by $\overline{A}^{\vee}$, i.e., if $\overline{A}^{\vee}(x) = p$ for an $x \in X$, then let

$$\varphi(p) := \overline{A}^{\wedge}(x) \in \mathcal{B}_2.$$

If $p \in \mathcal{B}_1$ is not the image of any $x \in X$, then let

$$\varphi(p) := \phi(p).$$

We consider the bisemilattice $\mathcal{B} = (B, \leq_\vee, \leq_\wedge)$, such that $B = \mathcal{B}_1$, $p \leq_\vee q$ is the same as in $\mathcal{B}_1$ and $p \leq_\wedge q$ in $\mathcal{B}$ if and only if $\varphi(p) \leq_\wedge \varphi(q)$ in $\mathcal{B}_2$.

Now, the required B-fuzzy set (i.e., such that its level sets are collections $\mathcal{B}_1$ and $\mathcal{B}_2$), is $\overline{A} : X \longrightarrow B$, defined by:

$$\overline{A}(x) := \overline{A}^{\vee}(x).$$

By the theorem of synthesis for semilattices [8], we have that $A_p^{\vee} = p$, for $p \in \mathcal{B}_1$. Hence $A_B^{\vee} = \mathcal{B}_1$.

Further on, $x \in A_p^{\wedge}$ if and only if $\overline{A}(x) \geq_{\wedge} p$ if and only if $\overline{A}^{\vee}(x) \geq_{\wedge} p$ if and only if $\varphi(\overline{A}^{\vee}(x)) \geq_{\wedge} \varphi(p)$ in $\mathcal{B}_2$ if and only if $\varphi(\overline{A}^{\vee}(x)) \subseteq \varphi(p)$ if and only if $\overline{A}^{\wedge}(x) \subseteq \varphi(p)$.

Since $x \in \overline{A}^{\wedge}(x)$, the preceding formula implies that $x \in \varphi(p)$.

Conversely, if $x \in \varphi(p)$, then $\overline{A}^{\wedge}(x) \subseteq \varphi(p)$, by the definition of $\overline{A}^{\wedge}(x)$.

Hence, $A_p^{\wedge} = \varphi(p)$, for $p \in B$, and since φ is a bijection, we have that $A_B^{\wedge} = \mathcal{B}_2$. □

The following example illustrates the theorem.

Example 2.

Let $X = \{1,2,3,4,5\}$ and let $\Pi = \{\{1,2\},\{3\},\{4,5\}\}$. Π is a partition on X, induced by the following two families of subsets of X:

$$\mathcal{B}_1 = \{\emptyset,\{3\},\{1,2\},\{4,5\}\},\ \mathcal{B}_2 = \{\{3\},\{1,2,3\},\{3,4,5\},\{1,2,3,4,5\}\}.$$

It is not difficult to see that the conditions *(i) - (v)* of Theorem 3 are satisfied. Further on, we have that:

$$\overline{A}^{\vee} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \{1,2\} & \{1,2\} & \{3\} & \{4,5\} & \{4,5\} \end{pmatrix}$$

and

$$\overline{A}^{\wedge} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \{1,2,3\} & \{1,2,3\} & \{3\} & \{3,4,5\} & \{3,4,5\} \end{pmatrix}.$$

By the construction, partitions induced by semilattice-valued fuzzy sets $\overline{A}^{\vee}$ and $\overline{A}^{\wedge}$ coincide with Π. The bijection $\varphi : B_1 \longrightarrow B_2$ is:

$$\varphi = \begin{pmatrix} \{1,2\} & \{3\} & \{4,5\} & \emptyset \\ \{1,2,3\} & \{3\} & \{3,4,5\} & \{1,2,3,4,5\} \end{pmatrix}.$$

These collections are represented in Figure 2. By the construction described in Theorem 3, we obtain the following fuzzy set:

$$\overline{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & a & b & c & c \end{pmatrix},$$

where we denote $\{1,2\}$ by a, $\{3\}$ by b, $\{4,5\}$ by c and $\emptyset$ by d.

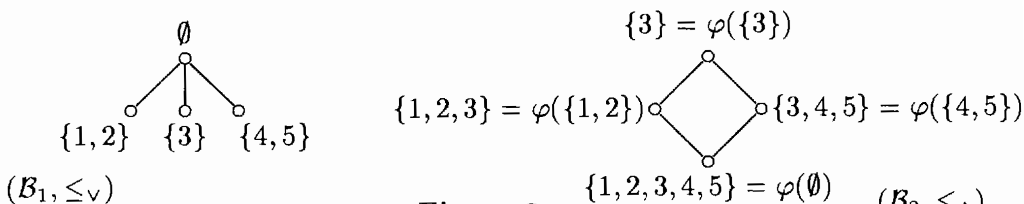

Figure 2

The obtained fuzzy set is $\overline{A} : X \longrightarrow B$, where $B = \{a,b,c,d\}$, and the bisemilattice $\mathcal{B}$ is given in Figure 3. $\overline{A}$ obviously has $\mathcal{B}_1$ and $\mathcal{B}_2$ as the families of level subsets.

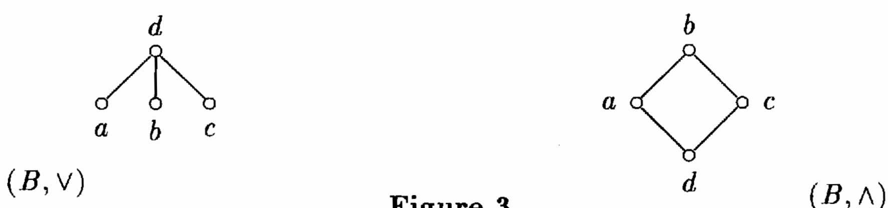

Figure 3

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