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MEDIAL CYCLIC n-QUASIGROUPS

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Abstract

An n-quasigroup (Q, f) is cyclic if $f(x_1, \ldots, x_n) = x_{n+1} \Leftrightarrow f(x_2, \ldots, x_{n+1}) = x_1$ for all $x_1, \ldots, x_{n+1} \in Q$, and it is called medial if $f(y_1, \ldots, y_n) = f(z_1, \ldots, z_n)$, where $y_i = f(x_{i1}, \ldots, x_{in})$, $z_j = f(x_{1j}, \ldots, x_{nj})$, for all $x_{ij} \in Q$, $i, j \in \{1, \ldots, n\}$. Some properties of medial cyclic n-quasigroups and n-loops are determined, and a complete description of medial cyclic n-quasigroups is given. Some sufficient conditions for the existence of self-orthogonal medial cyclic n-quasigroups are obtained. It is proved that every medial cyclic n-loop is a commutative n-group.

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1. Introduction and Definitions

A quasigroup is called semisymmetric if it satisfies the identity $x(yx) = y$, and it is called idempotent if the identity $xx = x$ holds. Cyclic n-quasigroups defined in [8] represent a generalization of semisymmetric quasigroups. Some classes of cyclic n-quasigroups are equivalent to Mendelsohn designs.

A quasigroup satisfying the identity $(xy)(uv) = (xu)(yv)$ is called medial. Mediality is an affine property and it serves as a characterization of affine geometries among Steiner systems. Mediality is also related to mean-value theory [1]. Generalizations of mediality to n-quasigroups were considered by Belousov V. D. [3],[4].

In this paper, medial cyclic n-quasigroups and n-loops will be considered. We shall give characterizations of such n-quasigroups and n-loops and obtain some results on the existence of some classes of these n-quasigroups.

First we give some basic definitions and notations.

The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by x_m^n or $\{x_i\}_{i=m}^n$. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \ldots, x$ (m times) will be denoted by $\overset{m}{x}$. If $m \leq 0$, then $\overset{m}{x}$ will be considered empty.

An n-ary groupoid (n-groupoid) (Q, f) is called an n-quasigroup if the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \ldots, n\} = \mathbb{N}_n$.

An n-quasigroup (Q, f) is called idempotent if for every $x \in Q$ $f(\overset{n}{x}) = x$.

An n-quasigroup (Q, f) is called an n-loop if there exists an element $e \in Q$ such that $f(\overset{i-1}{e}, x, \overset{n-i}{e}) = x$ for all $x \in Q$ and all $i \in \mathbb{N}_n$, and e is called a unit of that n-loop.

An n-quasigroup (Q, f) is called (i,j)-associative if the following identity holds

$$f(x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-1}) = f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

An n-quasigroup which is (i,j)-associative for all $i, j \in \mathbb{N}_n$ is called an n-group.

An n-quasigroup (Q, f) is medial if $f(y_1^n) = f(z_1^n)$, where $y_i = f(\{x_{ij}\}_{j=1}^n)$, $z_j = f(\{x_{ij}\}_{i=1}^n)$ for all $x_{ij} \in Q$, $i, j \in \mathbb{N}_n$.

By S_n we denote the symmetric group of degree n.

If (Q, f) is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup f^σ defined by

$$f^\sigma(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of f.

An n-quasigroup (Q, f) is called

a) totally symmetric if $f = f^\sigma$ for all $\sigma \in S_{n+1}$,

b) cyclic if $f = f^\sigma$ for all $\sigma \in C$, where C is a subgroup of S_{n+1} generated by the cycle $(12\dots n+1)$ ([8],[10]).

c) commutative if $f = f^\sigma$ for all $\sigma \in S_{n+1}$ such that $\sigma(n+1) = n+1$.

An n-quasigroup (Q, f) is cyclic iff the following identities hold

$$f(x_{i+1}^n, f(x_1^n), x_1^{i-1}) = x_i,\ i = 1, \dots, n.$$

The transpose of a quasigroup $(Q, \cdot)$ is the quasigroup $(Q, *)$ where $*$ is the binary operation defined by $x * y = yx$. A quasigroup $(Q, \cdot)$ is self-orthogonal if it is orthogonal to its transpose, that is, for all $a, b \in Q$ the system $xy = a,\ xy = b$ has a unique solution. The self-orthogonality of semisymmetric quasigroups was considered in [2],[5],[6],[7], and some of these results were generalized in [10].

The set $\{(Q, f_1), \dots, (Q, f_n)\}$ of n-quasigroups is said to be orthogonal if for each $(a_1^n) \in Q^n$, there exist a unique $(b_1^n) \in Q^n$ such that $f_i(b_1^n) = a_i,\ i = 1, \dots, n$. If (Q, f) is an n-quasigroup such that the set $\{f, f_1, \dots, f_{n-1}\}$ is orthogonal, where $f_i,\ i = 1, \dots, n-1$ are the parastrophes of f defined by $f_i(x_1^n) = f(x_{i+1}^n, x_1^i),\ \ i = 1, \dots, n-1$, then (Q, f) is called a self-orthogonal n-quasigroup.

By ε we denote the identity mapping of the set Q.

2. Medial cyclic n-quasigroups

Theorem 1. *Let (Q, f) be an n-quasigroup. (Q, f) is a medial cyclic n-quasigroup if and only if there exists a commutative group $(Q, +)$ such that*

$$f(x_1^n) = \varphi x_1 - \varphi^2 x_2 + \varphi^3 x_3 - \dots + (-1)^{n-1}\varphi^n x_n + b, \tag{1}$$

where φ is an automorphism of the group $(Q, +)$, $\varphi^{n+1} = \varepsilon$ when n is odd, $\varphi^{n+1} = -\varepsilon$ when n is even, and $\varphi b = -b$.

Proof. Let (Q, f) be a medial cyclic n-quasigroup. Since (Q, f) is medial by [3] it follows that there exists a commutative group $(Q, +)$ such that

$$f(x_1^n) = \sum_{i=1}^{n} \theta_i x_i + b,$$

where θ_i, $i = 1, \dots, n$, are automorphisms of the group $(Q, +)$, $\theta_i\theta_j = \theta_j\theta_i$ for all $i, j \in \mathbb{N}_n$ and b is a fixed element from Q.

Since (Q, f) is cyclic it satisfies the identities $f(x_{i+1}^n, f(x_1^n), x_1^{i-1}) = x_i$, $i = 1, \dots, n$, hence

$$\theta_1(\theta_1 x_1 + \dots + \theta_n x_n + b) + \theta_2 x_1 + \dots + \theta_n x_{n-1} + b = x_n, \tag{2}$$

$$\theta_1 x_2 + \dots + \theta_{n-1} x_n + \theta_n(\theta_1 x_1 + \dots + \theta_n x_n + b) + b = x_1, \tag{3}$$

and

$$\begin{aligned} \theta_1 x_{i+1} + \dots + \theta_{n-i} x_n + \theta_{n-i+1}(\theta_1 x_1 + \dots + \theta_n x_n + b) \\ + \theta_{n-i+2} x_1 + \dots + \theta_n x_{i-1} + b = x_i, \end{aligned} \tag{4}$$

for $i = 2, \dots, n-1$.

Putting $x_1 = \dots = x_n = 0$ in (2),(3) and (4), we get

$$\theta_{n-i+1} b = -b,$$

that is, $\theta_i b = -b$ for all $i \in \mathbb{N}_n$.

From (2) for $x_2 = \dots = x_n = 0$ it follows $\theta_1^2 = -\theta_2$, and for $x_1 = x_3 = \dots = x_n = 0$ we get $\theta_1\theta_2 = -\theta_3$. By a similar procedure, from (2),(3) and (4) it follows that

$$\theta_i\theta_j = -\theta_k \tag{5}$$

for all $i, j \in \mathbb{N}_n$, where $k = i + j$ for $i + j \leq n$, $k = i + j - (n+1)$ for $i + j > n + 1$, and $\theta_k = -\varepsilon$ for $i + j = n + 1$.

From (5) we get that $\theta_2 = -\theta_1^2$, $\theta_3 = \theta_1^3$, $\theta_4 = -\theta_1^4, \dots, \theta_n = (-1)^{n-1}\theta_1^n$ and also that $\theta_1^{n+1} = \varepsilon$ for n odd, $\theta_1^{n+1} = -\varepsilon$ for n even.

Hence we have obtained that

$$f(x_1^n) = \varphi x_1 - \varphi^2 x_2 + \varphi^3 x_3 - \dots + (-1)^n \varphi^{n-1} x_n + b$$

where $\varphi = \theta_1$.

The converse part of the theorem is straightforward. □

Theorem 2. *Every medial cyclic n-quasigroup (Q, f) defined by*

$$f(x_1^n) = \varphi x_1 - \varphi^2 x_2 + \varphi^3 x_3 - \dots + (-1)^{n-1} \varphi^n x_n$$

where $(Q, +)$ is a commutative group, φ an automorphism of the group $(Q, +)$, $\varphi^{n+1} = \varepsilon$ when n is odd, $\varphi^{n+1} = -\varepsilon$ when n is even, and $\varphi + \varepsilon$ is a bijection, is idempotent.

Proof. If n is odd, then $\varphi^{n+1} - \varepsilon = 0$. Since

$$\varphi^{n+1} - \varepsilon = (\varphi + \varepsilon)(-\varepsilon + \varphi - \varphi^2 + \cdots - \varphi^{n-1} + \varphi^n)$$

and $\varphi + \varepsilon$ is a bijection, we get that

$$\varphi - \varphi^2 + \cdots - \varphi^{n-1} + \varphi^n = \varepsilon.$$

Hence for every $a \in Q$

$$f(\overset{n}{a}) = \varphi a - \varphi^2 a + \cdots - \varphi^{n-1}a + \varphi^n a = (\varphi - \varphi^2 + \cdots - \varphi^{n-1} + \varphi^n)a = a.$$

If n is even, then $\varphi^{n+1} + \varepsilon = 0$, but since

$$\varphi^{n+1} + \varepsilon = (\varphi + \varepsilon)(\varepsilon - \varphi + \varphi^2 - \cdots + \varphi^n)$$

we obtain $\varphi - \varphi^2 + \cdots - \varphi^n = \varepsilon$, which gives that (Q, f) is idempotent. □

Since in every commutative group $(Q, +)$ the automorphism φ defined by $\varphi(x) = -x$ satisfies all conditions of Theorem 1, we get the following theorem.

Theorem 3. *For every $v \in \mathbb{N}$ there exists a medial cyclic n-quasigroup of order v.*

In [10], self-orthogonal cyclic n-quasigroups were investigated. In Theorem 1 from [10] n-quasigroups defined by (1) with $b = 0$ were considered, but the proof of that theorem applies also to n-quasigroups with $b \neq 0$. So, Theorem 1 from [10] implies the following theorem.

Theorem 4. *Every medial cyclic n-quasigroup (Q, f) defined by*

$$f(x_1^n) = \varphi x_1 - \varphi^2 x_2 + \varphi^3 x_3 - \cdots + (-1)^n \varphi^{n-1} x_n + b,$$

where $(Q, +)$ is a commutative group, φ an automorphism of the group $(Q, +)$, $\varphi b = -b$, $\varphi^{n+1} = \varepsilon$ when n is odd, $\varphi^{n+1} = -\varepsilon$ when n is even, and $\varphi + \varepsilon$ is a bijection, is self-orthogonal.

Some results on the existence of self-orthogonal medial cyclic n-quasigroups can also be obtained from [10]. These results are given in the next two theorems.

Theorem 5. *If $n, v \geq 3$ are odd numbers, then there exists a self-orthogonal medial cyclic n-quasigroup of order v.*

If n is an even positive integer, $p_1, \dots, p_m$ primes and $\alpha_1, \dots, \alpha_m$ positive integers such that $p_i^{\alpha_i} \equiv 1 \pmod{s_i}$, where $s_i > 1$ is a divisor of $n+1$, $i = 1, \dots, m$, then for arbitrary nonnegative integers β_i, $i = 1, \dots, m$, there exists a self-orthogonal medial cyclic n-quasigroup of the order

$$v = p_1^{\alpha_1 \beta_1} \dots p_m^{\alpha_m \beta_m}.$$

Theorem 6. *Let $n \geq 2$ be a positive integer and $p_1, \dots, p_s$ primes such that $p_i > n+1$, $i = 1, \dots, s$. Then there are positive integers $k_1, \dots, k_s$, $1 \leq k_i \leq n$, $i = 1, \dots, s$, such that, for all positive integers α_i, $i = 1, \dots, s$ there is a self-orthogonal medial cyclic n-quasigroup of the order*

$$v = p_1^{k_1 \alpha_1} \dots p_s^{k_s \alpha_s}.$$

3. Medial cyclic n-loops

Now we shall consider medial cyclic n-loops.

Theorem 7. *Every medial n-loop is commutative.*

Proof. Let (Q, f) be a medial n-loop and e a unit of that n-loop. We shall prove that $f = f^{(ij)}$ for every $i, j \in \mathbb{N}_n$.

Since (Q, f) is medial we have the following identity

$$(6) \quad f(f(\{x_{1i}\}_{i=1}^n), \dots, f(\{x_{ni}\}_{i=1}^n)) = f(f(\{x_{i1}\}_{i=1}^n), \dots, f(\{x_{in}\}_{i=1}^n))$$

If $i, j \in \mathbb{N}_n$, $i < j$, and if in (6) we replace by e all variables except x_{ij}, x_{ji} and x_{kk}, $k = 1, \dots, i-1, i+1, \dots, j-1, j+1, \dots, n$, we get

$$f(x_{11}, \dots, x_{i-1,i-1}, x_{ij}, x_{i+1,i+1}, \dots, x_{j-1,j-1}, x_{ji}, x_{j+1,j+1}, \dots, x_{nn}) =$$

$$f(x_{11}, \dots, x_{i-1,i-1}, x_{ji}, x_{i+1,i+1}, \dots, x_{j-1,j-1}, x_{ij}, x_{j+1,j+1}, \dots, x_{nn}).$$

This means that $f = f^{(ij)}$ for every $i, j \in \mathbb{N}_n$, hence (Q, f) is commutative. □

Corollary 1. *Every medial cyclic n-loop is totally symmetric.*

Proof. If (Q,f) is a medial cyclic n-loop, by the preceding theorem it follows that (Q,f) is commutative. Combining this and the cyclicity of (Q,f) we get that (Q,f) is totally symmetric. □

Theorem 8. *If (Q,f) is a medial cyclic n-loop, then (Q,f) is $(1,n)$-associative and $(i,i+1)$-associative for all $i \in \mathbb{N}_{n-1}$.*

Proof. Let (Q,f) be a medial cyclic n-loop and e a unit of that n-loop. Then

$$f(f(x_1^n), f(\overset{n-1}{e}, x_{n+1}), \ldots, f(\overset{n-1}{e}, x_{2n-1})) =$$

$$f(f(x_1, \overset{n-1}{e}), \ldots, f(x_{n-1}, \overset{n-1}{e}), f(x_n^{2n-1})),$$

that is,

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^{n-1}, f(x_n^{2n-1})),$$

hence (Q,f) is $(1,n)$-associative. (Q,f) is also cyclic and from Theorem 1 of [9] it follows that (Q,f) is $(i,i+1)$-associative for all $i,j \in \mathbb{N}_{n-1}$. □

Theorem 2 from [9] implies the following corollary.

Corollary 2. *Every medial cyclic n-loop is an n-group.*

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