NOVI SAD J. MATH.
VOL. 27, NO. 2, 1997, 49-56

OSCILLATION OF SECOND ORDER NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS

Mirko Budinčević
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

We are concerned with the second order neutral differential equations with variable coefficients and give some sufficient conditions such that all solutions are oscillatory or else tend to zero.

AMS Mathematics Subject Classification (1991): 34K15
Key words and phrases: oscillation, neutral nonlinear differential equation.

1. Introduction

In this paper we are concerned with the oscillatory behaviour of the solutions of nonlinear second order neutral differential equations of the form

$$(1)\quad (a(x)(p_0(x)y(x) + \sum_{i=1}^{k} p_i(x)y(x-x_i))')' + q(x)f(y(x-x_0)) = 0, x \geq 0,$$

where $a(x), a'(x)$ are continuous functions such that $a(x) > 0$, $q(x) \geq 0$ is continuous and not identically equal to zero in any neighbourhood of infinity. Functions $p_i(x)$ are twice differentiable and $\lim_{x\to\infty} p_i(x) = p_i \geq 0$ for $i = 0, 1, ..., k$. The function f satisfies the condition

(0) f is cont., differentiable, nondecreasing, and $xf(x) > 0$ for $x \neq 0$.

Constants x_i, $i = 0, 1, ..., k$ are positive.

A real function $g(x)$ has some property **eventually** if there exists $T \geq 0$ such that $g(x)$ has this property for $x \geq T$.

A nontrivial solution of (1) is said to be **oscillatory** if $y(x)$ is not of the same sign eventually. Otherwise, y is said to be **nonoscillatory.** An equation is called **oscillatory** if its every solution is oscillatory. Otherwise it is called **nonoscillatory.**

For a neutral differential equation the highest derivative of the unknown function appears with the argument x (the present state of the system) as well as with the one or more retarded arguments (the past state of the system). Investigations of such equations or systems, besides of their theoretical interest, have some importance in applications (see [1] and [2]).

A first paper concerning oscillations of neutral differential equations was published in 1980 ([3]). There is much current interest in oscillatory theory of such kind of equations ([4], [5], [6] and [7]). However, not much has been done in the nonlinear case and continuous coefficients. This was the motivation for our paper.

2. Preliminaries

In what follows we shall use the following lemmas which give useful information about the bonds for nonoscillatory solutions of the next equation:

$$(a(x)z'(x))' + q(x)f(z(x)) = 0, \quad x \geq 0. \tag{2}$$

Lemma 1. *([8]) Consider (2) subject to the conditions (0),*

$$q(x) \geq 0,\ q(x) \text{ is continuous and not eventually zero,} \tag{3}$$

$a(x)$ is positive and continuous and

$$\int_0^\infty \frac{dx}{a(x)} < \infty. \tag{4}$$

Then, every nonoscillatory solution y of (2) satisfies eventually the following estimate

$$A\rho(x) \leq |y(x)| \leq B$$

for some positive constants A and B (depending on y), where

$$\rho(x) = \int_x^\infty \frac{dt}{a(t)}.$$

Lemma 2. *([9]) Consider (2) subject to conditions (0), (3), $a(x) > 0$ and*

$$\int_0^\infty \frac{dx}{a(x)} = \infty. \tag{5}$$

Then, every nonoscillatory solution y of (2) satisfies eventually the following estimate

$$C \le |y(x)| \le DR(x)$$

for some positive constants C and D (depending on y), where

$$R(x) = \int_0^x \frac{dt}{a(t)}.$$

In addition to these a priori estimates, we need

Lemma 3. *Suppose that $y(x) > 0$ eventually, $\lim_{x\to\infty} p_i(x) = p$, $i = 0, 1, ..., k$ and define*

$$z(x) = p_0(x)y(x) + \sum_{i=1}^{k} p_i(x)y(x - x_i). \tag{6}$$

If $p_0 > \sum_{i=1}^{k} p_i = p > 0$, then $\lim_{x\to\infty} z(x) = c \ge 0$ if and only if $\lim_{x\to\infty} y(x) = \dfrac{c}{p_0 + p}$.

Proof. Suppose that $\lim_{x\to\infty} z(x) = c$. Let

$$\overline{\lim_{x\to\infty}} y(x) = \lim_{n\to\infty} y(\bar{x}_n) = \frac{c + q_1}{p_0 + p}$$

and

$$\underline{\lim}_{x\to\infty} y(x) = \lim_{n\to\infty} y(\underline{x}_n) = \frac{c - q_2}{p_0 + p}.$$

We shall prove that $q_1 = q_2 = 0$.

a) Suppose that $q_1 \geq q_2 \geq 0$ and $q_1 > 0$. Taking $x = \bar{x}_n$, (6) implies that

$$c = \frac{c+q_1}{p_0+p} + \sum_{i=1}^{k} p_i \lim_{n\to\infty} y(\bar{x}_n - x_i) \geq$$

$$\geq \frac{c+q_1}{p_0+p} p_0 + \sum_{i=1}^{k} p_i \frac{c-q_2-\varepsilon}{p_0+p} \Rightarrow q_2 p + \varepsilon p \geq q_1 p_0.$$

Choosing $\varepsilon = \dfrac{(p_0-p)q_1}{2p}$ we get

$$q_1 p_0 \leq q_2 p + \frac{p_0 q}{2} - \frac{p q_1}{2} \Rightarrow \frac{q_1 p_0}{2} \leq \frac{p(2q_2 - q_1)}{2} < \frac{p_0 q_2}{2},$$

contradiction.

b) Suppose that $q_2 \geq q_1$ and $q_2 > 0$. Taking $x = \underline{x}_n$, (6) implies that

$$c = \frac{c+q_2}{p_0+p} p_0 + \sum_{i=1}^{k} p_i \lim_{n\to\infty} y(\underline{x}_n - x_i) \leq$$

$$\leq \frac{c-q_2}{p_0+p} p_0 + \sum_{i=1}^{k} p_i \frac{c+q_1+\varepsilon}{p_0+p} \Rightarrow q_2 p_0 \leq p q_1 + \varepsilon p.$$

Choosing $\varepsilon = \dfrac{(p_0-p)q_2}{2p}$ we get $q_2 < q_1$, an immediate contradiction.

As the existence of $\lim_{x\to\infty} y(x)$ implies the existence of $\lim_{x\to\infty} y(x-x_i)$, the proof in the opposite direction is obvious.

Remark 1. If $p_0 = p$, $y(x) = 2 + \sin x$ for appropriate x_i could be a counter - example to the assertion of Lemma 3.

Remark 2. The assertion of Lemma 3 is the same if instead of $y(x - x_i)$ in (6) we put $y(x - x_i(x))$ where $x_i(x) \geq 0$ and $x - x_i(x) \to \infty$, $x \to \infty$. The proof is similar.

3. Oscillation

Consider the second order neutral differential equation (1). We can prove

Theorem 1. *If*

$$\int_0^\infty q(x)dx = \infty \quad \text{and} \quad \int_0^\infty \frac{dx}{a(x)} \int_0^x q(t)dt = \infty$$

then every solution $y(x)$ of (1) is either oscillatory or else $y(x) \to 0$ as $x \to \infty$.

Proof. Let $y(x)$ be a nonoscillatory solution of (1). Suppose, without loss of generality, that $y(x) > 0$ eventually. This implies that $y(x - x_i) > 0$ eventually for $i = 1, 2, ..., k$. From (6) it follows that $z(x) > p_0(x)y(x)$ eventually.

According to (1) we have $(a(x)z'(x))' \leq 0$ eventually. Thus, either $z'(x) \geq 0$ or $z'(x) \leq 0$ eventually.

a) Assume that $z'(x) \geq 0$ eventually. It follows that $z'(x - x_0) \geq 0$ eventually and by (6)

$$p_0(x - x_0)y(x - x_0) = z(x - x_0) - \sum_{i=1}^{k} p_i(x - x_0)y(x - x_i - x_0) \geq$$

$$\geq z(x-x_0) - \sum_{i=1}^{k} \frac{p_i(x - x_0)z(x - x_i - x_0)}{p_0(x - x_i - x_0)} \geq z(x-x_0) - \sum_{i=1}^{k} \frac{p_i(x - x_0)z(x - x_0)}{p_0(x - x_i - x_0)}.$$

It implies that

$$y(x - x_0) \geq p^* z(x - x_0) \tag{7}$$

eventually, for some constant $p^* > 0$. Define a positive function $w(x)$ such that

$$w(x) = \frac{a(x)z'(x)}{f(p^* z(x - x_0))},$$

then

$$w'(x) = \frac{(a(x)z'(x))'}{f(p^* z(x - x_0))} - \frac{a(x)z'(x)f'(p^* z(x - x_0))p^* z'(x - x_0)}{f^2(p^* z(x - x_0))}.$$

Condition (0) together with (7) yield

$$w'(x) \leq -q(x),$$

which, after integration from t_0 to x, gives

$$w(x) \leq w(t_0) - \int_{t_0}^{x} q(t)dt.$$

Letting $x \to \infty$ we get that $w(x) \to -\infty$, which is a contradiction.

b) Assume that $z'(x) \leq 0$ eventually. Then $\lim_{x\to\infty} z(x) = c$ and suppose that $c > 0$. According to Lemma 3 $\lim_{x\to\infty} y(x - x_0) = \dfrac{c}{p_0 + p}$ which implies that $y(x - x_0) \geq \dfrac{c}{2p_0 + p}$ eventually. Thus

$$(a(x)z'(x))' \leq -q(x)f(\frac{c}{2p_0 + p}) \equiv -c_1 q(x) \quad (c_1 > 0),$$

eventually. Integrating the above inequality from t_0 to x, we get

$$a(x)z'(x) \leq a(t_0)z'(t_0) - c_1 \int_{t_0}^{x} q(t)dt \leq -c_1 \int_{t_0}^{x} q(t)dt.$$

Dividing by $a(x)$ and integrating again from t_0 to x, we get

$$z(x) \leq z(t_0) - c_1 \int_{t_0}^{x} \frac{dt}{a(t)} \int_{t_0}^{t} q(s)ds.$$

Letting $x \to \infty$, the right side of the last inequality tends to $-\infty$ which is a contradiction to the fact that $z(x) \to c > 0$, and the proof is complete.

The next theorem shows when equation (1) is oscillatory.

Theorem 2. *If*

$$\int_{0}^{\infty} q(x)dx = \infty \quad \textit{and} \quad \int_{0}^{\infty} \frac{dx}{a(x)} = \infty$$

then the equation (1) is oscillatory.

Proof. Let $y(x)$ be a nonoscillatory solution of (1). Without loss of generality we may suppose that $y(x) > 0$ eventually. As was shown in [9], the second condition of the theorem implies that $z'(x) \geq 0$. Now, the proof follows the same line as in case a) Theorem 1, and it will be omitted.

The question is what happens when $\int_0^\infty \frac{dx}{a(x)}$ converges. The answer gives

Theorem 3. *If*

$$\int_0^\infty \frac{dx}{a(x)} < \infty \quad \text{and} \quad \int_0^\infty \frac{dx}{a(x)} \int_0^x q(s)ds = \infty,$$

then every solution $y(x)$ of (1) is either oscillatory or else $y(x) \to 0$ as $x \to \infty$.

Proof. As in the proof of Theorem 1 we introduce $z(x)$ and distinguish two cases:

a) Assume that $z'(x) \geq 0$ eventually. According to Lemma 1 we have that $\lim_{x\to\infty} z(x) = c > 0$, which by Lemma 3 gives $\lim_{x\to\infty} y(x) = \frac{c}{p_0 + p}$ and the estimate (7). The conditions of our theorem imply that $\int_{t_0}^\infty q(x)dx = \infty$, and we can proceed as in the proof of case a) of Theorem 1.

b) Assume that $z'(x) \leq 0$ eventually. According to the observation given in case a) the proof follows the same line as the proof of case b) of Theorem 1, and it will be omitted.

Remark 3. In the light of Remark 2 we are able to prove all above theorems in the case of more general differential equation than (1).

Theorems 1 and 2 are improvements and generalizations of Theorems 1 and 2 in the paper [4].

References

[1] Hale, J., Theory of functional differential equations, Springer - Verlag, New York, 1977.

[2] Győri, I., Ladas, G., Oscillation theory of delay differential equations, Clarendon Press, Oxford, 1991.

[3] Zahariev, A.I., Bainov, D.D., Oscillating properties of the solutions of a class of neutral type functional differential equations, Bulletin of the Australian Mathematical Society 22 (1980), 365-372.

[4] Grace, S.R., Lalli, B.S., Oscillation and asymptotic behaviour of certain second order neutral differential equations, Radovi Mat. 5 (1989), 121-126.

[5] Erbe, L.H., Kong, Q., Oscillation results for second order neutral differential equations, Funk. Ekvacioj Vol. 35, No 3 (1992), 545-557.

[6] Lin, X.Z., Yu, J.S., Zhang, B.G., Oscillation and nonoscillation for a class of neutral differential equations, Differential Equations Dynam. Systems 1, No. 3 (1993), 197-204.

[7] Ladas, G., Qian C., Yan, J., Oscillations of higher order neutral differential equations, Portugaliae Mathematica, Vol. 48, Fasc. 3 (1991), 291-307.

[8] Budinčević, M., Kulenović, M.R.S., Asymptotic analysis of nonlinear second order difference equation II, Zbornik radova PMF u Novom Sadu, Vol 12 (1982), 73-92.

[9] Kulenović, M.R.S., Budinčević, M., Asymptotic analysis of nonlinear second order difference equation, An. Stiin. Univ. "Al. I. Cuza", Tom XXX (1984), 39-52.

Received by the editors May 16, 1996.