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ON CONGRUENCES ON n-GROUPS

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Abstract

Let θ be an equivalence relation in a set Q, and let (Q,F) be an m-groupoid, $m \in N$. Then: a) θ is a congruence relation of the m-groupoid (Q,F) iff for all $a,b \in Q$ and for every sequence c_1^{m-1} over Q (:1.1) the following statement holds

$$\bigwedge_{i=1}^{m} (a\theta b \Rightarrow F(c_1^{i-1}, a, c_i^{m-1})\theta F(c_1^{i-1}, b, c_i^{m-1})); \text{ and}$$

b) θ is a normal congruence of the m-groupoid (Q,F) iff for all $a,b \in Q$ and for every sequence c_1^m over Q the following statement holds

$$\bigwedge_{i=1}^{m} (a\theta b \Leftrightarrow F(c_1^{i-1}, a, c_i^{m-1})\theta F(c_1^{i-1}, b, c_i^{m-1})); \text{ (:1.5).}$$

Further on, let (Q,A) be an n-group (:1.2), e its $\{1,n\}$-neutral operation (:1.3) and f its inversing operation (1.4). The main result of the paper is: If θ is a congruence relation of the n-groupoid (Q,A), then: 1) θ is a normal congruence of the n-groupoid (Q,A) for every $n \geq 2$; 2) θ is a normal congruence of the $(n-2)$-groupoid (Q,e) for every $n \geq 3$; 3) θ is a congruence of the $(n-1)$-groupoid (Q,f) for every $n \geq 2$; and 4) θ is a normal congruence of the $(n-1)$-groupoid (Q,f) for $n=2$.

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1. Preliminaries

1.1. About the expression a_p^q

Let $p \in N$, $q \in N \cup \{0\}$ and let a be the maping of the set $\{i | i \in N \wedge i \geq p \wedge i \leq q\}$ into the set S ; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, ..., a_q \; ; & p < q \\ a_p \; ; & p = q \\ \text{empty sequence } (= \emptyset) \; ; & p > q. \end{cases}$$

For example:

$$A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1}), \; j \in \{1, ..., n\}, \; n \in N \backslash \{1, 2\}, \text{ for } j = n$$

stands for

$$A(a_1, ..., a_{n-1}, A(a_n, ..., a_{2n-1})).$$

Besides, in some situations *instead of* a_p^q *we write* $(a_i)_{i=p}^q$ (briefly: $(a_i)_p^q$).

For example:

$$(\forall x_i \in Q)_1^q$$

for $q > 1$ stands for

$$\forall x_1 \in Q ... \forall x_q \in Q$$

[usually, we write: $(\forall x_1 \in Q)...(\forall x_q \in Q)$],
for $q = 1$ stands for

$$\forall x_1 \in Q$$

[usually, we write: $(\forall x_1 \in Q)$],
and for $q = 0$ it stands for an empty sequence $(= \emptyset)$.

In *some cases*, instead of a_p^q only, we write: sequence a_p^q (sequence a_p^q over a set S). For example: ... for every sequence a_p^q over a set S ... . And if $p \leq q$, we usually write: $a_p^q \in S$.

1.2. About n-groups

Let A:$Q^n \to Q$ and $n \in N\backslash\{1\}$. Then:

1) (Q, A) is said to be an n-semigroup iff for every $i \in \{2, ..., n\}$ and for every $x_1^{2n-1} \in Q$ the equality

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1})$$

is satisfied;

2) (Q, A) is said to be an n-quasigroup iff for every $i \in \{1, ..., n\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds; and

3) (Q, A) is said to be an n-group iff it is both n-semigroup and n-quasigroup. For $n = 2$ it is a group. The notion of an n-group has been introduced in [1].

1.3. On a $\{1, n\}$-neutral operation in an n-groupoid

Let $(Q,\ A)$ be an n-groupoid and $n \in N\ \backslash\{1\}$. Let also $\mathbf{e}$ be an $(n-2)$-ary operation in Q; for $n = 2$ this is a nullary operation. We say that $\mathbf{e}$ is a $\{1, n\}$-neutral operation in the n-groupoid $(Q,\ A)$ iff the following holds:

(1) $$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q) \\ (A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x \wedge A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x).$$

For $n = 2$, $\mathbf{e}(a_1^0)(= \mathbf{e}(\emptyset)) = \mathrm{e} \in Q$ is a neutral element of the groupoid $(Q,\ A)$. The notion of an $\{i, j\}$-neutral operation of an n-groupoid (: $n \in N\ \backslash\{1\}$, $\{i, j\} \subseteq \{1, ..., n\}$, $i \neq j$) has been introduced in [3]. The following propositions hold:

1.3.1 [3]: *In an n-groupoid ($n \in N\ \backslash\{1\}$) there is at most one $\{1, n\}$-neutral operation;*

1.3.2 [3]: *In every n-group there is a $\{1, n\}$-neutral operation*[1];

1.3.3 [3]: *For $n \geq 3$, an n-semigroup $(Q,\ A)$ is an n-group iff $(Q,\ A)$*

[1]The cases $\{i, j\} \neq \{1, n\}$ $(n \geq 3)$ were described in [5]

has a $\{1,n\}$*-neutral operation*

Therefore, an algebra $(Q,\{A,\mathbf{e}\})$ satisfying (1) and

$$(2)\quad (\forall x_i \in Q)_1^{2n-1}\Big(\bigwedge_{j=2}^{n} A(A(x_1^n),x_{n+1}^{2n-1}) = A(x_1^{j-1},A(x_j^{j+n-1}),x_{j+n}^{2n-1})\Big),$$

for $n \geq 3$, is also taken to be an n-group.

1.4 t On inversing operation in an n-group

The following proposition holds:

1.4.1 [4]: *Let* $(Q,\ A)$ *be an* n*-semigroup and* $n \in N\setminus\{1\}$*. Then:*

a) There is at most one $(n-1)$*-ary operation* f *in* Q *such that the following formulas hold*

$$(3)\quad (\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)$$
$$A(f(a_1^{n-2},a),a_1^{n-2},A(a,a_1^{n-2},x)) = x$$

and

$$(4)\quad (\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)$$
$$A(A(x,a_1^{n-2},a),a_1^{n-2},f(a_1^{n-2},a)) = x\ ;$$

b) If there is an $(n-1)$*-ary operation* f *in* Q *such that the formulas* (3) *and* (4) *are satisfied, then* $(Q,\ A)$ *is an* n*-group; and*

c) If $(Q,\ A)$ *is an* n*-group , then there is an* $(n-1)$*-ary operation* f *in* Q *such that the formulas* (3) *and* (4) *hold.*[2]

Therefore, an algebra $(Q,\{A,f\})$ satisfying (2), (3) and (4) is also taken to be an n-group.

As for the case $n=2$ we say that the operation f is an *inversing operation in the* n*-group* (Q,A); [4]. The following propositions hold:

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[2] $f(a_1^{n-2},a) \overset{def}{=} \mathsf{E}(a_1^{n-2},a,a_1^{n-2})$, where E is a $\{1,2n-1\}$-neutral operation of a $(2n-1)$-group $(Q,\overset{2}{A})$; $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n),x_{n+1}^{2n-1})$. We note that for $n=2$, this is the inversing in a group.

1.4.2 [4]: *Let (Q, A) be an n-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation, f its inversing operation and $n \in N\backslash\{1\}$. Then the following formula holds:*

$$(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(A(f(a_1^{n-2}, a), a_1^{n-2}, a) =$$
$$\mathbf{e}(a_1^{n-2}) \wedge A(a, a_1^{n-2}, f(a_1^{n-2}, a)) = \mathbf{e}(a_1^{n-2}));$$

and

1.4.3 [4] *Let (Q, A) be an n-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation, f its inversing operation and $n \in N\backslash\{1\}$. Then the formula holds:*

$$(\forall x \in Q)(\forall y \in Q)(\forall a_i \in Q)_1^{n-2}(\forall b_i \in Q)_1^{n-2}$$
$$A(x, b_1^{n-2}, y) = A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(b_1^{n-2}))), a_1^{n-2}, y)$$ [3]

1.5. On congruences in an m-groupoid

Let (Q, F) be an m-groupoid and $m \in N$. Let also Θ be an equivalence relation in the set Q. Then, Θ is a *congruence relation* on the m-groupoid (Q, F) iff the following holds:

$$(\forall a_j \in Q)_1^m(\forall b_j \in Q)_1^m((\bigwedge_{i=1}^{m} a_i \Theta b_i) \Longrightarrow F(a_1^m)\Theta F(b_1^m)).$$

The following proposition is true: Θ is a *congruence* on an m-groupoid (Q, F) iff the following holds:

$$(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{m-1}$$
$$(\bigwedge_{i=1}^{m} (a\Theta b \Longrightarrow \; F(c_1^{i-1}, a, c_i^{m-1})\Theta F(c_1^{i-1}, b, c_i^{m-1}))).$$

A congruence relation Θ on an m-groupoid (Q, F) is said to be *normal* iff the following holds:

$$(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{m-1}$$
$$(\bigwedge_{i=1}^{m} (F(c_1^{i-1}, a, c_i^{m-1})\Theta F(c_1^{i-1}, b, c_i^{m-1}) \Longrightarrow a\Theta b)).$$

Thus, an equivalence relation Θ in a set Q is a *normal congruence relation* on an m-groupoid (Q, F) iff the following holds:

$$(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{m-1}$$

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[3] For $n = 2 : (\forall x \in Q)(\forall y \in Q)A(x, y) = A(x, y)$.

$$(\bigwedge_{i=1}^{m}(a\Theta b \Longleftrightarrow F(c_1^{i-1},a,c_i^{m-1})\Theta F(c_1^{i-1},b,c_i^{m-1})))^4.$$

2. Main result

Theorem 2.1. *Let (Q,A) be an n-group, $\mathbf{e}$ its $\{1,n\}$-neutral operation, f its inversing operation and $n \in N\setminus\{1,2\}$. Let also Θ be an equivalence relation on Q satisfying:*

(0) $(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{n-1}$

$$(\bigwedge_{i=1}^{n}(a\Theta b \Longrightarrow A(c_1^{i-1},a,c_i^{n-1})\Theta A(c_1^{i-1},b,c_i^{n-1}))).^5$$

Then, the following statements hold:

(1) $(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{n-1}$

$$(\bigwedge_{i=1}^{n}(a\Theta b \Longleftrightarrow A(c_1^{i-1},a,c_i^{n-1})\Theta A(c_1^{i-1},b,c_i^{n-1})));$$

(2) $(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{n-2}$

$$(a\Theta b \Longleftrightarrow f(c_1^{n-2},a)\Theta f(c_1^{n-2},b));$$

(3) $(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{n-3}$

$$(\bigwedge_{i=1}^{n-2}(a\Theta b \Longleftrightarrow \mathbf{e}(c_1^{i-1},a,c_i^{n-3})\Theta \mathbf{e}(c_1^{i-1},b,c_i^{n-3})));\ \ \textit{and}$$

(4) $(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{n-2}$

$$(\bigwedge_{i=1}^{n-1}(a\Theta b \Longrightarrow f(c_1^{i-1},a,c_i^{n-2})\Theta f(c_1^{i-1},b,c_i^{n-2}))).$$

Proof.

1) The following holds:

(0') $(\forall a \in Q)(\forall b \in Q)(\forall c_j \in Q)_1^{n-1}$

$$(\bigwedge_{i=1}^{n}(A(c_1^{i-1},a,c_i^{n-1})\Theta A(c_1^{i-1},b,c_i^{n-1}) \Longrightarrow a\Theta b)).$$

Indeed:

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[4]For $m=2$ e.g. in [2].

[5] To see : 1.5.

Let a, b, c_1^{n-1} be arbitrary elements of the set Q such that

$$A(c_1^{i-1}, a, c_i^{n-1})\Theta A(c_1^{i-1}, b, c_i^{n-1}); i \in \{1, ..., n\}.$$

We shall consider, respectively, the cases: $i = 1$, $i = n$ and $i \in \{1, ..., n\}\backslash\{1, n\}$.

$i = 1$: By the assumption (0) and 1.4.1, we have the following sequence of implications

$$A(a, c_1^{n-1})\Theta A(b, c_1^{n-1}) \Longrightarrow$$
$$A(A(a, c_1^{n-1}), c_1^{n-2}, f(c_1^{n-1}))\Theta A(A(b, c_1^{n-1}), c_1^{n-2}, f(c_1^{n-1})) \Longrightarrow$$
$$a\Theta b,$$

hence

$$A(a, c_1^{n-1})\Theta A(b, c_1^{n-1}) \Longrightarrow a\Theta b,$$

$i = n$: By the assumption (0) and 1.4.1, we have the following implications:

$$A(c_1^{n-1}, a)\Theta A(c_1^{n-1}, b) \Longrightarrow$$
$$A(f(c_2^{n-1}, c_1), c_2^{n-1}, A(c_1^{n-1}, b))\Theta A(f(c_2^{n-1}, c_1), c_2^{n-1}, A(c_1^{n-1}, b)) \Longrightarrow$$
$$a\Theta b,$$

and thereby

$$A(c_1^{n-1}, a)\ \Theta\ A(c_1^{n-1}, b)) \Longrightarrow\ a\Theta b.$$

$i \in \{1, ..., n\}\backslash\{1, n\}$: By the assumption (0), and since (Q, A) is an $n-$semigroup, and also by (0') for $i = 1$ and $i = n$, we have the implications

$$A(c_1^{i-1}, a, c_i^{n-1})\Theta A(c_1^{i-1}, b, c_i^{n-1}) \Longrightarrow$$
$$A(d_i^{n-1}, A(c_1^{i-1}, a, c_i^{n-1}), d_1^{i-1})\Theta A(d_i^{n-1}, A(c_1^{i-1}, b, c_i^{n-1}), d_1^{i-1}) \Longrightarrow$$
$$A(A(d_i^{n-1}, c_1^{i-1}, a), c_i^{n-1}, d_1^{i-1})\Theta A(A(d_i^{n-1}, c_1^{i-1}, b), c_i^{n-1}, d_1^{i-1}) \Longrightarrow$$
$$A(d_i^{n-1}, c_1^{i-1}, a)\Theta A(d_i^{n-1}, c_1^{i-1}, b) \Longrightarrow$$
$$a\Theta b,$$

and hence

$$A(c_1^{i-1}, a, c_i^{n-1})\Theta A(c_1^{i-1}, b, c_i^{n-1}) \Longrightarrow a\Theta b.$$

Since the conjuction of (0) and (0') is equivalent with (1), we conclude that (1) holds.

2) By (just proved) proposition (1), by 1.2, by 1.4 (:Proposition 1.4.2), and by 1.3, the following sequence of equivalences hold

$$f(c_1^{n-2}, a)\Theta f(c_1^{n-2}, b) \Longleftrightarrow$$
$$A(a, c_1^{n-2}, f(c_1^{n-2}, a))\Theta A(a, c_1^{n-2}, f(c_1^{n-2}, b)) \Longleftrightarrow$$
$$A(A(a, c_1^{n-2}, f(c_1^{n-2}, a)), c_1^{n-2}, b)\Theta A(A(a, c_1^{n-2}, f(c_1^{n-2}, b)), c_1^{n-2}, b) \Longleftrightarrow$$
$$A(A(a, c_1^{n-2}, f(c_1^{n-2}, a)), c_1^{n-2}, b)\Theta A(a, c_1^{n-2}, A(f(c_1^{n-2}, b), c_1^{n-2}, b)) \Longleftrightarrow$$
$$A(\mathbf{e}(c_1^{n-2}), c_1^{n-2}, b)\Theta A(a, c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \Longleftrightarrow$$
$$b\Theta a$$

for all $a, b, c_1^{n-2} \in Q$, and hence (2) holds.

3) By (1), by 1.4.3 and by (2), we have the following sequence of equivalences

$$a\Theta b \Longleftrightarrow A(x, c_1^{i-1}, a, c_i^{n-3}, y)\Theta A(x, c_1^{i-1}, b, c_j^{n-3}, y)^6 \Longleftrightarrow$$
$$A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(c_1^{i-1}, a, c_i^{n-3}))), a_1^{n-2}, y)\Theta$$
$$A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}))), a_1^{n-2}, y) \Longleftrightarrow$$
$$A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(c_1^{i-1}, a, c_i^{n-3})))\Theta A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}))) \Longleftrightarrow$$
$$f(a_1^{n-2}, \mathbf{e}(c_1^{i-1}, a, c_i^{n-3}))\Theta f(a_1^{n-2}, \mathbf{e}(c_1^{i-1}, b, c_i^{n-3})) \Longleftrightarrow$$
$$\mathbf{e}(c_1^{i-1}, a, c_i^{n-3})\Theta \mathbf{e}(c_1^{i-1}, b, c_i^{n-3}),$$

and hence, (3) holds.

4) By (3) and since

$$f(a_1^{n-2}, a) \stackrel{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2}),$$

where E is a $\{1, 2n-1\}$-neutral operation of the $(2n-1)$-group $(Q, \overset{2}{A})$

[6] $i \in \{1, \ldots, n-2\}$; $n \in N\backslash\{1, 2\}$.

(:footnote at 1.4.1), we have the following implications

$$a\Theta b \Longrightarrow$$
$$\mathsf{E}(c_1^{i-1}, a, c_i^{n-3}, c, c_1^{i-1}, a, c_i^{n-3})^7\Theta$$
$$\mathsf{E}(c_1^{i-1}, b, c_i^{n-3}, c, c_1^{i-1}, a, c_i^{n-3}) \text{ and}$$

$$a\Theta b \Longrightarrow$$
$$\mathsf{E}(c_1^{i-1}, b, c_i^{n-3}, c, c_1^{i-1}, a, c_i^{n-3})\Theta$$
$$\mathsf{E}(c_1^{i-1}, b, c_i^{n-3}, c, c_1^{i-1}, b, c_i^{n-3})$$

for all $a, b, c, c_1^{n-3} \in Q$, and hence we have the implication

$$a\Theta b \Longrightarrow f(c_1^{i-1}, a, c_i^{n-3}, c)\Theta f(c_1^{i-1}, b, c_i^{n-3}, c)$$

for every $i \in \{1, ..., n-2\}$ and for every sequence a, b, c, c_1^{n-3} over a set Q, i.e. (4) holds. □

3. Example

The groupoid $(\{1,2,3,4\}, \cdot)$ represented in Table 1 is a cyclic group. $(\{1,2,3,4\}, A)$, where

$$A(x_1^3) \stackrel{def}{=} x_1 \cdot x_2 \cdot x_3 \cdot 2$$

for all $x_1^3 \in \{1,2,3,4\}$, is a 3-group: tables $2_1 - 2_4$; $A_1(x,y) \stackrel{def}{=} 1 \cdot x \cdot y \cdot 2 = 2 \cdot (x \cdot y)$,

$A_2(x,y) \stackrel{def}{=} 2 \cdot x \cdot y \cdot 2 = x \cdot y, A_3(x,y) \stackrel{def}{=} 3 \cdot x \cdot y \cdot 2 = 4 \cdot (x \cdot y)$,

$A_4(x,y) \stackrel{def}{=} 4 \cdot x \cdot y \cdot 2 = 3 \cdot (x \cdot y)$.

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

Table 1

A_1	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	1	2
4	3	4	2	1

Table 2_1

[7] $i \in \{1, ..., n-2\}$; $n \in N \setminus \{1,2\}$.

A_2	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

Table 2_2

A_3	1	2	3	4
1	4	3	1	2
2	3	4	2	1
3	1	2	3	4
4	2	1	4	3

Table 2_3

A_4	1	2	3	4
1	3	4	2	1
2	4	3	1	2
3	2	1	4	3
4	1	2	3	4

Table 2_4

The equivalence relation Θ in the set $\{1,2,3,4\}$ given by

$$\{1,2,3,4\}/\Theta = \{\{1,2\},\{3,4\}\}$$

is a congruence on the 3-group $(\{1,2,3,4\},A)$; 1.5. The coresponding factor 3-groupoid $(\{1,2,3,4\}/\Theta,\mathbf{A})$ is given in Tables 3_1 and 3_2.

$A_{\{1,2\}}$	$\{1,2\}$	$\{3,4\}$
$\{1,2\}$	$\{1,2\}$	$\{3,4\}$
$\{3,4\}$	$\{3,4\}$	$\{1,2\}$

Table 3_1

$A_{\{3,4\}}$	$\{1,2\}$	$\{3,4\}$
$\{1,2\}$	$\{3,4\}$	$\{1,2\}$
$\{3,4\}$	$\{1,2\}$	$\{3,4\}$

Table 3_2

A_1	1	2
1	2	1
2	1	2

Table 4_1

A_2	1	2
1	1	2
2	2	1

Table 4_2

A_3	3	4
3	3	4
4	4	3

Table 5_1

A_4	3	4
3	4	3
4	3	4

Table 5_2

$(\{1,2\},A)$ and $(\{3,4\},A)$ are 3-subgroups of the 3-group $(\{1,2,3,4\},A)$; respectively Tables 4_1 - 4_2 and Tables 5_1-5_2.

f	1	2	3	4
1	1	2	4	3
2	1	2	4	3
3	2	1	3	4
4	2	1	3	4

Table 6

The inversing operation f in the 3-group $(\{1,2,3,4\},A)$ is represented in Table 6. (See footnote at 1.4; here:
$f(x,y) = \mathsf{E}(x,y,x), \mathsf{E}(x_1^3) = (x_1 \cdot x_2 \cdot x_3)^{-1}, \overset{2}{A}(x_1^5) = A(A(x_1^3),x_4^5) = (x_1 \cdot x_2 \cdot x_3 \cdot 2) \cdot x_4 \cdot x_5 \cdot 2 = x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5$.)

4. Remark

4.1: By 1.5, the part of Theorem 2.1 can be formulated also in the following way: Let (Q,A) be an n-group, $\mathbf{e}$ its $\{1,n\}$-neutral operation, Θ its congruence and $n \in N\backslash\{1,2\}$. Then Θ is a normal congruence relation on the algebra $(Q,\{A,\mathbf{e}\})$.

4.2: Theorem 2.1 is proved under the assumption that $n \geq 3$. However, on analizing the proof, one can easily see that (1) and (2) hold also for $n = 2$. Therefore, the following proposition holds: Let (Q,A) be an n-group, f its inversing operation, Θ its congruence and $n \in N\backslash\{1\}$.Then, for $n = 2$, Θ is a normal congruence relation on the algebra $(Q,\{A,f\})$.

4.3: From Table 6 we can see that the following proposition holds:

$$f(1,3)\Theta f(3,3) \wedge \rceil 1\Theta 3.$$

Hence, Θ is not a normal congruence of the groupoid $(\{1,2,3,4\},f)$. (Note that Θ is a congruence on the groupoid $(\{1,2,3,4\},\ f)$ indeed.)

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