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INTEGRATED SEMIGROUPS, RELATIONS WITH GENERATORS

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Abstract

If $S_n = T * f_n$ and T is a C_0 - semigroup then the corresponding generator A is determined by $Ax = (n+1)! \lim_{h \downarrow 0} \frac{S_n(h)x - \frac{h^n}{n!}x}{h^{n+1}}$, $x \in D(A)$.

The same holds for a general n-times integrated exponentially bounded semigroup if its infinitesimal generator is densely defined.

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1. Introduction

Integrated exponentially bounded semigroup of operators on Banach spaces were introduced by Arendt [2] and applied on abstract Cauchy problems with generators which do not generate C_0 semigroups (cf. [5], [7], [11], [16]). In [8], [9], [10] we studied this class of semigroups in relation to distributional semigroups. In this paper we analyse relations between an integrated semigroup and the corresponding infinitesimal generator. Using the fact that the n-th distribution derivative of an n-times integrated semigroup, with densely defined infinitesimal generator, is a distribution semigroup, we find A as a limit given in Abstract.

2. Preliminaries from the semigroup theory

We denote by E a Banach space with the norm $\|\cdot\|$; $L(E) = L(E,E)$ is a space of bounded linear operators from E into E. A family $(T(t))_{t\geq 0}$ in $L(E)$ is a *semigroup of bounded linear operators on* E if

(i) $T(t)T(s) = T(t+s)$, for any $t, s \geq 0$,

(ii) $T(0) = I$, where I is the identity operator on E.

If for a semigroup $(T(t))_{t\geq 0}$ the following condition holds:

(iii) $\lim\limits_{t\downarrow 0} T(t)x = x$, for any $x \in E$,

then $(T(t))_{t\geq 0}$ is said to be a *strong continuous semigroup* or, simply, a C_0- *semigroup.* A linear operator A, defined on the set

$$D(A) = \left\{ x \in E: \quad \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \quad \text{exists} \right\}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \left. \frac{d^+T(t)x}{dt} \right|_{t=0}, \quad x \in D(A),$$

is *the infinitesimal generator* of the semigroup $(T(t))_{t\geq 0}$; $D(A)$ is the domain of A.

Let A be a linear operator E and let $(T(t))_{t\geq 0}$ be a C_0- semigroup. It is well known that A is the infinitesimal generator of this semigroup iff there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R : \{\lambda \in \mathbb{C};\ Re\lambda > \omega\} \to L(E)$, defined by $R(\lambda) = (\lambda I - A)^{-1} = \mathcal{L}(T)(\lambda)$, $Re\lambda > \omega$, where $\mathcal{L}(T)$ is the Laplace transformation of $(T(t))_{t\geq 0}$.

3. Preliminaries from the distribution theory

For the properties of spaces $\mathcal{D}(\mathbb{R}), \mathcal{E}(\mathbb{R}), \mathcal{S}(\mathbb{R})$, their strong duals and $\mathcal{S}'(E) = L(\mathcal{S}(\mathbb{R}), E)$ we refer to [14], [15], [17] and for the space $S_+ = \{\varphi : | t^k \varphi^{(\nu)}(t) | < C_{k,\nu}, t \in [0,\infty),\ k, \nu \in \mathbb{N}_0\}$, $(\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ and its dual $\mathcal{S}'_+$, which consists of tempered distributions supported by $[0,\infty)$, we refer to [18].

Let $\mathcal{S}'(E) = L(\mathcal{S}, E)$ us denote the space of continuous linear functions $\mathcal{S} \to E$ with respect to the topology of uniform convergence on bounded sets of $\mathcal{S}$. Denote $\mathcal{S}'_+(E) = L(\mathcal{S}_+, E)$. It is a subspace of $\mathcal{S}'(E)$ with elements supported by $[0, \infty)$. There holds:

Proposition 1. *$\mathcal{S}'(E) = \mathcal{S}'(\mathbb{R})\hat{\otimes}E = L(\mathcal{S}, E)$ where the symbol $\hat{\otimes}$ means the completion of tensor product with respect to the $\varepsilon-$ topology (which is equal to the $\pi-$ topology since $\mathcal{S}'(\mathbb{R})$ is nuclear). Also, $\mathcal{S}'_+(E) = \mathcal{S}'_+\hat{\otimes}E$.*

Proof. It follows from [17], pp. 533-534.

The convolution of $f \in \mathcal{S}'_+(E)$ and $g \in \mathcal{S}'_+$ is defined by $\langle f * g, \varphi\rangle = \langle f, \check{g} * \varphi\rangle$, $\varphi \in \mathcal{S}(\mathbb{R})$, $(\check{g}(t) = g(-t))$. One can prove that $f * g = g * f \in \mathcal{S}'_+(E)$.

4. Distribution semigroups

Denote by $\mathcal{D}_-$ a subset of $C^\infty(\mathbb{R})$ which consists of elements φ with supp$\varphi \subset (-\infty, a]$, $a \in \mathbb{R}$. By $\mathcal{D}'_+(E)$ is denoted the space $L(\mathcal{D}_-, E)$. Denote by $\mathcal{D}_0$ a subset of C_0^∞ whose elements are supported by $[0, \infty)$.

J.Lions (cf. [6]) introduced the notion of a distribution semigroup. Recall, an $L(E)$ valued distribution G is a *distribution semigroup* or SGD, if the following conditions are satisfied:

(D1.) $G \in \mathcal{D}'_+(L(E))$,

(D.2) $G(\varphi * \psi, \cdot) = G(\varphi, G(\psi, \cdot))$, $\varphi, \psi \in \mathcal{D}_0$,

(D.3) $\bigcap\limits_{\varphi \in \mathcal{D}_0} N(G(\varphi, \cdot)) = \{0\}$,

(D.4) The linear hull $\Re$ of $\bigcup\limits_{\varphi \in \mathcal{D}_0} R(G(\varphi, \cdot))$ is dense in E,

(D.5) For every $x \in \Re$ there exists a function $u : \mathbb{R} \to E$ such that supp$u \subset [0, \infty)$, $u(0) = x$ and u is continuous for $t \geq 0$ and $G(\varphi, x) = \int\limits_0^\infty \varphi(t)u(t)dt$ for any $\varphi \in \mathcal{D}_0$.

If, in addition, there exists $\xi_0 \in \mathbb{R}$ such that

(D.6) $e^{-\xi t}G \in \mathcal{S}'_+(L(E))$, for $\xi > \xi_0$,

then it is called the *exponential distribution semigroup or SGDE.*

5. Integrated semigroups

We involve the family of distributions

$$f_n(t) = \begin{cases} \dfrac{H(t)t^{n-1}}{(n-1)!}, & n \in \mathbb{N}, \\ f^{(n_1)}_{n+n_1}(t), & -n \in \mathbb{N}_0,\ n_1 \in \mathbb{N},\ n+n_1 > 0,\ t \in \mathbb{R}, \end{cases}$$

where H is Heaviside's function.

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with the infinitesimal generator A. Put

$$(1)\quad S_n(t) = (T(s) * f_n(s))(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}T(s)ds,\ t \geq 0,\ n \in \mathbb{N},$$

where the integral is taken in Bochner's sense.

Then, we have

$$\mathcal{L}(S_n)(\lambda) = \mathcal{L}(T)(\lambda)\cdot\mathcal{L}(f_n)(\lambda) = \frac{1}{\lambda^n}\int_0^\infty e^{-\lambda t}T(t)dt, \quad Re\lambda > \omega,$$

which gives

$$\mathcal{L}(S_n)(\lambda) = \frac{1}{\lambda^n}R(\lambda, A), \qquad Re\lambda > \omega.$$

Following W.Arendt [2], a strongly continuous family $(S_n(t))_{t\geq 0} \subset L(E)$ is called the n-times integrated semigroup, $n \in \mathbb{N}$, if

$$(2)\qquad S_n(t, S_n(s,x)) = \frac{1}{(n-1)!}\left[\int_t^{t+s}(t+s-r)^{n-1}S_n(r,x)dr - \right.$$

$$\left. -\int_0^s (t+s-r)^{n-1}S_n(r,x)dr\right],\ t,s \geq 0 \text{ and } S_n(0,x) = 0,\ x \in E.$$

It is non-degenerate if $S_n(t,x) = 0$ for all $t \geq 0$ implies $x = 0$. It is exponentially bounded if there exists $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S_n(t)\| \leq Me^{\omega t}$, $t \geq 0$.

In particular, if $(T(t))_{t\geq 0} \subset L(E)$ is a C_0 semigroup, then S_n defined by (1) satisfies (2).

Let $(S_n(t))_{t\geq 0}$ be an n-times integrated semigroup, where $n \in \mathbb{N}$. Let $R(\lambda) = \lambda^n \mathcal{L}(S_n)$, where $Re\lambda > \omega$. Then, by the resolvent equation, $\ker R(\lambda)$ is independed of $Re\lambda > \omega$. Hence, by the uniqueness theorem, $R(\lambda)$ is injective iff $(S_n(t))_{t\geq 0}$ is non-degenerate. In this case there exists a unique operator A satisfying $(\omega, \infty) \subset \rho(A)$ such that $R(\lambda) = (\lambda I - A)^{-1}$ for all λ with $Re\lambda > \omega$. This operator is called the *generator* of $(S_n(t))_{t\geq 0}$. We put this in the following definition.

Definition 1 *Let $n \in \mathbb{N}$. An operator A is the generator of an n-times integrated semigroup $(S_n(t))_{t\geq 0}$ iff $(a, \infty) \subset \rho(A)$ for some $a \in \mathbb{R}$ and the function $\lambda \to \dfrac{(\lambda I - A)^{-1}}{\lambda^n} = \mathcal{L}(S_n)(\lambda)$, $Re\lambda > a$, is injective.*

6. Relations between A and $S_n(t)$

First we consider an n-times integrated semigroup of the form $T * f_n$ where T is a C_0 - semigroup.

Theorem 1 *(Special form) Let $(S_n(t))_{t\geq 0} \subset L(E)$ be an n-times integrated exponentially bounded semigroup where*

$$S_n(t) = (T(s) * f_n(s))(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} T(s) ds, \quad t \geq 0, \quad n \in \mathbb{N},$$

(the integral is taken in Bochner's sense) and $(T(t))_{t\geq 0}$ is a C_0 semigroup. Let A be the densely defined generator of $(S_n(t))_{t\geq 0}$. Then

$$Ax = (n+1)! \lim_{h \downarrow 0} \frac{S_n(h)x - \frac{h^n}{n!}x}{h^{n+1}}, \qquad x \in D(A).$$

Proof. Let $x \in E$. We have to show

$$A\frac{R(\lambda)}{\lambda^n}x = \lambda \frac{R(\lambda)}{\lambda^n}x - \frac{1}{\lambda^n}x,$$

where

$$\frac{R(\lambda)}{\lambda^n}x = \int_0^\infty e^{-\lambda t} S_n(t)x dt, \quad Re\lambda > \omega.$$

Then

$$(n+1)!\frac{S_n(h) - \frac{h^n}{n!}}{h^{n+1}} \int_0^\infty e^{-\lambda t} S_n(t)x \, dt$$

$$= \frac{(n+1)!}{h^{n+1}} \int_0^\infty e^{-\lambda t} S_n(h) S_n(t) x dt - \frac{n+1}{h} \int_0^\infty e^{-\lambda t} S_n(t) x dt$$

$$= \frac{(n+1)!}{h^{n+1}} \int_0^\infty e^{-\lambda t} \frac{1}{(n-1)!} \left[\int_h^{h+t} (h+t-r)^{n-1} S_n(r) dr - \int_0^t (h+t-r)^{n-1} S_n(r) dr \right] x dt$$

$$- \frac{n+1}{h} \frac{R(\lambda)}{\lambda^n} = \frac{(n+1)n}{h^{n+1}} \int_0^\infty e^{-\lambda t} \int_h^{h+t} (h+t-r)^{n-1} S_n(r) x dr dt$$

$$- \frac{(n+1)n}{h^{n+1}} \int_0^\infty e^{-\lambda t} \int_0^t (h+t-r)^{n-1} S_n(r) x dr dt - I_3 = I_1 - I_2 - I_3.$$

$$I_1 = \frac{(n+1)n}{h^{n+1}} \int_0^\infty e^{-\lambda t} \int_h^{h+t} (h+t-r)^{n-1} S_n(r) x dr dt$$

$$= \frac{(n+1)n}{h^{n+1}} \int_h^\infty S_n(r) \int_{r-h}^\infty (h+t-r)^{n-1} e^{-\lambda t} x dt dr$$

$$= \left| \begin{matrix} h+t-r = u \\ dt = du \end{matrix} \right| = \frac{(n+1)n}{h^{n+1}} \int_h^\infty S_n(r) \int_0^\infty u^{n-1} e^{-\lambda(u+r-h)} x du dr$$

$$= \frac{(n+1)n}{h^{n+1}} e^{\lambda h} \int_h^\infty e^{-\lambda r} S_n(r) \int_0^\infty u^{n-1} e^{-\lambda u} x du dr$$

$$= \frac{(n+1)n e^{\lambda h}}{h^{n+1}} \frac{(n-1)!}{\lambda^n} \int_h^\infty e^{-\lambda r} S_n(r) x dr.$$

$$I_2 = \frac{(n+1)n}{h^{n+1}} \int_0^\infty e^{-\lambda t} \int_0^t (h+t-r)^{n-1} S_n(r) x dr dt$$

$$= \frac{(n+1)n}{h^{n+1}} \int_0^\infty S_n(r) \int_h^\infty (h+t-r)^{n-1} e^{-\lambda t} x dt dr = \begin{vmatrix} h+t-r=u \\ dt=du \end{vmatrix}$$

$$= \frac{(n+1)n}{h^{n+1}} \int_0^\infty S_n(r) \int_h^\infty u^{n-1} e^{-\lambda(u-h+r)} x du dr$$

$$= \frac{(n+1)!}{h^{n+1}} e^{\lambda h} \int_0^\infty e^{-\lambda r} S_n(r) \int_h^\infty u^{n-1} e^{-\lambda u} x du dr$$

$$= \frac{(n+1)ne^{\lambda h}}{h^{n+1}} \frac{R(\lambda)}{\lambda^n} \int_h^\infty u^{n-1} e^{-\lambda u} x du.$$

Therefore,

$$(n+1)! \frac{S_n(h) - \frac{h^n}{n!}}{h^{n+1}} \int_0^\infty e^{-\lambda t} S_n(t) x dt$$

$$= \frac{(n+1)!}{h^{n+1}} \frac{e^{\lambda h}}{\lambda^n} \int_h^\infty e^{-\lambda r} S_n(r) x dr - \frac{(n+1)ne^{\lambda h}}{h^{n+1}} \frac{R(\lambda)}{\lambda^n} \int_h^\infty u^{n-1} e^{-\lambda u} x du - \frac{n+1}{h} \frac{R(\lambda)}{\lambda^n}$$

$$= \frac{(n+1)! e^{\lambda h} \int_h^\infty e^{-\lambda r} S_n(r) x dr - (n+1)ne^{\lambda h} R(\lambda) \int_h^\infty u^{n-1} e^{-\lambda u} x du - (n+1)h^n R(\lambda) x}{h^{n+1} \lambda^n}.$$

Now by by L'Hospital rule, if $h \downarrow 0$, we obtain

$$A \frac{R(\lambda)}{\lambda^n} x = \lambda \frac{R(\lambda)}{\lambda^n} x - \frac{1}{\lambda^n} x.$$

For $x \in D(A)$, using $S_n(t)S_n(s) = S_n(s)S_n(t),\ t, s \geq 0$, one can prove

$$\frac{R(\lambda)}{\lambda^n} Ax = \lambda \frac{R(\lambda)}{\lambda^n} x - \frac{1}{\lambda^n} x,$$

and A is the generator of an integrated semigroup $(S_n(t))_{t\geq 0} \subset L(E)$.

Let T be a distribution semigroup. Recall the definition of $T(f, x),\ x \in D(T(f))$ where $f \in \mathcal{E}'(\mathbb{R})$, $\operatorname{supp} f \subset [0, \infty)$ ([6]). The domain of $T(f, \cdot),\ D(T(f)) \subset$

E, is the set of x for which there exists a sequence $\{\rho_\nu\}$ in C_0^∞, with $\text{supp}\rho_\nu \subset [0,\infty)$, $\nu \in \mathbb{N}$, such that

$\rho_\nu \to \delta$, $\nu \to \infty$ and $T(\rho_\nu, x) \to x$, $T(f * \rho_\nu, x)$ converges when $\nu \to \infty$.

Then, $\lim\limits_{\nu\to\infty} T(f * \rho_\nu, x)$ does not depended on ρ_ν and it is the value of $T(f,\cdot)$ at x. As usual we define $\overline{T(f,\cdot)}$ as the closure of $T(f,\cdot)$ (cf. [6]). Note $D(T(f))$ is dense in E. We have

$$\overline{T(-\delta', x)} = Ax, \;\; x \in D(A), \quad \overline{T(\delta, x)} = x, \;\; x \in E.$$

Proposition 2. *Let $f \in \mathcal{K}'_{1+}(E)$. Then, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, there exist a strongly continuous function $F_n : \mathbb{R} \to E$, $\text{supp}F_n \subset [0,\infty)$, and positive constants m_n and C_n, such that*

$\|F_n(t)\| \leq C_n e^{m_n t}, \quad t \geq 0, \quad f = F_n^{(n)} \quad$ *($^{(n)}$ is the distribution n-th derivative).*

Let $S_n(\cdot, x) = T(\cdot, x) * f_n$, $n \in \mathbb{N}$,

$$\langle S_n(s,x), \varphi(s)\rangle = \langle T(s,x), (\check{f}_n * \varphi)(s)\rangle, \qquad \varphi \in \mathcal{D}_-,\; x \in E,$$

where

$$(\check{f}_n * \varphi)(x) = \frac{1}{(n-1)!}\int\limits_x^\infty (t-x)^{n-1}\varphi(t)dt, \quad x \in \mathbb{R}.$$

Using Proposition 2 one can simply prove

$$S_n(t,x) = \lim_{\nu\to\infty}\langle S_n(s,x), \rho_\nu(t-s)\rangle, \;\; t > 0,$$

$$S_n(\varphi^{(n)}, x) = (-1)^n T(\varphi, x), \qquad \varphi \in \mathcal{D}_0, \;\; x \in E.$$

As above, we define $D(S_n(f))$ for an $f \in \mathcal{E}'(\mathbb{R})$, $\text{supp}f \subset [0,\infty)$. Then, $D(T(f)) = D(S_n(f^{(n)}))$ and

$$\overline{S_n(f^{(n)}, x)} = (-1)^n \overline{T(f,x)}, \; x \in D(T(f)),$$

$$\overline{S_n(h,x)} = \overline{S_n(\delta(t-h), x)}, \;\; x \in E.$$

In particular

$$(-1)^n \overline{S_n(\delta^{(n)}, x)} = x, \qquad x \in E,$$

$$(-1)^n \overline{S_n(-\delta^{(n+1)}, x)} = Ax, \qquad x \in D(A).$$

Arendt (see [2]) remarked that a densely defined operator A is the generator of an exponentially bounded distribution semigroup if and only if A generates an n-times integrated, non-degenerated, exponentially bounded semigroup for same $n \in \mathbb{N}_0$. This follows from his results in [2] and Sova's results in [13].

Now we will consider the general case.

Theorem 2. *(General) Let $(S_n(t))_{t\geq 0}$ be an n-times integrated exponentially bounded semigroup, $n \in \mathbb{N}_0$ and A be its densely defined generator. Then*

$$Ax = (n+1)! \lim_{h\downarrow 0} \frac{S_n(h)x - \frac{h^n}{n!}}{h^{n+1}}, \qquad x \in D(A).$$

Proof. Let $\varphi \in \mathcal{D}$. Since,

$$\frac{(n+1)!}{h^{n+1}} \left(\varphi(h) - \frac{h^n}{n!} \varphi^{(n)}(0) \right) \to \varphi^{(n+1)}(0), \quad \text{as} \quad h \to 0,$$

it follows

$$\frac{(n+1)!}{h^{n+1}} \left\langle \delta(t-h) - \frac{h^n}{n!} (-1)^n \delta^{(n)}(t), \varphi(t) \right\rangle \to (-1)^{n+1} \left\langle \delta^{(n+1)}(t), \varphi(t) \right\rangle, \quad \varphi \in \mathcal{D} \ \text{ as } h \to 0.$$

Then, we have,

$$(n+1)! \lim_{h\downarrow 0} \frac{S_n(h,x) - \frac{h^n}{n!}x}{h^{n+1}} = (n+1)! \lim_{h\downarrow 0} \frac{\overline{S_n(\delta(t-h), x)} - \overline{S_n\left(\frac{h^n}{n!}(-1)^n \delta^{(n)}(t), x\right)}}{h^{n+1}}$$

$$= (n+1)! \lim_{h\downarrow 0} \frac{\overline{S_n\left(\delta(t-h) - \frac{h^n}{n!}(-1)^n \delta^{(n)}(t), x\right)}}{h^{n+1}}$$

$$= \overline{S_n((-1)^{n+1}\delta^{(n+1)}(t), x)} = \overline{S_0(-\delta, x)} = Ax.$$

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