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TRIANGULAR SOLUTIONS OF BOOLEAN EQUATIONS

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Abstract

We give an algorithm which determines the formulas of general reproductive solutions of a given Boolean equation in triangular form . The algorithm also makes simpler formulas of these solutions.

This algorithm also dos a simplification of the formulas of these solutions.

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Our basic terminology, related to Boolean equations, follows Rudeanu's book [6]. For everything about Boolean equations, not given here, see also [6].

Let $\mathcal{B} = (B, \cup, \cap, ', 0, 1)$ be a Boolean algebra, n be a natural number and $p = 2^n - 1$. Further, let $\{A_0, A_1, \ldots, A_p\} = \{0,1\}^n$. If $A \in \{0,1\}^n$ then the $j-$th coordinate of A will be denoted by $(A)_j$ i.e. $A = ((A)_1, \ldots, (A)_n)$. We shall also use the notation:

$$T_k = (t_1, \ldots, t_k) \ \ (k = 1, \ldots, n).$$

Especially, if $k = n$ we shall use the notation:

$$T = (t_1, \ldots, t_n) \ \text{ and } \ X = (x_1, \ldots, x_n).$$

Definition 1. *Let $f, \Phi_1, \ldots, \Phi_n : B^n \to B$ be Boolean functions and $\Phi = (\Phi_1, \ldots, \Phi_n)$. The formula*

$$X = \Phi(T)$$

or, in a scalar form

$$x_j = \Phi_j(t_1, \ldots, t_n) \quad (j = 1, \ldots, n)$$

defines a general solution of the consistent Boolean equation $f(X) = 0$ if and only if

$$(\forall X \in B^n) f(\Phi(X)) = 0 \wedge (\forall X \in B^n)(f(X) = 0 \Rightarrow (\exists T \in B^n) X = \Phi(T)).$$

Definition 2. *Let $x \in B$. Then*

$$x^1 = x, \quad x^0 = x'.$$

If $X = (x_1, \ldots, x_n) \in B^n$ and $A = (a_1, \ldots, a_n) \in \{0,1\}^n$ then

$$X^A = x_1^{a_1} \cdots x_n^{a_n}.$$

Theorem 1. *[6] The function $f : B^n \to B$ is Boolean if and only if it can be written in the canonical disjunctive form*

$$f(X) = \bigcup_{k=0}^{p} f(A_k) X^{A_k}.$$

Theorem 2. *[6] Let $f : B^n \to B$ be a Boolean function. The equation $f(X) = 0$ is consistent if and only if*

$$\prod_{k=0}^{p} f(A_k) = 0.$$

Theorem 3. *[2] Let $f : B^n \to B$ be a Boolean function. If $f(X) = 0$ is consistent, then the formula*

$$\begin{aligned} X = & \bigcup_{k=0}^{p} (f'(A_{i_{k,0}}) A_{i_{k,0}} \cup f(A_{i_{k,0}}) f'(A_{i_{k,1}}) A_{i_{k,1}} \cup \\ (1) \qquad & \cup f(A_{i_{k,0}} f(A_{i_{k,1}}) f'(A_{i_{k,2}}) A_{i_{k,2}} \\ & \cdots \cup f(A_{i_{k,0}}) f(A_{i_{k,1}}) f(A_{i_{k,2}}) \cdots f'(A_{i_{k,p}}) A_{i_{k,p}}) T^{A_k} \end{aligned}$$

defines a general solution of $f(X) = 0$*, under conditions*
$(i_{k,0}, i_{k,1}, \ldots, i_{k,p})$ *are permutations of* $\{0, 1, \ldots, p\}$
and
$(i_{0,0}, i_{1,0}, \ldots, i_{p,0})$ *is a permutation of* $\{0, 1, \ldots, p\}$.

Bearing in mind the method of successive eliminations, it is known that every consistent Boolean equation $f(x_1, \ldots, x_n) = 0$ has a triangular general reproductive solution of the form

$$\begin{aligned} x_1 &= g_1(t_1) \\ x_2 &= g_2(t_1, t_2) \\ &\vdots \\ x_n &= g_n(t_1, t_2, \ldots, t_n). \end{aligned}$$

We shall prove that the vector $A_{i_{k,j}}$ in Theorem 3 can be choosen such that the solution (1) is triangular.

Definition 3. *Let* $A = (a_1, \ldots, a_n) \in \{0, 1\}^n$*. We define a sequance*

$$B(A) = B_0(A),\ B_1(A), \ldots, B_p(A)$$

in the following way:

(I) $B_0(A) = (a_1, \ldots, a_n)$

$B_1(A) = (a_1, \ldots, a_{n-1}, a'_n)$

(II) *for every* $k \in \{1, \ldots, n-1\}$

$B_{2^k}(A) = (a_1, \ldots, a'_{n-k}, D_k(0))$

$B_{2^k+1}(A) = (a_1, \ldots, a'_{n-k}, D_k(1))$

$\vdots$

$B_{2^k+2^k-1}(A) = (a_1, \ldots, a'_{n-k}, D_k(2^k - 1))$

where $D_k(s)$ *is the* $k-$*tuple of the binary digits of the number* s $(s \in \{0, 1, \ldots, 2^k - 1\})$ *in binary expansion.*

Lemma 1. *Let* $A = (e_1, \ldots, e_k, e_{k+1}, \ldots, e_n) = (E_k, F_{n-k}) \in \{0, 1\}^n$*. Then for* $r \geq 0$

$$B_{2^{n-k}+r}(E_k, F_{n-k})$$

does not depend on F_{n-k}.

Proof. Bearing in mind Definition 1 we have

$$B_{2^{n-k}+r}(E_k, F_{n-k}) = (e_1, \ldots, e'_m, D_{n-m}(s))$$

(for some m and some s, where $m \leq k$ and $0 \leq s \leq 2^{n-m} - 1$)

$$= B_{2^{n-m}+s}(e_1, \ldots, e'_m, D_{n-m}(s)). \quad \square$$

Theorem 4. *Let $f(X) = 0$ be a consistent Boolean equation. The formula*

$$(2) \qquad \begin{aligned} X = \bigcup_A [&f'(B_0(A))B_0(A) \cup f(B_0(A))f'(B_1(A))B_1(A) \cup \cdots \\ &\cup f(B_0(A))f(B_1(A)) \cdots f'(B_p(A))B_p(A)]\, T^A \end{aligned}$$

defines a general solution of $f(X) = 0$.

Proof. Since

$$\{B_0(A), B_1(A), \ldots, B_p(A)\} = \{0,1\}^n$$

and

$$\{B_0(A) |\ A \in \{0,1\}^n\} = \{A |\ A \in \{0,1\}^n\} = \{0,1\}^n$$

the conditions of Theorem 3 are fulfilled.

Remark 1. Since the equation $f(X) = 0$ is consistent i.e.

$$f(B_0(A))f(B_1(A)) \cdots f(B_p(A)) = 0$$

we can omit $f'(B_p(A))$ from $F(B_0(A))f(B_1(A)) \cdots f'(B_p(A))$ because of $ab = 0 \Rightarrow ab' = a$.

Remark 2. The formula (1) can be written as

$$X = \bigcup_A \left[\bigcup_{i=0}^{p} f'(B_i(A))B_i(A) \prod_{j=0}^{i-1} f(B_j(A)) \right] T^A$$

i.e. we can write

$$(3) \quad x_k = \bigcup_A \left[\bigcup_{i=0}^{p} f'(B_i(A))(B_i(A))_k \prod_{j=0}^{i-1} f(B_j(A)) \right] T^A \quad (k = 1, \ldots, n)$$

(we assume that $\prod_{j=0}^{-1} f(B_j) = 1$).

Lemma 2. *Let* $A = (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) = (E_k, F_{n-k}) \in \{0,1\}^n$. *Then*

$$\bigcup_{i=0}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})) \tag{4}$$

does not depend on F_{n-k}.

Proof. The union (4) can be written as the union of two unions

$$\bigcup_{i=0}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})) =$$
$$= \bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))$$
$$\cup \bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k})).$$

Note that

$$i < 2^{n-k} \Rightarrow (B_i(E_k, F_{n-k}))_k = e_k, \tag{5}$$

because of Definition 3. The first union can be written as

$$\bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k}))(B_i(E_k, F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))$$
$$= e_k \bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k})) \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))$$

(because of (5))

$$= e_k \bigcup_{i=0}^{2^{n-k}-1} f'(B_i(E_k, F_{n-k}))$$

(because of $a'_1 \cup a_1 a'_2 \cup \cdots \cup a_1 a_2 \cdots a'_q = a'_1 \cup a'_2 \cup \cdots \cup a'_q$)

$$= e_k \cup_{i=0}^{2^{n-k}-1} f'(E_k, D_{n-k}(i))$$

(by Definition 3).

The second union can be written as

$$\bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k,F_{n-k}))(B_i(E_k,F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k,F_{n-k}))$$

$$= \bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k,F_{n-k}))(B_i(E_k,F_{n-k}))_k \cdot$$

$$\cdot \prod_{j=0}^{2^{n-k}-1} f(B_j(E_k,F_{n-k})) \prod_{j=2^{n-k}}^{i-1} f(B_j(E_k,F_{n-k}))$$

$$\text{(we assume that } \prod_{j=2^{n-k}}^{2^{n-k}-1} f(B_j) = 1)$$

$$= \prod_{j=0}^{2^{n-k}-1} f(E_k, D_{n-k}(j)) \bigcup_{i=2^{n-k}}^{p} f'(B_i(E_k,F_{n-k}))(B_i(E_k,F_{n-k}))_k \cdot$$

$$\cdot \prod_{j=2^{n-k}} f(B_j(E_k,F_{n-k})).$$

Since the latter union contains only the members of the form

$$B_{2^{n-k}+r}(E_k, F_{n-k})$$

where $r \geq 0$, it does not depend of F_{n-k}, because of Lemma 1.

Therefore (4) does not depend on F_{n-k}.

Remark 3. Since

$$\bigcup_{i=0}^{p} f'(B_i(E_k,F_{n-k}))(B_i(E_k,F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k,F_{n-k}))$$

does not depend on F_{n-k}, we have

$$\bigcup_{i=0}^{p} f'(B_i(E_k,F_{n-k}))(B_i(E_k,F_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k,F_{n-k}))$$

$$= \bigcup_{i=0}^{p} f'(B_i(E_k,G^*_{n-k}))(B_i(E_k,G^*_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k,G^*_{n-k})),$$

where G^*_{n-k} is an arbitrary but fixed element from the set $\{0,1\}^{n-k}$.

Theorem 5. *Let $f(X) : B^n \to B$ be a Boolean function. If the equation $f(X) = 0$ is consistent, then the formulas*

$$x_k = \bigcup_{E_k\in\{0,1\}^k} \left[\bigcup_{i=0}^{p} f'(B_i(E_k, G^*_{n-k}))(B_i(E_k, G^*_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k, G^*_{n-k}))\right] T_k^{E_k}$$

$$(k = 1, \dots, n)$$

defines a general solution of $f(X) = 0$, where G^*_{n-k} are arbitrary but fixed elements from the sets $\{0,1\}^{n-k}$ $(k = 1, \dots, n)$.

Comment. Specifically, we can take $G^*_{n-k} = (0, \dots, 0)$.

Proof. In accordance with (3) we have for $k = 1, \dots, n$

$$\begin{aligned}
x_k &= \bigcup_{A} \left[\bigcup_{i=0}^{p} f'(B_i(A))(B_i(A))_k \prod_{j=0}^{i-1} f(B_j(A))\right] T^A \\
&= \bigcup_{E_k\in\{0,1\}^k} \bigcup_{F_{n-k}\in\{0,1\}^{n-k}} \left[\bigcup_{i=0}^{p} f'(B_i(E_k, F_{n-k}A))(B_i(E_k, F_{n-k}))_k \cdot \right. \\
&\qquad \left. \cdot \prod_{j=0}^{i-1} f(B_j(E_k, F_{n-k}))\right] T_k^{E_k} T_{n-k}^{F_{n-k}} \\
&\qquad\qquad (T_k = (t_1, \dots, t_k),\ T_{n-k} = (t_{k+1}, \dots, t_n)) \\
&= \bigcup_{E_k\in\{0,1,\}^k} \bigcup_{F_{n-k}\in\{0,1\}^{n-k}} \left[\bigcup_{i=0}^{p} f'(B_i(E_k, G^*_{n-k}))(B_i(E_k, G^*_{n-k}))_k \cdot \right. \\
&\qquad \left. \cdot \prod_{j=0}^{i-1} f(B_j(E_k, G^*_{n-k}))\right] T_k^{E_k} T_{n-k}^{F_{n-k}} \\
&= \bigcup_{E_k\in\{0,1\}^k} \left[\bigcup_{i=0}^{p} f'(B_i(E_k, G^*_{n-k}))(B_i(E_k, G^*_{n-k}))_k \prod_{j=0}^{i-1} f(B_j(E_k, G^*_{n-k}))\right] T_k^{E_k} \\
&\qquad (\text{because} \bigcup_{F_k\in\{0,1\}^{n-k}} T_{n-k}^{F_{n-k}} = 1). \quad \square
\end{aligned}$$

Definition 4. *If $f : B^n \to B$ be a Boolean function and $A \in \{0,1\}^n$, then the term $S_k(f, A)$ is defined by the following algorithm:*

for $i = 0$ **to** p **do**

if $(B_i(A))_k = 0$ **then**

if $(B_{i+1}(A))_k = 0$ **then** write $(f(B_i(A))$

else write $f(B_i(A))($

else if $(\exists m > i)(B_m(A))_k = 1$ **then** write $f'(B_i(A)) \cup$

else write $f'(B_i(A)) \ldots)$

("write" $f'(B_i(A)) \ldots)$" means "write $f'(B_i(A))$ and close all brackets").

Comment. The term $S_k(f, (E_k, G^*_{n-k}))$ contains every member

$$f(A_0),\ f(A_1), \ldots, f(A_p)$$

at most once, because of Definition 4 and

$$\{B_0(A), B_1(A), \ldots, B_p(A)\} = \{0,1\}^n.$$

Lemma 3. *If $f(X) : B^n \to B$ be a Boolean function and $A \in \{0,1\}^n$ then*

$$\bigcup_{i=0}^{p} f'(B_i(A))(B_i(A))_k \prod_{j=0}^{i-1} f(B_j(A)) = S_k(f, A).$$

The proof follows from Definition 4, distributive law and

$$a' \cup ab = a' \cup b.$$

Theorem 6. *Let $f(X) : B^n \to B$ be a Boolean function. If the equation $f(X) = 0$ is consistent then the formulas*

$$x_k = \bigcup_{E_k \in \{0,1\}^k} S_k(f, (E_k, G^*_{n-k}))T_k^{E_k} \quad (k = 1, \ldots, n)$$

define a general solution of $f(X) = 0$, where G^*_{n-k} are arbitrary but fixed elements from the set $\{0,1\}^{n-k}$ $(k = 1, \ldots, n)$.

Proof. The proof follows from Theorem 5 and Lemma 3.

Comment. In accordance with Definition 4, it can be remarked that the algorithm described in Theorem 6 simplifies of the formulas of be solutions given in Theorem 3. Namely, the coefficient $A_{i_{k,j}}$ appears in the term $S_k(f, A)$ at most once.

Example 2. Determine a triangular general solution of the consistent Boolean equation $f(x_1, x_2) = 0$.

$$\begin{aligned} x_1 &= S_1(f,(0,G_1^*))t_1' \cup S_1(f,(1,G_1^*))t_1 = \\ &= S_1(f,(0,0))t_1' \cup S_1(f,(1,0))t_1 = \quad \text{(we take } G_1^* = 0) \\ &= (f(0,0)f(0,1)(f'(1,0) \cup f'(1,1)))t_1' \\ &\quad \cup (f'(1,0) \cup f'(1,1))t_1 \\ x_2 &= S_2(f,(0,0))t_1't_2' \cup S_2(f,(0,1))t_1't_2 \\ &\quad \cup S_2(f,(1,0))t_1t_2' \cup S_2(f,(1,1))t_1t_2 \\ &= (f(0,0)(f'(0,1) \cup f(1,0)(f'(1,1)))t_1't_2' \\ &\quad \cup (f'(0,1) \cup f(0,0)f(1,0)(f'(1,1)))t_1't_2 \\ &\quad \cup (f(1,0)(f'(1,1) \cup f(0,0)f'(0,1)))t_1t_2' \\ &\quad \cup (f'(1,1) \cup f(1,0)f(0,0)(f'(0,1)))t_1t_2. \end{aligned}$$

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