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SPECTRAL APPROXIMATION FOR INNER AND OUTER SOLUTION OF SOME SPP

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Abstract

Certain type of layer problem described by the second order differential equations with the small parameter multiplying the highest derivative will be considered. The solution inside the layer, as well as the solution out of the layer, will be approximated by two different truncated orthogonal series. The character of the layer and the degree of the spectral approximation of the inner solution will determine the domain decomposition, constructed by the use of the appropriate resemblance function.

Numerical results will be compared to those obtained earlier by the author, where the reduced solution was used to approximate the exact solution out of the layer.

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1. Introduction

We shall consider the selfadjoint singularly perturbed problem

$$(1) \qquad L_\varepsilon y \equiv -\varepsilon^2 y''(x) + g(x)y(x) = h(x) \quad 0 \le x \le 1.$$

$$(2) \qquad y(0) = A, \quad y(1) = B,$$

where the function $g(x)$ satisfies the condition

$$(3) \qquad g(x) \geq K^2 > 0, \quad K \in \mathbf{R}.$$

It is known that under the assumption (3) the selfadjoint problem has the unique solution $y(x) \in \mathbf{C}^2[0,1]$. In general, the solution displays two boundary layers of order $O(\varepsilon)$. If we denote by $y_R(x)$ the solution of the reduced problem

$$g(x)y_R(x) = h(x), \quad 0 \leq x \leq 1,$$

then, if

$$(4) \qquad y_R(0) \neq A$$

the boundary layer occures at $x = 0$, and if

$$y_R(1) \neq B$$

we have the boundary layer at the endpoint $x = 1$.

The problem (1),(2) represents mathematical model of the large number of phenomena in sciences such as conduction and diffusion in fluid dynamics, theory of semiconductors and catalytic processes in chemistry and biology. It is of great interest to describe the behavior of the exact solution of these problems, especially inside the layers.

Standard numerical methods give unsatisfactory results, so that various special procedures such as special grids, introduction of relaxation parameters, and special discretizations were constructed.

Standard spectral approximation also fails when applied to singularly perturbed problems. In several papers (see e.g. [2]) the author has developed the modification of standard spectral approximation, which assumes the division of the basic interval $[0,1]$ by the special procedure in such a way that the length of the layer intervals is addapted to the truncated orthogonal series which is used to approximate the exact solution inside the layers. Out of the layers the reduced solution $y_R(x) = \frac{f(x)}{g(x)}$ was used to approximate the exact solution. This procedure has given very accurate results for small values of the perturbation parameter, even when low degree orthogonal polynomials were used.

The error of that approximation depends on the error at the division point, which is pretty large when ε is not very small and the degree of the truncated orthogonal series is low. This paper gives an improvement in such cases.

In the first part of the paper the problem (1),(2) will be transformed in such a way that we divide the exact solution in three pieces: left and right layer solutions (inner solution) and central solution (outer solution).

In the second part of the paper the procedure for the appropriate division will be described and the spectral approximation to both the inner and outer solution will be constructed.

In the third part of the paper a numerical example illustrating the advantage of the presented method, will be considered.

2. Transformation of the problem

Let us represent the exact solution in the form

$$y(x) = \begin{cases} u_l(x) & 0 \le x \le x_0 \\ u_c(x) & x_0 \le x \le 1 - x_1 \\ u_r(x) & 1 - x_1 \le x \le 1, \end{cases} \tag{5}$$

where $u_l(x)$ and $u_r(x)$ are the left and right layer solutions. They are determined by the boundary value problems

$$L_\varepsilon u_l \equiv -\varepsilon^2 u_l''(x) + g(x) u_l(x) = h(x), \quad 0 \le x \le x_0, \tag{6}$$

$$u_l(0) = A, \ u_l(x_0) = u_c(x_0). \tag{7}$$

and

$$L_\varepsilon u_r \equiv -\varepsilon^2 u_r''(x) + g(x) u_r(x) = h(x), \quad 1 - x_1 \le x \le 1, \tag{8}$$

$$u_r(1 - x_1) = u_c(1 - x_1), \ u_r(1) = B. \tag{9}$$

The central solution $u_c(x)$ satisfies the differential equation

$$L_\varepsilon u_c \equiv -\varepsilon^2 u_c''(x) + g(x) u_c(x) = h(x), \quad x_0 \le x \le 1 - x_1 \tag{10}$$

and the continuity conditions at the endpoints, given in (7) and (9)

$$u_c(x_0) = u_l(x_0), \ u_c(1 - x_1) = u_r(1 - x_1).$$

Here $x_0, x_1 \in (0, \frac{1}{2})$ denote the values that are going to be determined in the next section in such a way that the layer solutions can be approximated in the best possible way by the truncated orthogonal series.

3. Spectral approximation

The idea is to perform the domain decomposition, using the division points x_0 and $1 - x_1$, in such a way that both the inner solutions $u_l(x)$, $u_r(x)$ and outer solution $u_c(x)$ can be satisfactorily represented in the form of truncated low-degree orthogonal series, due to some orthogonal polynomial basis.

The main problem is how to determine the values x_0 and x_1. Numerical examples show that a very small change of these values may cause the error of the spectral approximation increases hundred times, or even more.

For the construction of these values we shall take into account the asymptotic behavior of the exact solution. The procedure will be carried out for the left layer solution, and we shall state the appropriate results for the right layer solution.

As the layer length is of the order $O(\varepsilon)$, we shall construct the division point x_0 in the form $x_0 = c\varepsilon$. The parameter c is determined by the special procedure, in such a way that it depends on the degree n of the spectral approximation for the layer solution $u_l(x)$, which is represented in the form of the truncated orthogonal series

$$u_n(x) = \sum_{k=0}^{n}{}' a_k T_k^*(x). \tag{11}$$

Here $T_k^*(x)$ denote the Chebyshev polynomials of the first kind upon $[0, x_0]$, and the notation $'a_k$ means that the summation involves $\frac{1}{2}a_0$ instead of a_0.

Instead of the Chebyshev polynomials we can use any other orthogonal polynomial basis.

The procedure for determining the division point x_0 bases on the introduction of the *resemblance function* for the layer solution $u_l(x)$ and, similary to the procedure described in [1], it is given by the following definition, lemma and theorem:

Definition 1. *The resemblance function for the left layer solution is the polynomial $p_n(x)$ of the degree $n \geq 2$, such that*

1. $p_n(0) = A$ *and* $p_n(x_0) = y_R(0)$,

2. *x_0 is the stationary point for $p_n(x)$,*

3. *$p_n(x)$ is concave if $A > y_R(0)$ and convex if $A < y_R(0)$.*

Lemma 1 *The resemblance function for the left layer solution is given by*

$$p_n(x) = y_R(0) + (A - y_R(0))\left(1 - \frac{x}{x_0}\right)^n, \; n \geq 2. \tag{12}$$

Proof. We have to verify the conditions from Definition 1.

1.
$$p_n(0) = y_R(0) + (A - y_R(0))\left(1 - \frac{0}{x_0}\right)^n = A$$

and
$$p_n(x_0) = y_R(0) + (A - y_R(0))\left(1 - \frac{x_0}{x_0}\right)^n = y_R(0).$$

2.
$$p_n'(x) = -\frac{n(A - y_R(0))}{x_0}\left(1 - \frac{x}{x_0}\right)^{n-1}.$$

If the left layer exists, according to (4), $A - y_R(0) \neq 0$, and we have that $p_n'(x) = 0$ only for $x = x_0$, so x_0 is the stationary point.

3.
$$p_n''(x) = \frac{n(n-1)(A - y_R(0))}{x_0^2}\left(1 - \frac{x}{x_0}\right)^{n-2},$$

so that
$$\operatorname{sgn} p_n''(x) = \operatorname{sgn}(A - y_R(0)).$$

If $A < y_R(0)$ we see that $p_n''(x) < 0$, which means that $p_n(x)$ is convex, and if $A > y_R(0)$ we see that $p_n''(x) > 0$, which means that $p_n(x)$ is concave.

In order to determine the division point x_0 we shall ask that the resemblance function has to satisfy the differential equation at the layer point $x = 0$. This will give us

Theorem 1. *The value c that determines the division point $x_0 = c\varepsilon$ is given by*

$$c = \sqrt{\frac{n(n-1)}{g(0)}}. \tag{13}$$

Proof. If we introduce (12) into the differential equation (6), at the point $x = 0$ we obtain

$$-\varepsilon^2 \frac{n(n-1)}{c^2\varepsilon^2}(A - y_R(0)) + g(0)A = h(0).$$

Whit respect to $h(0) = g(0)y_R(0)$, the solution of the above equation for c, $c > 0$ will give us (13).

The existence of the square rooth in (13) is provided by the assumption (3).

The same procedure for the right layer solution gives us that the value $x_1 = d\varepsilon$ in the expression for the division point $1 - x_1$ is determined by

$$d = \sqrt{\frac{n(n-1)}{g(1)}}. \tag{14}$$

Once the division points x_0 and $1 - x_1$ are determined, we introduce the streaching variables

$$t = \frac{2x}{x_0} - 1, \quad t = \frac{2x-2}{x_1} + 1, \quad t = \frac{2(x - x_0)}{1 - (x_0 + x_1)} - 1,$$

which transform the layer subintervals $[0, x_0]$ and $[1 - x_1, 1]$ and the central subinterval $[x_0, 1 - x_1]$ into $[-1, 1]$.

Thus, transforming (6)-(10), we come to the problems

$$L_l w \equiv -\frac{4}{c^2} w''(t) + G_l(t)w(t) = H_l(t), \ -1 \le t \le 1, \tag{15}$$

$$w(-1) = A, \quad w(1) = v(-1), \tag{16}$$

$$L_c v \equiv -\frac{4\varepsilon^2}{1 - (c+d)^2\varepsilon^2} v''(t) + G_c(t)v(t) = H_c(t), \ -1 \le t \le 1, \tag{17}$$

$$v(-1) = w(1), \quad v(1) = z(-1), \tag{18}$$

$$L_r z \equiv -\frac{4}{d^2} z''(t) + G_r(t)z(t) = H_r(t), \ -1 \le t \le 1, \tag{19}$$

$$z(-1) = v(1), \quad z(-1) = B. \tag{20}$$

In these equations we have used the notation

$$G_l(t) = g\left(\frac{x_0(t+1)}{2}\right), \quad H_l(t) = h\left(\frac{x_0(t+1)}{2}\right),$$

$$G_c(t) = g\left(\frac{(1-x_0-x_1)(t+1)+2x_0}{2}\right),$$
$$H_c(t) = h\left(\frac{(1-x_0-x_1)(t+1)+2x_0}{2}\right),$$
$$G_r(t) = g\left(\frac{x_1(t-1)+2}{2}\right), \quad H_r(t) = h\left(\frac{x_1(t-1)+2}{2}\right).$$

The obtained problems can now be solved by the use of standard spectral technique, approximating $w(t)$ $v(t)$ and $z(t)$ by

$$(21)\ w_n(t) = \sum_{k=0}^{n}{}' a_k T_k(t), \quad v_n(t) = \sum_{k=0}^{n}{}' b_k T_k(t), \quad z_n(t) = \sum_{k=0}^{n}{}' c_k T_k(t).$$

$T_k(t)$ denote the classical Chebyshev polynomials of the first kind upon $[-1,1]$.

The coefficients a_k, b_k and c_k, $k = 0,1,\ldots,n$, can be determined by collocation method. Thus, we come to the following theorem:

Theorem 2. *The coefficients a_k, b_k and c_k, $k = 0,1,\ldots,n$ represent the solution of the system of $3n+3$ linear equations*

$$(22) \sum_{k=0}^{n}{}'(-\frac{4g(0)}{n(n-1)}T_k''(t_j) + G_l(t_j)T_k(t_j))a_k = H_l(t_j),\ j = 1,\ldots,n-1$$

$$(23) \qquad \sum_{k=0}^{n}{}'(-1)^k a_k = A, \quad \sum_{k=0}^{n}{}' a_k = \sum_{k=0}^{n}{}'(-1)^k b_k,$$

$$\sum_{k=0}^{n}{}'(-\frac{4\varepsilon^2 g(0)g(1)}{g(0)g(1)-\varepsilon^2 n(n-1)(g(0)+g(1))}T_k''(t_j) + G_c(t_j)T_k(t_j))b_k = H_c(t_j),$$
$$(24) \qquad j = 0,\ldots,n$$

$$(25) \sum_{k=0}^{n}{}'(-\frac{4g(1)}{n(n-1)}T_k''(t_j) + G_r(t_j)T_k(t_j))c_k = H_r(t_j),\ j = 1,\ldots,n-1$$

$$(26) \qquad \sum_{k=0}^{n}{}'(-1)^k c_k = \sum_{k=0}^{n}{}' b_k, \quad \sum_{k=0}^{n}{}' c_k = B,$$

where

$$t_j = \cos\frac{j\pi}{n}, \quad j = 1,\ldots,n-1$$

are the Gauss-Lobatto nodes.

Proof. The first $n-1$ equations (22) are obtained approximating $w(t)$ in (15) by the truncated Chebyshev series $w_n(t)$ defined in (21) and making use of (13). From the request that the obtained equality has to be satisfied at the Gauss-Lobatto nodes we come to (22).

The next two equations (23) are obtained directly from the boundary conditions (16), approximating $w(t)$ by $w_n(t)$ with respect to $T_k(-1) = (-1)^k$ and $T_k(1) = 1$.

The equations (24) are obtained by approximating $v(t)$ by $v_n(t)$, defined in (21), in the differential equation (17), making use of (13) and (14), and collocating the obtained equality, not only at the Gauss-Lobatto nodes, but at the endpoints -1 and 1, which are obtained for $j = 0$ and $j = n$. This has given us the next $n+1$ equations.

Finaly, the $n-1$ equations (25) are obtained by approximating $z(t)$ in the differential equation (19) by the truncated Chebyshev series $z_n(t)$, defined in (21), making use of (14) and collocating the obtained equality at the Gauss-Lobatto nodes.

The last two equations (26) are obtained by approximating $z(t)$ by $z_n(t)$ in the boundary conditions (20).

Let us remark that the boundary conditions (18) imply the second equation in (23) and the first equation in (26).

The collocation of the central solution at the endpoints is of the vital importance for the accuracy of the proposed method because it enables the accurate boundary conditions at the division points.

4. Numerical example

As a numerical example we shall consider the boundary layer problem, from [4]

$$-\varepsilon^2 y''(x) + y(x) = -\cos^2 \pi x - 2(\varepsilon\pi)^2 \cos 2\pi x,\ 0 \le x \le 1, \tag{27}$$

$$y(0) = 0, \quad y(1) = 0. \tag{28}$$

Here, the reduced solution is $y_R(x) = -\cos^2 \pi x$, so we have boundary layers at both endpoints. The results are given only for the left layer, and for the right layer they are identical.

For the numerical results we have tested several values for ε and the low-degree truncated orthogonal series with n=4. The layer subinterval, according to (13) is $[0, x_0] = [0, 2\sqrt{3}\varepsilon]$.

The error inside the layer is evaluated by the method presented in this paper and compared to the error obtained by the previously constructed method, where the reduced solution was used. The results are given in the following Table:

ε	2^{-2}	2^{-4}	2^{-8}
presented method	0.014	0.03128	0.031301112
use of reduced solution	0.042	0.03130	0.031301113

If ε is smaller both methods give the results of the same accuracy.

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