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SOME NONLINEAR SPP AND SPECTRAL APPROXIMATION

Nevenka Adžić
Faculty of Technical Sciences, University of Novi Sad
Trg D. Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

We shall consider the selfadjoint singularly perturbed problem described by the second order diferrential equation. The solution inside the layer is approximated by Newton's iteration represented in the form of truncated orthogonal series due to the Chebyshev basis. For that purpose, the domain decomposition is performed according to the suitable resemblance function. The coefficients of the spectral approximation are determined by the collocation method at Gauss-Lobatto nodes. The error function is estimated according to the principle of inverse monotonicity, using the asymptotic behavior of the exact solution. Numerical results show high accuracy of the presented method.

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1. Introduction

We shall consider the singularly perturbed problem

(1) $$Tu(x) \equiv -\varepsilon^2 u''(x) + b(x,u) = 0, \quad x \in [0,1]$$

$$(2) \qquad u(0) = 0, \quad u(1) = 0,$$

where $\varepsilon > 0$ is a small parameter, $b_u(x,u) \geq \beta^2 > 0$, $(x,u) \in [0,1] \times R$ and $b(x,u) \in C^2([0,1] \times R)$.

It is well known (see e.g. [5]) that under the above assumptions the problem (1), (2) has two boundary layers, where the values of the exact solution change extremely rapidly. The problems of this kind are involved in mathematical models of diffusion-convection phenomena and it has been recognized that certain difficulties arise when standard spectral approximations are applied in the cases where ε is very small. The author has already developed a modification of the standard spectral methods for some linear singularly perturbed problems of selfadjoint type (see e.g. [1]). In that paper the author has introduced the domain decomposition by using the so-called *resemblance function*, and the coefficients of the spectral approximation were evaluated directly from the obtained linear system. In this paper the author adapts these ideas to the given nonlinear case. The domain decomposition is performed using a modified resemblance function, and the approximate solution inside the layer is represented in the form of truncated orthogonal series due to the Chebyshev orthogonal basis. The approximation is evaluated iteratively by Newton's method, which allows the coefficients of the spectral approximation to be evaluated for each iteration as the solution of the appropriate linear system of equations. Such system is obtained by the collocation method using Gauss-Lobatto nodes. This combination gives a highly accurate method using only a small number of terms in the appropriate truncated series. This is confirmed by the numerical results.

In Section 2, the original problem will be transformed and the layer subintervals will be determined using the same idea as in the case of self-adjoint linear problems. In Section 3, the approximate solution will be constructed using Newton's iterative method. In Section 4, the upper bound function for the error estimate will be constructed, and in Section 5 the theoretical results will be illustrated by a numerical example.

2. Transformation of the problem

It was shown in [4] that for the problem (1),(2), under the given assumptions, we have the unique exact solution $u_\varepsilon(x) \in C^4[0,1]$, for which the following estimate holds

$$|u_\varepsilon(x) - z(x)| \le M_0 e^{\frac{-\beta x}{\varepsilon}} + M_1 e^{\frac{-\beta(1-x)}{\varepsilon}} + M_2\varepsilon^2, \tag{3}$$

where $z(x)$ represents the solution of reduced problem $b(x,u) = 0$ and M_i, $i = 0,1,2$ are constants independent of x and ε, such that

$$M_0 \ge |z(0)|,\ M_1 \ge |z(1)|,\ M_2 \ge \frac{z''(x)}{\beta^2},\ x \in [0,1].$$

From (3) we can see that, in general, we have two boundary layers of the order $O(\varepsilon)$.

It is known that under the given assumptions there exist the unique reduced solution $z(x)$ which belongs to the class $C^2[0,1]$.

We are going to approximate the exact solution of (1),(2) by

$$\tilde{u}(x) = \begin{cases} u_l(x) & x \in [0, c\varepsilon] \\ z(x) & x \in [c\varepsilon, 1-c\varepsilon] \\ u_r(x) & x \in [1-c\varepsilon, 1] \end{cases}. \tag{4}$$

The function $u_l(x)$ approximates the left layer solution and it represents the solution of the boundary value problem

$$Tu_l(x) = 0\ ,\ u_l(0) = 0\ ,\ u_l(c\varepsilon) = z(c\varepsilon) \tag{5}$$

and $u_r(x)$ approximates the right layer solution and represents the solution of the problem

$$Tu_r(x) = 0\ ,\ u_r(1-c\varepsilon) = z(1-c\varepsilon)\ ,\ u_r(1) = 0.$$

All further investigations will be carried out for the left layer solution.

The value c, which determines the division points $c\varepsilon$ and $1-c\varepsilon$ is going to be determined by the same technique as in the linear case (see [1]), using the following definition and lemma

Definition 1. *A sum of the reduced solution and a function $p_m(x) \in C^2[0,c\varepsilon]$ is called a resemblance function for the problem (5) if*

1. *it satisfies the boundary conditions in (5),*

2. $x = c\varepsilon$ *is the stationary point for* $p_m(x)$,

3. $p_m(x)$ *is concave for* $z(0) < 0$ *and convex for* $z(0) > 0$.

Lemma 1. *The function*

$$r(x) = z(x) + p_m(x) = z(x) - z(0)\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^m \ , \ m \in N \ , \ m \geq 2. \tag{6}$$

is a resemblance function for the problem (5).

Proof. We have to verify the conditions from Def. 1.

1. $r(0) = z(0) - z(0)\left(\frac{c\varepsilon - 0}{c\varepsilon}\right)^m = 0$
   and
   $r(c\varepsilon) = z(c\varepsilon) - z(0)\left(\frac{c\varepsilon - c\varepsilon}{c\varepsilon}\right)^m = z(c\varepsilon)$.

2. $p_m'(x) = \frac{m z(0)}{c\varepsilon}\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^{m-1} = 0$ only for $x = c\varepsilon$.

3. $p_m''(x) = -\frac{m(m-1)z(0)}{c\varepsilon}\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^{m-2}$, so that $\text{sgn} p_m''(x) = - \ \text{sgn} z(0)$.

The resemblance function enables us to determine the value c in the expression for the division point, in such a way that it depends on the degree m of the truncated orthogonal series, which is going to be used for the approximation of the layer solution. The value c is obtained from the request that the resemblance function satisfies the differential equation in (5) at the layer point $x = 0$. Geometrically, this will show us how far from the layer point $x = 0$ should we go if we want to provide that the approximate solution, represented as a sum of the reduced solution and truncated orthogonal series of degree m, resembles the layer solution $u_l(x)$.

Theorem 1. *The value* c*, which determines the division point* $c\varepsilon$*, is*

$$c = \sqrt{\frac{-z(0)m(m-1)}{b(0,0)}}, \tag{7}$$

when ε *is sufficiently small.*

Proof. Introducing (6) into the differential equation in (5) we obtain

$$-\varepsilon^2 z''(x) + \frac{m(m-1)z(0)}{c^2}\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^{m-2} + b(x, r(x)) = 0.$$

At the layer point $x = 0$, with respect to $r(0) = 0$, the above equality becomes

$$-\varepsilon^2 z''(0) + \frac{m(m-1)z(0)}{c^2} + b(0,0) = 0.$$

For ε sufficiently small, this will give us the quadratic equation in c

$$\frac{m(m-1)z(0)}{c^2} + b(0,0) = 0.$$

The positive solution of this equation is given by (7).

The existence of the square root in (7) is provided by the fact that for $z(0) > 0$ the layer solution is convex, ie. $u''(x) < 0$ for $x \in [0, c\varepsilon]$, and from (1) we can see that $\mathrm{sgn} b(x,u) = \mathrm{sgn} u''(x)$. Thus, $b(0,0) < 0$. If $z(0) < 0$ the layer solution is concave, which implies that $b(0,0) > 0$.

Now we can proceed to construct the approximate solution for the problem (5) using Newton's iteration method combined with the spectral approximation.

3. Costruction of the approximate solution

We shall look for the approximate solution of the problem (5) in the form

$$v_n(x) = y_n(x) + \frac{z(c\varepsilon)x}{c\varepsilon}, \tag{8}$$

where $y_n(x)$ is obtained using the n-th iteration in Newton's method, representing it as

$$y_n(x) = \sum_{k=0}^{m}{}' a_k T_k\left(\frac{2x}{c\varepsilon} - 1\right), \tag{9}$$

i. e. the truncated orthogonal series of the degree m, due to the Chebysev orthogonal basis. (The notation $'a_k$ means that the summation involves $\frac{1}{2}a_0$ rather than a_0.)

In order to obtain the coefficients a_k, $k = 0, 1, ..., m$ we have to transform first the layer subinterval $[0, c\varepsilon]$ into $[-1, 1]$, using the streching variable $t = \frac{2x}{c\varepsilon} - 1$. Thus, the finite series (9) becomes

$$(10) \qquad w_n(t) = \sum_{k=0}^{m}{}' a_k T_k(t), \;\; T_k(t) = \cos(k \cdot \arccos t), \;\; k = 0, 1, ...$$

and it represents the n-th iteration of Newton's method for the problem

$$(11) \qquad w''(t) + c(t, w) = 0, \;\; w(-1) = 0, \;\; w(1) = 0,$$

where

$$c(t, w) = -\frac{c^2}{4} \cdot b\left(\frac{c\varepsilon}{2}(t+1), u + \frac{z(c\varepsilon)}{2}(t+1)\right).$$

If we assume that the function $c(t, w)$ is continuous on $[-1, 1] \times R$ and satisfies

a) general Lipschitz condition

$$K_1(v - \omega) \leq c(t, v) - c(t, \omega) \leq K_2(v, \omega), \;\; v - \omega \geq 0, \;\; K_1, K_2 \in R, \;\; K_2 \leq \frac{\pi^2}{4},$$

b) $c_w(t, w)$ exists, is continous and concave, or convex, i. e.

$$c(t, \omega) - c(t, v) \leq (\omega - v) \cdot c_w(t, v) \;\; \text{or} \;\; c(t, \omega) - c(t, v) \geq (\omega - v) \cdot c_w(t, v)$$

then Newton's iteration sequence $w_n(t)$, defined by

$$(12) \qquad w_n''(t) + (w_n(t) - w_{n-1}(t))c_w(t, w_{n-1}(t)) + c(t, w_{n-1}(t)) = 0,$$

$$w_n(-1) = 0, \;\; w_n(1) = 0$$

converges monotonically and uniformly to the exact solution of the problem (11), starting with $w_0(t) \equiv 0$. (For the proof see [2].)

If we introduce (10) into (12) at each iteration, and ask that the obtained equation is satisfied at Gauss-Lobatto nodes $t_i = \cos\frac{i\pi}{m}$, $i = 1, 2, ..., m-1$, we come to the system of $m+1$ equations with $m+1$ unknown coefficients a_k, $k = 0, 1, ...m$. The solution of this system gives the approximate solution (8).

4. The error estimate

Out of the boundary layer, the exact solution of the problem (1), (2) is approximated by the solution of the reduced problem. The error estimate is given by (3).

Let us now estimate the error upon the layer subinterval $(0, c\varepsilon]$. The error function, according to (8) is

$$d(x) = |u_\varepsilon(x) - v_n(x)| \le |u_\varepsilon(x) - u_l(x)| + |u_l(x) - v_n(x)|. \tag{13}$$

In order to estimate it, we have, first, to prove the following lemma:

Lemma 2. *Let $b(x,u) \in C^2([0,c\varepsilon] \times R)$ and $b_u(x,u) \ge \beta^2$, $\beta \in R$ for $x \in (0, c\varepsilon]$. Then*

$$|u_\varepsilon(x) - u_l(x)| \le C(\varepsilon^2 + e^{-\beta c}) \text{ for } x \in (0, c\varepsilon]. \tag{14}$$

Proof. The function $u_\varepsilon(x)$ satisfies the boundary value problem

$$Tu_\varepsilon(x) = 0,\ x \in (0, c\varepsilon],\ u_\varepsilon(0) = 0,\ u_\varepsilon(c\varepsilon) = u_0. \tag{15}$$

Subtracting (5) from (15) we obtain

$$T(u_\varepsilon - u_l)(x) = 0,\ (u_\varepsilon - u_l)(0) = 0,\ (u_\varepsilon - u_l)(c\varepsilon) = u_\varepsilon(c\varepsilon) - z(c\varepsilon).$$

Under the given assumptions the operator T is inverse monotone. So, by the principle of inverse monotonicity we can conclude that

$$|u_\varepsilon(x) - u_l(x)| \le |u_\varepsilon(c\varepsilon) - z(c\varepsilon)|. \tag{16}$$

Using the estimate (3) for $x = c\varepsilon$, as $e^{\frac{\beta(1-c\varepsilon)}{\varepsilon}}$ tends to zero when ε is sufficiently small, we obtain (14).

Theorem 2. *Let $\omega_n(t)$ represent the solutions of the equation*

$$\omega_n''(t) + K_2\omega(t) = K_2 w_n(t) - c(t, w_n(t)) \quad t \in (-1,1), \tag{17}$$
$$\omega_n(-1) = 0,\ \omega_n(1) = 0$$

where K_2 is the constant under generalized Lipshitz condition a) and $w_n(t)$ is the n-th Newton's iteration. Then the error $d(x)$, defined by (13), can be estimated as

$$d(x) \le C(\varepsilon^2 + e^{c\beta}) + \left| y_n(x) - \omega\left(\frac{2x}{c\varepsilon} - 1\right)\right|, \quad x \in (0, c\varepsilon]. \tag{18}$$

Proof. The proof will be carried out for the case when $c(t,w)$ is concave. Then Newton's iteration sequence $w_n(t)$ converges monotonically and uniformly downwards to the exact solution of the problem (11), so that $w_n(t) \geq w(t)$. This implies that

$$(w(t) - \omega_n(t))'' + K_2(w(t) - \omega_n(t)) \leq 0$$

and so, by the principle of inverse monotonicity, we conclude $w(t) \geq \omega_n(t)$.

If we define

$$v_n(x) = \omega_n\left(\frac{2x}{c\varepsilon} - 1\right) + \frac{z(c\varepsilon)x}{c\varepsilon}$$

then, the following estimate holds

$$v_n(x) \leq u_l(x) \leq v_n(x),$$

which implies

$$(19) \qquad |u_l(x) - v_n(x)| \leq |v_n(x) - v_n(x)| = \left|\omega_n\left(\frac{2x}{c\varepsilon} - 1\right) - y_n(x)\right|$$

Thus, using (14) and (18) in (13) we obtain (17).

5. Numerical example

We shall use the following test example, given in [6]:

$$-\varepsilon^2 u'' + (1+u)(1+(1+u)^2) = 0\ ,$$
$$u(0) = 0\ , \quad u(1) = 0.$$

It can be easily seen that

$$b_u(x,u) = 1 + 3(1+u)^2 \geq 1,$$

which provides the existence and uniqness of the exact solution and the error estimate at the division point.

The reduced solution is $z(x) \equiv -1$.

The following results are given for $\varepsilon = 2^{-16}$, using the truncated Chebyshev series of degree $m = 4$, $m = 6$ and $m = 8$. The calculation is performed in three iterations. The maximal difference between the exact solution and the asymptotic one, together with the appropriated values of the number c, which determines the layer subinterval, are presented in Table 1.

	m=4	m=6	m=8
c	2.49	3.87	5.29
n=1	0.04	0.06	0.1
n=2	0.06	0.04	0.037
n=3	0.06	0.04	0.04

Tabela 1.

The maximal difference between the two iterations $y_{n+1}(x) - y_n(x)$ is presented in Table 2.

	m=4	m=6	m=8
c	2.49	3.87	5.29
n=2	0.00015	0.002	0.006
n=3	6 E-9	1.4 E-7	1.1 E-5
n=4		6 E-13	3.5 E-11

Tabela 2.

These results show that the presented method is highly accurate and that, using only a small number of iterations, we can achieve the best possible accuracy, which is limited by the error in the boundary data at the division point $c\varepsilon$.

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