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ON CONGRUENCE CLASSES OF n-GROUPS

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Abstract

The following proposition is well known in the group theory: if (Q, A) is a group (2-group) and θ its congruence relation, then there is exactly one $C_a \in Q/\theta$ such that (C_a, A) is a subgroup of the group (Q, A). However, for $n \geq 3$, for instance, there are n-groups (Q, A) and their congruences θ such that for any $C_a \in Q/\theta$ the pair (C_a, A) is not an n-group (:4.1, 4.3). The main results of the paper are Theorems 3.1, 3.2 and 5.1.

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1. Preliminaries

1.1. About the expression a_p^q

Let $p \in \mathbf{N}$, $q \in \mathbf{N} \cup \{0\}$, and let a be a mapping of the set $\{i|\ i \in \mathbf{N} \wedge i \geq p \wedge i \leq q\}$ into the set S; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, \ldots, a_q; & p < q \\ a_p; & p = q \\ \text{empty sequence } (= \emptyset); & p > q. \end{cases}$$

For example:
$A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1})$, $j \in \{1, \ldots, n\}$, $n \in \mathbf{N} \backslash \{1,2\}$, for $j = n$ stands for

$$A(a_1, \ldots, a_{n-1}, A(a_n, \ldots, a_{2n-1})).$$

Besides, in some situations **instead of a_p^q we write** $(a_i)_{i=p}^q$ (briefly: $(a_i)_p^q$).

For example:

$$(\forall x_i \in Q)_1^q$$

for $q > 1$ stands for

$$\forall x_1 \in Q \ldots \forall x_q \in Q$$

[usually, we write: $(\forall x_1 \in Q) \ldots (\forall x_2 \in Q)$],
for $q = 1$ it stands for

$$\forall x_1 \in Q$$

[usually, we write: $(\forall x_1 \in Q)$],
and for $q = 0$ it stands for an empty sequence ($= \emptyset$).

In **some cases**, instead of a_p^q only, we write: sequence a_p^q (sequence a_p^q over a set S). For example: ... for every sequence a_p^q over a set S ... . And if $p \leq q$, we usually write: $a_p^q \in S$.

If a_p^q is a sequence over a set S, $p \leq q$ and the equalities $a_p = \ldots = a_q = b$ ($\in S$) are satisfied, then

$$a_p^q \textbf{ is denoted by } \overset{q-p+1}{b}.$$

In connection with this, if $q - p + 1 = r$ (when we assume that there would be no missunderstanding),

$$\textbf{instead of } \overset{q-p+1}{b} \textbf{ we write } \overset{r}{b}.$$

In addition, we denote **the empty sequence over** S with $\overset{0}{b}$, where b is an arbitrary element from S.

1.2. About n-groups

Let $A : Q^n \to Q$ and $n \in \mathbf{N} \setminus \{1\}$. Then:

1) (Q, A) is said to be an n-semigroup iff for every $x_1^{2n-1} \in Q$ and for every $i \in \{2, \dots, n\}$ the equality

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1})$$

is satisfied;

2) (Q, A) is said to be an n-quasigroup iff for every $i \in \{1, \dots, n\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds; and

3) (Q, A) is said to be an n-group iff it is both, an n-semigroup and an n-quasigroup. (For $n = 2$ it is a group. The notion of an n-group has been introduced in [1].)

1.3. On a $\{1, n\}$-neutral operation in an n-groupoid

Let (Q, A) be an n-groupoid and $n \in \mathbf{N} \setminus \{1\}$. Let also $\mathbf{e}$ be an $(n-2)$-ary operation in Q; for $n = 2$ this is a nullary operation. We say that $\mathbf{e}$ is a $\{1, n\}$-neutal operation in the n-groupoid (Q, A) iff the following holds:

$$(1) \quad \begin{aligned} &(\forall a_i \in Q)_1^{n-2} \, (\forall x \in Q) \, (A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x \wedge \\ &\wedge A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x). \end{aligned}$$

For $n = 2$, $\mathbf{e}(a_1^0)$ $(= \mathbf{e}(\emptyset)) = e \in Q$ is a neutral element of the groupoid (Q, A). The notion of an $\{i, j\}$-neutral operation of an n-groupoid $(:n \in \mathbf{N} \setminus \{1\}, \{i, j\} \subseteq \{1, \dots, n\}, i \neq j)$ has been introduced in [3]. The following propositions hold:

1.3.1 [3]: *In an n-groupoid* $(n \in \mathbf{N} \setminus \{1\})$ *there is at most one* $\{1, n\}$*-neutral operation;*

1.3.2 [3]: *In every n-group,* $n \in \mathbf{N} \setminus \{1\}$*, there is a* $\{1, n\}$*-neutral operation*[1]*;*

1.3.3 [3]: *For* $n \geq 3$*, an n-semigroup* (Q, A) *is an n-group iff* (Q, A) *has a* $\{1, n\}$*-neutral operation.* □

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[1] The cases $\{i, j\} \neq \{1, n\}$ $(n \geq 3)$ were described in [5].

By virtue of Proposition 1.3.3, the **universal algebra** $(Q,\{A,\mathbf{e}\})$ which satisfies (1) and

$$(2)\qquad (\forall x_i \in Q)_1^{2n-1}\,(\bigwedge_{j=2}^{n} A(A(x_1^n),x_{n+1}^{2n-1}) = A(x_1^{j-1},A(x_j^{j+n-1}),x_{j+n}^{2n-1})),$$

for $n \geq 3$, is considered to be an n-**group**.

1.4. On inversing operation in an n-group

The following **proposition** holds [4]:

1.4.1: *Let (Q,A) be an n-semigroup and $n \in \mathbf{N}\setminus\{1\}$. Then:*

a) There is at most one $(n-1)$-ary operation f in Q such that the following formulas hold

(1)

$$(\forall a_i \in Q)_1^{n-2}\,(\forall a \in Q)\,(\forall x \in Q)\;\; A(f(a_1^{n-2},a),a_1^{n-2},A(a,a_1^{n-2},x)) = x$$

and

(2)

$$(\forall a_i \in Q)_1^{n-2}\,(\forall a \in Q)\,(\forall x \in Q)\;\; A(A(x,a_1^{n-2},a),a_1^{n-2},f(a_1^{n-2},a)) = x;$$

b) If there is an $(n-1)$-ary operation f in Q such that the formulas (1) and (2) are satisfied, then (Q,A) is an n-group; and

c) If (Q,A) is an n-group, then there is an $(n-1)$-ary operation f in Q such that the formulas (1) and (2) hold[2]. □

Therefore, a **universal algebra** $(Q,\{A,f\})$ satisfying (1), (2) and

$$(3)\qquad (\forall x_i \in Q)_1^{2n-1}\,(\bigwedge_{j=2}^{n} A(A(x_1^n),x_{n+1}^{2n-1}) = A(x_1^{j-1},A(x_j^{j+n-1}),x_{j+n}^{2n-1}))$$

is also taken to be an n-**group**.

As for the case $n = 2$ we say that the operation f is an **inversing operation in the** n-**group** (Q,A); [4].

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[2] $f(a_1^{n-2},a) \stackrel{def}{=} \mathsf{E}(a_1^{n-2},a,a_1^{n-2})$, where E is a $\{1,2n-1\}$-neutral operation of a $(2n-1)$-group $(Q,\overset{2}{A})$; $\overset{2}{A}(x_1^{2n-1}) \stackrel{def}{=} A(A(x_1^n),x_{n+1}^{2n-1})$. We note that for $n=2$, this is the inversing in a group.

1.5. On superpositions of an n-semigroup operation

1.5.1: *Let (Q, B) be an n-groupoid and $n \in \mathbf{N} \setminus \{1\}$. Then: 1) $\overset{1}{B} \overset{def}{=} B$; and 2) for every $k \in \mathbf{N}$ and for every $x^{(k+1)(n-1)+1} \in Q$*

$$\overset{k+1}{B}(x_1^{(k+1)(n-1)+1}) \overset{def}{=} B(\overset{k}{B}(x_1^{k(n-1)+1}), x_{k(n-1)+2}^{(k+1)(n-1)+1}).$$

1.5.2: *Let (Q, B) be an n-***semigroup***, $n \in \mathbf{N} \setminus \{1\}$ and $(i, j) \in \mathbf{N}^2$. Then, for every $x^{(i+j)(n-1)+1} \in Q$ and for every $t \in \{1, \ldots, i(n-1)-1\}$, the following equality holds*

$$\overset{i+j}{B}(x_1^{(i+j)(n-1)+1}) = \overset{i}{B}(x_1^{t-1}, \overset{j}{B}(x_t^{t+j(n-1)}), x_{t+j(n-1)+1}^{(i+j)(n-1)+1}).$$

An immediate consequence of Proposition 1.5.2 is the following proposition:

1.5.3: *If (Q, B) is an n-semigroup (n-group) and $k \in \mathbf{N} \setminus \{1\}$, then $(Q, \overset{k}{B})$ is a $(k(n-1)+1)$-semigroup ($(k(n-1)+1)$-group).*

Remark: More about superpositions of an n-semigroup operation (with different notations) can be found in [2].

1.6. On congruences of m-groupoids

1.6.1: *Let (Q, Φ) be an m-groupoid and $m \in \mathbf{N}$. Further on, let θ be an equivalence relation in Q. Then we say that θ is a* **congruence** *relation on the m-groupoid (Q, Φ) iff the following statement holds*

$$(1) \qquad (\forall a_j \in Q)_1^m \, (\forall b_j \in Q)_1^m \, ((\bigwedge_{i=1}^{m} a_i \theta b_i) \Rightarrow \Phi(a_1^m) \theta \Phi(b_1^m)).$$

The statement (1) is equivalent to the statement

$$(2) \quad (\forall a \in Q)\,(\forall b \in Q)\,(\forall c \in Q)_1^m \quad (\bigwedge_{i=1}^{m} (a \theta b \Rightarrow \Phi(c_1^{i-1}, a, c_i^{m-1}) \theta \Phi(c_1^{i-1}, b, c_i^{m-1}))).$$

Moreover, the following statement holds: If

$$(3) \qquad \Phi(C_{x_1},\ldots,C_{x_m}) \overset{def}{=} C_{\Phi(x_1^m)}$$

for every $C_{x_1},\ldots,C_{x_m} \in Q/\theta$, *then* $(Q/\theta,\Phi)$ *is an* m*-groupoid. We say that* $(Q/\theta,\Phi)$ *is a* **factor** m**-groupoid of the** m**-groupoid** (Q,Φ) **over the congruence** θ.

θ *is a congruence of a universal algebra* (Q,Ω) *iff* θ *is a congruence of the* m_i*-groupoid* (Q,Φ_i) *for every* $\Phi_i \in \Omega$. □

The following propositions hold:

1.6.2: *If a universal algebra* $(Q,\{A,f\})$ *is an* n*-group (:1.4),* $n \in \mathbf{N}\setminus\{1\}$, *and* θ *its congruence relation, then* $(Q/\theta,\{\mathbf{A},\mathbf{F}\})$, *where*

$$(3_1) \qquad \mathbf{A}(C_{x_1},\ldots,C_{x_n}) \overset{def}{=} C_{A(x_1^n)}$$

and

$$(3_2) \qquad \mathbf{F}(C_{x_1},\ldots,C_{x_{n-1}}) \overset{def}{=} C_{f(x_1^{n-1})},$$

is also an n*-group.*

[The sketch of the part of the proof:

$$\mathbf{A}(\mathbf{F}(C_{x_1},\ldots,C_{x_{n-2}},C_a),C_{x_1},\ldots,C_{x_{n-2}},\mathbf{A}(C_a,C_{x_1},\ldots,C_{x_{n-2}},C_x)) =$$
$$\mathbf{A}(C_{f(x_1^{n-2},a)},C_{x_1},\ldots,C_{x_{n-2}},C_{A(a,x_1^{n-2},x)}) =$$
$$C_{A(f(x_1^{n-2},a),x_1^{n-2},A(a,x_1^{n-2},x))} = C_x \ (:1.4).]$$

1.6.3: *If a universal algebra* $(Q,\{A,\mathbf{e}\})$ *is an* n*-group (:1.3),* $n \in \mathbf{N}\setminus\{1,2\}$, *and* θ *its congruence relation, then* $(Q/\theta,\{\mathbf{A},\mathbf{E}\})$, *where* $\mathbf{A}$ *is defined by* (3_1) *and*

$$(3_3) \qquad \mathbf{E}(C_{x_1},\ldots,C_{x_{n-2}}) \overset{def}{=} C_{\mathbf{e}(x_1^{n-2})},$$

is also an n*-group.*

1.6.4 [6]: *Let* $n \in \mathbf{N}\setminus\{1\}$, (Q,A) *an* n*-group,* f *its inversing operation (:1.4) and* $\mathbf{e}$ *its* $\{1,n\}$*-neutral operation (:1.3). Then, the following statements hold: (i) if* θ *is a congruence on the* n*-groupoid* (Q,A), *then* θ *is also a congruence of the* $(n-1)$*-groupoid* (Q,f)*; and (ii) is* θ *is a congruence on the* n*-groupoid* (Q,A) *and* $n \geq 3$, *then* θ *is also a congruence of the* $(n-2)$*-groupoid* $(Q,\mathbf{e})$.

2. Auxiliary statements

Proposition 2.1. *Let $n \in \mathbf{N} \setminus \{1\}$, let (Q, A) be an n-group and f its inversing operation (:1.4). Then, for every $a \in Q$ the following implication holds:*

$$A(\overset{n}{a}) = a \Rightarrow f(\overset{n-1}{a}) = a.$$

Proof.

Let a be an arbitrary element of the set Q such that the following equality holds

(1) $$A(\overset{n}{a}) = a.$$

Moreover, by proposition 1.4.1, we conclude that the following equality holds

$$A(f(\overset{n-2}{a}, a), \overset{n-2}{a}, A(a, \overset{n-2}{a}, a)) = a,$$

whence, by (1), we conclude that the following equality holds

(2) $$A(f(\overset{n-1}{a}), \overset{n-1}{a}) = a.$$

Finally, since (Q, A) is an n-quasigroup (:1.2), we conclude that (1) and (2) imply the following equality:

$$f(\overset{n-1}{a}) = a.$$

Proposition 2.2. *Let $n \in \mathbf{N} \setminus \{1, 2\}$, let (Q, A) be an n-group and* $\mathbf{e}$ *its $\{1, n\}$-neutral operation (:1.3). Then for every $a \in Q$ the following equivalence holds:*

$$A(\overset{n}{a}) = a \Leftrightarrow \mathbf{e}(\overset{n-2}{a}) = a.$$

Proof.

1) $\Rightarrow$:

Let a be an arbitrary element of the set Q such that the following equality holds:

(1) $$A(\overset{n}{a}) = a.$$

In addition, the following equality holds

$$A(\mathbf{e}(\overset{n-2}{a}), \overset{n-2}{a}, a) = a \quad (: 1.3). \tag{2}$$

Finally, since (Q, A) is an n-quasigroup (:1.2), we conclude that (1) and (2) imply the following equality:

$$\mathbf{e}(\overset{n-2}{a}) = a. \tag{3}$$

2) $\Leftarrow$:

Let a be an arbitrary element of the set Q such that the equality (3) holds. Since with (3) also hold (2) (:1.3), we conclude that (1) also holds.

Proposition 2.3. *Let Q be a* **finite set**, *$n \in \mathbb{N} \setminus \{1\}$ and (Q, A) an n-group. Then the following statement holds:*

$$(\forall a \in Q)\,(\exists k \in \mathbb{N})\ \overset{k}{A}(\overset{k(n-1)+1}{a}) = a$$

(:1.5.1).

Proof.

Let a be an arbitrary element of the set Q. Then,

$$\{\overset{t}{A}(\overset{t(n-1)+1}{a}) \mid t \in \mathbb{N}\} \subseteq Q \quad (: 1.5.1).$$

Hence, since Q is a finite set, there is $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that the following statements hold

$$(\exists k \in \mathbb{N})\ \ i + k = j \text{ and}$$

$$\overset{i}{A}(\overset{i(n-1)+1}{a}) = \overset{j}{A}(\overset{j(n-1)+1}{a}),$$

whence, by Proposition 1.5.2 and since (Q, A) is an n-quasigroup (:1.2), we conclude that the following series of implications hold

$$\overset{i}{A}(\overset{i(n-1)+1}{a}) = \overset{i+k}{A}(\overset{(i+k)(n-1)+1}{a}) \Rightarrow$$

$$\overset{i}{A}(\overset{i(n-1)+1}{a}) = \overset{i}{A}(\overset{k}{A}(\overset{k(n-1)+1}{a}), \overset{i(n-1)}{a}) \Rightarrow$$

$$\overset{k}{A}(\overset{k(n-1)+1}{a}) = a.$$

Proposition 2.4. *Let $n \in \mathbf{N} \setminus \{1\}$ and let (Q, A) be an n-group. Further on, let θ be a congruence of the n-group (Q, A) and let $(Q/\theta, \mathbf{A})$ be the factor n-group of the n-group (Q, A) over the congruence θ. Then, the $(k(n-1)+1)$-group $(Q/\theta, \overset{k}{\mathbf{A}})$ is a factor $(k(n-1)+1)$-group of the $(k(n-1)+1)$-group $(Q, \overset{k}{A})$ over the congruence θ for an arbitrary $k \in \mathbf{N}$.*

Proof.

For an arbitrary $k \in \mathbf{N}$ $(Q, \overset{k}{A})$ $[(Q/\theta, \overset{k}{\mathbf{A}})]$ is a $(k(n-1)+1)$-group (:1.5.3). The congruence θ of the n-group (Q, A) is also a congruence of a $(k(n-1)+1)$-group $(Q, \overset{k}{A})$ for an arbitrary $k \in \mathbf{N}$ (:1.5, 1.6, induction over k). Let $(Q/\theta, \mathbf{B})$ be a factor $(k(n-1)+1)$-group of the $(k(n-1)+1)$-group $(Q, \overset{k}{A})$ over the congruence θ. Then, for every $C_{x_1}, \ldots, C_{x_{k(n-1)+1}} \in Q/\theta$ the following equality holds

$$\mathbf{B}(C_{x_1}, \ldots, C_{x_{k(n-1)+1}}) = C_{\overset{k}{A}(x_1^{k(n-1)+1})} \tag{1}$$

(:1.6). On the other hand, for every $C_{x_1}, \ldots, C_{x_{k(n-1)+1}} \in Q/\theta$ the following equality holds

$$\overset{k}{\mathbf{A}}(C_{x_1}, \ldots, C_{x_{k(n-1)+1}}) = C_{\overset{k}{A}(x_1^{k(n-1)+1})}. \tag{2}$$

[The sketch of the proof: a) $\overset{2}{\mathbf{A}}(C_{x_1}, \ldots, C_{x_n}, C_{x_{n+1}}, \ldots, C_{x_{2n-1}}) = \mathbf{A}(\mathbf{A}(C_{x_1}, \ldots, C_{x_n}), C_{x_{n+1}}, \ldots, C_{x_{2n-1}}) = \mathbf{A}(C_{A(x_1^n)}, C_{x_{n+1}}, \ldots, C_{x_{2n-1}}) = C_{A(A(x_1^n), x_{n+1}^{2n-1})} = C_{\overset{2}{A}(x_1^{2n-1})}$; b) $\overset{t}{\mathbf{A}}(C_{x_1}, \ldots, C_{x_{t(n-1)+1}}) = C_{\overset{t}{A}(x_1^{t(n-1)+1})} \Rightarrow \overset{t+1}{\mathbf{A}}(C_{x_1}, \ldots, C_{x_{(t+1)(n-1)+1}}) = C_{\overset{t+1}{A}(x_1^{(t+1)(n-1)+1})}$ (:1.5, 1.6).]

Finally, since (1) and (2) hold for every $C_{x_1}, \ldots C_{x_{k(n-1)+1}} \in Q/\theta$, we conclude that $\mathbf{B} = \overset{k}{\mathbf{A}}$.

3. Main results

Theorem 3.1. *Let $n \in \mathbf{N} \setminus \{1\}$ and let $(Q, \{A, f\})$ be an n-group (:1.4).*

Further on, let θ be a congruence of the universal algebra (n-group) $(Q,\{A,f\})$[3] *and $(Q/\theta,\{\mathbf{A},\mathbf{F}\})$ the factor n-group of the n-group $(Q,\{A,f\})$ over the congruence θ (:1.6.1). Then, the following statement holds: for an arbitrary $C_a \in Q/\theta$, $a \in Q$, $(C_a,\{A,f\})$ is an n-group (n-subgroup of the n-group $(Q,\{A,f\})$ iff the following equality holds*

$$\mathbf{A}(\overset{n}{C_a}) = C_a.$$

Proof.

1) Let C_a be an arbitrary element of the set Q/θ. Then, since θ is a congruence of the n-group $(Q,\{A,f\})$, for every $x_1^n \in Q$, the following equalities hold

$$\mathbf{A}(C_{x_1},\ldots,C_{x_n}) = C_{A(x_1^n)}$$

and

$$\mathbf{F}(C_{x_1},\ldots,C_{x_{n-1}}) = C_{f(x_1^{n-1})} \quad (:1.6.1, 1.6.2),$$

whence we conclude that for every $x_1^n \in C_a$ the following equalities hold

(1) $$\mathbf{A}(\overset{n}{C_a}) = C_{A(x_1^n)}$$

and

(2) $$\mathbf{F}(\overset{n}{C_a}) = C_{f(x_1^{n-1})}.$$

2) $\Rightarrow$:

Let $(C_a,\{A,f\})$ be an n-group. Then, for every $x_1^n \in C_a$ the following equalities hold

$$C_{A(x_1^n)} = C_a \text{ and } C_{f(x_1^{n-1})} = C_a,$$

whence, by (1) [and (2)], we conclude that the equality

(3) $$\mathbf{A}(\overset{n}{C_a}) = C_a$$

holds [and also the equality

(4) $$\mathbf{F}(\overset{n-1}{C_a}) = C_a$$

[3] If $(Q,\{A,f\})$ is an n-group and θ a congruence of the n-groupoid (Q,A), then, θ is a congruence also on the $(n-1)$-groupoid (Q,f) (:1.6.4).

holds].

3) $\Leftarrow$:

Let (3) holds. Then, by Proposition 2.1, the equality (4) holds. Further on, by (1) and (2), we conclude that for every $x_1^n \in C_a$

$$A(x_1^n) \in C_a \text{ and } f(x_1^{n-1}) \in C_a.$$

Theorem 3.2. *Let $n \geq 3$ and let $(Q, \{A, \mathbf{e}\})$ be an n-group (:1.3). Further on, let θ be a congruence of the universal algebra (n-group) $(Q, \{A, \mathbf{e}\})$[4] and $(Q/\theta, \{\mathbf{A}, \mathbf{E}\})$ the factor n-group of the n-group $(Q, \{A, \mathbf{e}\})$ over the congruence θ (:1.6.1). Then, for arbitrary $C_a \in Q/\theta$, $a \in Q$, the following statements are equivalent:*

(i) $(C_a, \{A, \mathbf{e}\})$ is an n-group [n-subgroup of the n-group $(Q, \{A, \mathbf{e}\})$];

(ii) the following equality holds

$$\mathbf{A}(\overset{n}{C_a}) = C_a;$$

(iii) the following equality holds

$$\mathbf{E}(\overset{n-2}{C}_a) = C_a.$$

Proof.

Let $(Q, \{A, f\})$ be an n-group and $\mathbf{e}$ its $\{1, n\}$-neutral operation (:1.3). Then for $n \geq 3$ the universal algebras $(Q, \{A, f\})$ and $(Q, \{A, \mathbf{e}\})$ uniquely represent the n-group (Q, A) (:1.4, 1.3.3). In addition, if θ is a congruence of the n-groupoid (Q, A), θ is a congruence of the $(n-1)$-groupoid (Q, f) and $(n-2)$-groupoid $(Q, \mathbf{e})$ (:1.6.4). Whence, by Theorem 3.1, we conclude that for every $C_a \in Q/\theta$ the following equivalence holds

$$(i) \Leftrightarrow (ii).$$

Finally, by Proposition 2.2, we conclude that for every $C_a \in Q/\theta$ also the following equivalence holds

$$(ii) \Leftrightarrow (iii).$$

[4] If $(Q, \{A, \mathbf{e}\})$ is an n-group ($n \geq 3$) and θ a congruence of the n-groupoid (Q, A), then θ is also a congruence of the $(n-2)$-groupoid $(Q, \mathbf{e})$ (:1.6.4).

4. Two examples and a proposition

4.1. Example: Let $(\{1,2,3,4\},\cdot)$ be Klein's group: Table 1. Then, $(\{1,2,3,4\},A)$, where

$$A(x_1^3)\stackrel{def}{=}x_1\cdot x_2\cdot x_3\cdot 3$$

for every $x_1^3\in\{1,2,3,4\}$, is a 3-group;
Tables 2_1-2_4, $A_i(x,y)\stackrel{def}{=}A(x,i,y)$, $i\in\{1,2,3,4\}$ [:$A_1(x,y)=x\cdot y\cdot 3$, $A_2(x,y)=x\cdot y\cdot 4$, $A_3(x,y)=x\cdot y$, $A_4(x,y)=x\cdot y\cdot 2$].

·	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 1.

A_1	1	2	3	4
1	3	4	1	2
2	4	3	2	1
3	1	2	3	4
4	2	1	4	3

Table 2_1

A_2	1	2	3	4
1	4	3	2	1
2	3	4	1	2
3	2	1	4	3
4	1	2	3	4

Table 2_2

A_3	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 2_3

A_4	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

Table 2_4

The equivalence relation θ in the set $\{1,2,3,4\}$ defined by the equality

$$\{1,2,3,4\}/\theta=\{\{1,2\},\{3,4\}\}$$

is a congruence relation of the 3-group $(\{1,2,3,4\},A)$ (:Tables 2_1-2_4). The corresponding factor 3-group $(\{\{1,2\},\{3,4\}\},\mathbf{A})$ is represented in Tables 3_1-3_2[5]. We can see from Table 3_1-3_2 that $\{1,3\}$-neutral operation $\mathbf{E}$ of the 3-group $(\{\{1,2\},\{3,4\}\},\mathbf{A})$ is the following permutation of the set $\{\{1,2\},\{3,4\}\}$

$$\begin{pmatrix}\{1,2\} & \{3,4\}\\ \{3,4\} & \{1,2\}\end{pmatrix}.$$

$\mathbf{A}_{\{1,2\}}$	$\{1,2\}$	$\{3,4\}$
$\{1,2\}$	$\{3,4\}$	$\{1,2\}$
$\{3,4\}$	$\{1,2\}$	$\{3,4\}$

Table 3_1

Since $\mathbf{E}(\{1,2\})=\{3,4\}$ and $\mathbf{E}(\{3,4\})=\{1,2\}$, then by Theorem 3.2, we conclude that the pairs $(\{1,2\},A)$ and $(\{3,4\},A)$ are not 3-groups.

$\mathbf{A}_{\{3,4\}}$	$\{1,2\}$	$\{3,4\}$
$\{1,2\}$	$\{1,2\}$	$\{3,4\}$
$\{3,4\}$	$\{3,4\}$	$\{1,2\}$

Table 3_2

4.2. Example: Let $(\{1,2,3,4\},\cdot)$ be the Klein's group: Table 1. Then, $(\{1,2,3,4\},B)$, where

$$B(x_1^3)\stackrel{def}{=}x_1\cdot x_2\cdot x_3\cdot 2$$

[5] $\mathbf{A}_{\{1,2\}}(x,y)\stackrel{def}{=}\mathbf{A}(x,\{1,2\},y)$, $\mathbf{A}_{\{3,4\}}(x,y)\stackrel{def}{=}\mathbf{A}(x,\{3,4\},y)$; $x,y\in\{\{1,2\},\{3,4\}\}$.

for every $x_1^3 \in \{1,2,3,4\}$, is a 3-group; Table $4_1 - 4_4$, $B_i(x,y) \overset{def}{=} B(x,i,y)$, $i \in \{1,2,3,4\}$ [:$B_1(x,y) = x \cdot y \cdot 2$, $B_2(x,y) = x \cdot y$, $B_3(x,y) = x \cdot y \cdot 4$, $B_4(x,y) = x \cdot y \cdot 3$].

B_1	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

Table 4_1

B_2	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 4_2

B_3	1	2	3	4
1	4	3	2	1
2	3	4	1	2
3	2	1	4	3
4	1	2	3	4

Table 4_3

B_4	1	2	3	4
1	3	4	1	2
2	4	3	2	1
3	1	2	3	4
4	2	1	4	3

Table 4_4

The equivalence relation θ in the set $\{1,2,3,4\}$ defined by the equality

$$\{1,2,3,4\}/\theta = \{\{1,2\},\{3,4\}\}$$

is a congruence relation of the 3-group $(\{1,2,3,4\},B)$ (:Table $4_1 - 4_4$). The corresponding factor 3-group $(\{\{1,2\},\{3,4\}\},\mathbf{B})$ is represented in Table 5_1-5_2. Since $\mathbf{B}(\{1,2\},\{1,2\},\{1,2\}) = \{1,2\}$ (:Table 5_1) and $\mathbf{B}(\{3,4\},\{3,4\},\{3,4\}) = \{3,4\}$ (:Table 5_2), then by Theorem 3.1 (or Theorem 3.2) we conclude that the pairs $(\{1,2\},B)$ and $(\{3,4\},B)$ are 3-groups. They are represented, respectively in Table $6_1 - 6_2$ and Table $7_1 - 7_2$.

$\mathbf{B}_{\{1,2\}}$	$\{1,2\}$	$\{3,4\}$
$\{1,2\}$	$\{1,2\}$	$\{3,4\}$
$\{3,4\}$	$\{3,4\}$	$\{1,2\}$

Table 5_1

$\mathbf{B}_{\{3,4\}}$	$\{1,2\}$	$\{3,4\}$
$\{1,2\}$	$\{3,4\}$	$\{1,2\}$
$\{3,4\}$	$\{1,2\}$	$\{3,4\}$

Table 5_2

B_1	1	2
1	2	1
2	1	2

Table 6_1

B_2	1	2
1	1	2
2	2	1

Table 6_2

B_3	3	4
3	4	3
4	3	4

Table 7_1

B_4	3	4
3	3	4
4	4	3

Table 7_2

In the group theory ($n = 2$) the following proposition is well known: if (Q,A) is a group, $^{-1}$ its inverting operation and θ its congruence, then there is a normal subgroup (H,A) of (Q,A) such that

(1) $H \in Q/\theta$; and

(2) for every $a,b \in Q$ the equivalence

$$a\theta b \Leftrightarrow A(a^{-1},b) \in H$$

holds. By the cited proposition and by Example 4.1 and Example 4.2, we conclude that the following proposition holds:

Proposition 4.3. *a) Let (Q, A) be an arbitrary n-group, θ its arbitrary congruence and $n = 2$. Then, there is* **exactly one** *$C_a \in Q/\theta$ such that (C_a, A) is an n-group (:group). b) If $n \geq 3$, then*

(i) there exist an n-group (Q, A) and its congruence θ such that **for every** *$C_a \in Q/\theta$ the pair (C_a, A)* **is not** *an n-group; and*

(ii) there exist an n-group (Q, A) and its congruence θ such that **for every** *$C_a \in Q/\theta$ the pair (C_a, A)* **is** *an n-group.*

5. On n-groups with finite factor n-groups

Theorem 5.1. *Let $n \in \mathbf{N} \setminus \{1\}$, let (Q, A) be an n-group, θ its congruence relation and $(Q/\theta, \mathbf{A})$ the factor n-group of the n-group (Q, A) over the congruence θ. Then: if Q/θ is a finite set, then for every $C_a \in Q/\theta$ there is $k \in \mathbf{N}$ such that $(C_a, \overset{k}{A})$ is a $(k(n-1)+1)$-group.*

Proof.

Let C_a be an arbitrary element of the set Q/θ. Since $(Q/\theta, \mathbf{A})$ is a finite n-group, then by Proposition 2.3, there is $k \in \mathbf{N}$ such that

$$\overset{k}{\mathbf{A}}(\overset{k(n-1)+1}{C_a}) = C_a. \tag{1}$$

Further on, by Proposition 2.4, $(Q/\theta, \overset{k}{\mathbf{A}})$ is a factor $(k(n-1)+1)$-group of the $(k(n-1)+1)$-group $(Q, \overset{k}{A})$ over the congruence θ. Hence, by (1) and by Theorem 3.1, we conclude that $(C_1, \overset{k}{A})$ is a $(k(n-1)+1)$-group.

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