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ON SOME THEOREMS IN THERMOELASTICITY OF MICROPOLAR BODIES

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Abstract

This paper is concerned with some basic theorems of linear thermoelastodinamics for micropolar bodies. The uniqueness theorem and continuous dependence theorems are proved with the aid of Lagrange identity, with no definiteness assumptions on the thermoelastic coefficients

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1. Introduction

Previous papers on uniqueness and continuous dependence in elasticity or thermoelasticity have been based almost exclusivelly on the assumptions that the elasticity tensor or thermoelastic coefficients are positive definite. So, Weiner in [2] was the first to establish a uniqueness theorem in thermoelastodynamics of homogeneous and isotropic bodies. This result was extended in [3], to cover the anisotropic bodies. In other papers, the authors recourse to the energy conservation law, in order to derive the uniqueness or continuous dependence of the solutions. A uniqueness result was indicated in [4] by supplementing the restrictions arising from thermodynamics with certain definiteness assumptions. Rusu in [5] applies the theory of semigroups of linear operators to obtain the uniqueness of solutions for the initial-boundary value problems in thermoelasticity of materials with voids. Exceptions include a result of Brun which was the first to establish a uniqueness result in the isothermal theory. The Lagrange identity method has been used by Brun in conjuction with the energy conservation law. In our studies [6] and [7] we have extended these results in order to cover the thermoelasticity of bodies with microstructure. The this study objective is to examine by a new approach the initial-boundary value problem concerning thermoelasticity of micropolar bodies. The approach is developed on the basis of the Lagrange identity and its consequences. So, we establish the uniqueness and continuous dependence of the solutions with respect to the body forces, body couples, and heat supply, for previous problems. We also deduce the continuous dependence of solutions of our problems with respect to initial data and, at last, to thermoelastic coefficients. The results are obtained for the bounded regions of the Euclidian three-dimensional space. We point out, again, that the results are obtained without recourse to the energy conservation law, or to any boundedness assumptions on the thermoelastic coefficients, and avoid the use of definiteness assumptions on the thermoelastic coefficients.

2. Basic equations

Let at time $t = 0$ the body occupy a properly regular region B of the Euclidian three-dimensional space. We denote the closure of B with $\bar{B}$ and suppose that the boundary ∂B is a closed, bounded and piece-wise smooth surface. We use a fixed system of rectangular Cartesian axes and adopt the Cartesian tensor notation. The points of B are denoted by (x) and $t \in [0, T]$ is time. The usual summation and differentiation conventions are employed: Latin subscripts are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied, and a subscripts j preceded by a comma denotes partial differentiation with respect to the Cartesian coordinate x_j. The basic equations of the linear theory of micropolar thermoelasticity are

- the equations of motion

$$
(1) \qquad \begin{aligned} t_{ij,j} + \varrho F_i &= \varrho \ddot{u}_i; \\ m_{ij,j} + \varepsilon_{ijk} t_{jk} + \varrho M_i &= I_{ij} \ddot{\varphi}_j; \end{aligned}
$$

- the equation of energy

$$
(2) \qquad T_0 \dot{\eta} = q_{i,i} + \varrho r;
$$

- the constitutive equations

$$
(3) \qquad \begin{aligned} t_{ij} &= A_{ijmn} \varepsilon_{mn} + B_{ijmn} \gamma_{mn} - E_{ij} \theta, \\ m_{ij} &= B_{ijmn} \varepsilon_{mn} + C_{ijmn} \gamma_{mn} - D_{ij} \theta, \\ \eta &= E_{ij} \varepsilon_{ij} + D_{ij} \gamma_{ij} + \frac{a}{T_0} \theta, \quad q_i = k_{ij} \theta_j; \end{aligned}
$$

- the goemetrical equations

$$
(4) \qquad \varepsilon_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \quad \gamma_{ij} = \varphi_{j,i}.
$$

In these relations we have used the following notations: ϱ - the constant mass density; T_0 - the constant absolute temperature of the body in its reference state; u_i - components of the displacement; φ_i - components of the micro-rotation; θ - the temperature measured from the temperature T_0; $\varepsilon_{ij}, \gamma_{ij}$ - kinematic characteristics of the strain; t_{ij} - components of the stress tensor; m_i - components of couple-stress tensor; F_i - components of the body force; M_i - the components of the body couple; r - the heat supply per unit mass; η - the entropy per unit mass; q_i - components of heat flux; ε_{ijk} - alternative symbol; I_{ij} - coeficients of microinertia; $a, A_{ijmn}, B_{ijmn}, C_{ijmn}, E_{ij}, D_{ij}, k_{ij}$ - the characteristics of the material, and they obey the symmetry relations

$$
(5) \qquad A_{ijmn} = A_{mnij}, \quad C_{ijmn} = C_{mnij}, \quad k_{ij} = k_{ji}, \quad I_{ij} = I_{ji}.
$$

Assume that there exists a positive constant λ_0 such that $I_{ij}\xi_i\xi_j \geq \lambda_0 \xi_i \xi_i$, $\forall \xi_i$. Also, the spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion. The Second Law of Thermodynamics implies that $k_{ij}\theta_i\theta_j \geq 0$. Along with eqs. (1) to (4), we consider the initial conditions

$$
(6) \qquad \begin{gathered} u_i(x,0) = a_i(x), \quad \dot{u}_i(x,0) = b_i(x), \quad \varphi_i(x,0) = c_i(x), \\ \dot{\varphi}_i(x,0) = d_i(x), \quad \theta(x,0) = \theta_0(x), \ (x) \in B, \end{gathered}
$$

where the functions a_i, b_i, c_i, d_i and θ_0 are prescribed. Also, we adjoin the following standard boundary conditions

$$
(7)\qquad
\begin{aligned}
&u_i = \bar{u}_i \text{ on } \partial B_1 \times [0,T), && t_i \equiv t_{ij} n_j = \bar{t}_i \text{ on } \partial B_1^c \times [0,T),\\
&\varphi_i = \bar{\varphi}_i \text{ on } \partial B_2 \times [0,T), && m_i \equiv m_i n_j = \bar{m}_i \text{ on } \partial B_2^c \times [0,T),\\
&\theta = \bar{\theta} \text{ on } \partial B_3 \times [0,T), && q \equiv q_i n_i = \bar{q} \text{ on } \partial B_3^c \times [0,T),
\end{aligned}
$$

where n_i are the components of the outward unit normal vector on ∂B; ∂B_i with their complements ∂B_i^c that are subsets of ∂B such that

$$
\begin{aligned}
\partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \partial B_3 \cap \partial B_3^c = \emptyset,\\
\partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B_3 \cup \partial B_3^c = \partial B
\end{aligned}
$$

and the functions $\bar{u}_i$, $\bar{\varphi}_i$, $\bar{m}_i$, t_i, $\bar{\theta}$ and $\bar{q}$ are prescribed. By introducing (3) into (1) and (2), the following system of equations is obtained

$$
(8)\qquad
\begin{aligned}
\varrho \ddot{u}_i &= (A_{ijmn}\varepsilon_{mn})_{,j} + (B_{ijmn}\gamma_{mn})_{,j} - (E_{ij}\theta)_{,j} + \varrho F_i,\\
I_{ij}\ddot{\varphi}_i &= (B_{mnij}\varepsilon_{mn})_{,j} + (C_{ijmn}\gamma_{mn})_{,j} - (D_{ij}\theta)_{,j}+\\
&+\varepsilon_{ijk}(A_{jkmn}\varepsilon_{mn} + B_{jkmn}\gamma_{mn} - E_{jk}\theta) + \varrho M_i,\\
a\dot{\theta} &+ T_0(E_{ij}\dot{\varepsilon}_{ij} + D_{ij}\dot{\gamma}_{ij}) = (k_{ij}\theta_{,j})_{,i} + \varrho r.
\end{aligned}
$$

By a solution of the initial boundary value problem of the micropolar thermoelasticity in the cylinder $B \times [0,T)$, we mean an order array (u_i, φ_i, θ) which satisfies system (8) for all $(x,t) \in B \times [0,T)$, the initial conditions (6) and the boundary conditions (7).

3. Basic results

Throughout this paper it is assumed that a twice continuous differentiable solution (u_i, φ_i, θ) exists. Let $U_i(x,t)$ and $V_i(x,t)$ be functions assumed to be twice continuously differentiable with respect to the time variable. We have the following Lagrange identity

$$
(9)\qquad
\begin{aligned}
&\int_\beta \varrho(x)[U_i(x,t)\dot{V}_i(x,t) - \dot{U}_i(x,t)V_i(x,t)]dV =\\
&= \int_0^t \int_B \varrho(x)[U_i(x,s)\ddot{V}_i(x,s) - \ddot{U}_i(x,s)V_i(x,s)]dV ds+\\
&+ \int_B \varrho(x)[U_i(x,0)\dot{V}_i(x,0) - \dot{U}_i(x,0)V_i(x,0)]dV.
\end{aligned}
$$

Let us denote by $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\theta)}), (\alpha = 1, 2)$, solutions of the initial boundary value problem (8), (6), (8) corresponding to the same boundary and initial data but to the body forces, body couples and heat supplies $(F_i^{(\alpha)}, M_i^{(\alpha)}, r^{(\alpha)})$, $(\alpha = 1, 2)$, respectively. We introduce the notation $v_i = u_i^{(2)} - u_i^{(1)}$, $\psi_i = \varphi_i^{(2)} - \varphi_i^{(1)}$, $\chi = \theta^{(2)} - \theta^{(1)}$. Because of linearity, the difference (v_i, ψ_i, χ) represents the solution of an initial boundary value problem anallogous to (8), (6) and (7), in which, we have $G_i = F_i^{(2)} - F_i^{(1)}$, $L_i = M_i^{(2)} - M_i^{(1)}$, $P = r^{(2)} - r^{(1)}$, $\varepsilon_{ij} = \varepsilon_{ij}^{(2)} - \varepsilon_{ij}^{(1)}$, and so on, and relation (6) and (7) become homogeneous. By setting $U_i(x,s) = v_i(x,s)$, $V_i(x,s) = v_i(x, 2t-s)$, $s \in [0, 2t]$, $t \in [0, \frac{T}{2})$, then (9), after some straightforward calculus, becomes

$$(10) \quad 2\int_B \varrho\varphi_i(t)\dot{\varphi}_i(t)dV = \int_0^t \int_B \varrho[\varphi_i(2t-s)\ddot{\varphi}_i(s) - \ddot{\varphi}_i(2t-s)\varphi_i(s)]dV ds.$$

We shall eliminate the inertial terms on the right side of (10). In view of $(1)_1$, we have

$$\begin{aligned} &\varrho[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s)] = \\ &= [v_i(2t-s)t_{ij}(s) - v_i(s)t_{ij}(2t-s)]_{,j} + [t_{ij}(2t-s)v_{j,i}(s) - \\ &- t_{ij}(s)v_{j,i}(2t-s)] - \varrho[G_i(s)v_i(2t-s) - G_i(2t-s)v_i(s)], \end{aligned}$$

and then, with the aid of eqs. (4) and (3) we conclude

$$(11) \quad \begin{aligned} &\varrho[v_i(2t-s)v_i(s) - v_i(2t-s)v_i(s)] = \\ &= [v_i(2t-s)t_{ij}(s) - v_i(s)t_{ij}(2t-s)] + \varrho[G_i(s)v_i(2t-s) - \\ &- G_i(2t-s)v_i(s)] + A_{ijmn}\varepsilon_{jik}[\varepsilon_{mn}(s)\psi_k(2t-s) - \varepsilon_{mn}(2t-s)\psi_k(s)] + \\ &+ B_{ijmn}[\gamma_{mn}(2t-s)\varepsilon_{ij}(s) - \gamma_{mn}(s)\varepsilon_{ij}(2t-s)] + E_{ij}[\chi(s)\varepsilon_{ij}(2t-s) - \\ &- \chi(2t-s)\varepsilon_{ij}(s)] + B_{ijmn}\varepsilon_{jik}[\gamma_{mn}(s)\psi_k(2t-s) - \gamma_{mn}(2t-s)\psi_k(s)] + \\ &+ E_{ij}\varepsilon_{jik}[\chi(2t-s)\psi_k(s) - \chi(s)\psi_k(2t-s)]. \end{aligned}$$

On the other hand, because of symmetric relations of I_{ij} it follows

$$I_{i,j}\frac{d}{dt}[W_i(t)\dot{\psi}_j(t) - \dot{W}_i(t)\psi_j(t)] = I_{ij}[W_i(t)\ddot{\psi}_j(t) - \ddot{W}_i(t)\psi_j(t)].$$

Based on this equality, in view of the initial conditions, the following identity is obtained

$$(12) \quad \begin{aligned} &\int_B I_{ij}[W_i(t)\dot{\psi}_j(t) - \dot{W}_i(t)\psi_j(t)]dV = \\ &\int_0^t \int_B I_{ij}[W_i(s)\ddot{\psi}_j(s) - \ddot{W}_i(s)\psi_j(s)]dV ds. \end{aligned}$$

By substituting $W_i(s)$ with $\psi_i(2t-s)$, $s \in [0,2t]$, $t \in [0,\frac{T}{2})$, we get

$$
\begin{aligned}
&2\int_b I_{ij}\psi_i(t)\dot{\psi}_j(t)dV = \\
&= \int_0^t \int_B I_{ij}[\psi_i(2t-s)\ddot{\psi}_j(s) - \ddot{\psi}_i(2t-s)\psi_j(s)]dVds.
\end{aligned} \tag{13}
$$

We eliminate the inertial terms on the right side of relation (13) by means of $(1)_2$

$$
\begin{aligned}
&I_{ij}[\psi_i(2t-s)\ddot{\psi}_j(s) - \ddot{\psi}_i(2t-s)\psi_j(s)] = \\
&= [m_{ij}(s)\psi_i(2t-s) - m_{ij}(2t-s)\psi_i(s)]_{,j} + \varrho[L_i(s)\psi_i(2t-s) - \\
&-L_i(2t-s)\psi_i(s)] + m_{ij}(2t-s)\gamma_{ij}(s) - m_{ij}(s)\gamma_{ij}(2t-s) + \\
&+\varepsilon_{kij}[t_{ij}(s)\psi_k(2t-s) - t_{ij}(2t-s)\psi_k(s)].
\end{aligned} \tag{14}
$$

Next, with the aid of eqs. (4) and (3) we conclude

$$
\begin{aligned}
&I_{ij}[\psi_i(2t-s)\ddot{\psi}_i(s) - \ddot{\psi}_i(2t-s)\psi_i(s)] = \\
&= [m_{ij}(s)\psi_i(2t-s) - m_{ji}(2t-s)\psi_i(s)]_{,j} + \varrho[L_i(s)\psi_i(2t-s) - \\
&-L_i(2t-s)\psi_i(s)] + B_{ijmn}[\varepsilon_{ij}(2t-s)\gamma_{mn}(s) - \varepsilon_{ij}(s)\gamma_{mn}(2t-s)] + \\
&+D_{ij}[\chi(s)\gamma_{ij}(2t-s) - \chi(2t-s)\gamma_{ij}(s)] + A_{ijmn}\varepsilon_{kij}[\varepsilon_{mn}(s)\psi_k(2t-s) - \\
&-\varepsilon_{mn}(2t-s)\psi_k(s)] + B_{ijmn}\varepsilon_{kij}[\gamma_{mn}(s)\psi_k(2t-s) - \gamma_{mn}(2t-s)\psi_k(s)] + \\
&+E_{ij}\varepsilon_{kij}[\chi(2t-s)\psi_k(s) - \chi(s)\psi_k(2t-s)].
\end{aligned} \tag{15}
$$

We now integrate eqs. (11) and (15) over $B \times [0,t]$ and, with the aid of the divergence theorem and initial and boundary values, in view of (10) and (13), it follows

$$
\begin{aligned}
&\int_B (\varrho v_i \dot{v}_i + I_{ij}\psi_i\dot{\psi}_j)dV = \\
&= \int_0^t \int_B \{\varrho[G_i(s)v_i(2t-s) + L_i(s)\psi_i(2t-s) - G_i(2t-s)v_i(s) - \\
&-L_i(2t-s)\psi_i(s)] + E_{ij}[\chi(s)\varepsilon_{ij}(2t-s) - \chi(2t-s)\varepsilon_{ij}(s)] + \\
&+D_{ij}[\chi(s)\gamma_{ij}(2t-s) - \chi(2t-s)\gamma_{ij}(s)]\}dVds.
\end{aligned} \tag{16}
$$

In view of equations $(3)_3$ and (2), we obtain

$$
\begin{aligned}
&E_{ij}[\chi(s)\varepsilon_{ij}(s)(2t-s) - \chi(2t-s)\varepsilon_{ij}(s)] + D_{ij}[\chi(s)\gamma_{ij}(2t-s) \\
&-\chi(2t-s)\gamma_{ij}(s)] = \eta(2t-s)\chi(s) - \eta(s)\chi(2t-s).
\end{aligned} \tag{17}
$$

It is easy to verify that

$$
\begin{aligned}
&\eta(2t-s)\chi(s)-\eta(s)\chi(2t-s)= \\
&=\frac{1}{T_0}k_{ij}[\chi(s)\int_0^{2t-s}\chi_{,j}(\xi)d\xi-\chi(2t-s)\int_0^{s}\chi_{,j}(\xi)d\xi+ \\
&+\frac{1}{T_0}k_{ij}[\chi_{,i}(2t-s)\int_0^{s}\chi_{,j}(\xi)d\xi-\chi_{,i}(s)\int_0^{2t-s}\chi_{,j}(\xi)d\xi]+ \\
&+\frac{\varrho}{T_0}[\chi(s)\int_0^{2t-s}P(\xi)d\xi-\chi(2t-s)\int_0^{s}P(\xi)d\xi].
\end{aligned}
\tag{18}
$$

Based on the symmetry of k_{ij}, we can write

$$
\begin{aligned}
&\int_0^t\int_B\frac{1}{T_0}k_{ij}\frac{d}{ds}\left(\int_0^{s}\chi_i(\xi)d\xi\right)\left(\int_0^{2t-s}\chi_j(\xi)d\xi\right)dV\,ds= \\
&=\int_B\frac{1}{T_0}k_{ij}\left(\int_0^{2t-s}\chi_i(\xi)d\xi\right)\left(\int_0^{2t-s}\chi_j(\xi)d\xi\right)dV.
\end{aligned}
\tag{19}
$$

On the other hand, integrating by parts,

$$
\begin{aligned}
&\int_0^t\int_B\frac{1}{T_0}k_{ij}\left[\theta_{,i}(s)\int_0^{2t-s}\theta_{,j}(\xi)d\xi-\theta_{,j}(2t-s)\int_0^{2t-s}\theta_{,i}(\xi)d\xi\right]dV\,ds \\
&=\int_0^t\int_B\frac{1}{T_0}k_{ij}\frac{d}{ds}\left(\int_0^{2t-s}\theta_i(\xi)d\xi\right)\left(\int_0^{2t-s}\theta_j(\xi)d\xi\right)dV\,ds,
\end{aligned}
\tag{20}
$$

and then, with the aid of (19) we get

$$
\begin{aligned}
&\int_0^t\int_B\frac{1}{T_0}k_{ij}\left[\chi_{,j}(s)\int_0^{s}\chi_{,i}(\xi)d\xi-\chi_{,i}(2t-s)\int_0^{2t-s}\chi_{,j}(\xi)d\xi\right]dV\,ds \\
&=-\int_0^t\int_B\frac{1}{T_0}k_{ij}\left(\int_0^{t}\chi_{,i}(\xi)d\xi\right)\left(\int_0^{t}\chi_{,j}(\xi)d\xi\right)dV\,ds.
\end{aligned}
\tag{21}
$$

Thus (16), with the aid of eqs. (17)-(21), may be restated as follows

$$
\begin{aligned}
&2\int_B[\varrho v_i(t)\dot{v}_i(t)+I_{ij}\psi_i(t)\dot{\psi}_j(t)]dV+ \\
&+\int_B\frac{1}{T_0}k_{ij}\left(\int_0^{t}\chi_{,i}(\xi)d\xi\right)\left(\int_0^{t}\chi_{,j}(\xi)d\xi\right)dV= \\
&=\int_0^t\int_B\varrho[G_i(s)v_i(2t-s)+L_i(s)\psi_i(2t-s)- \\
&-G_i(2t-s)v_i(s)-L_i(2t-s)\psi_i(s)]dV\,ds+ \\
&+\int_0^t\int_B\frac{\varrho}{T_0}\left(\chi(s)\int_0^{2t-s}P(\xi)d\xi-\chi(2t-s)\int_0^{s}P(\xi)d\xi\right)dV\,ds.
\end{aligned}
\tag{22}
$$

Combining the above assertions, we obtain the following theorem

Theorem 1. *Let* $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\alpha)})$, $(\alpha = 1, 2)$ *be solutions of the problem defined by (8), (6), and (7) and their difference* $v_i = u_i^{(2)} - u_i^{(1)}$, $\psi_i = \varphi_i^{(2)} - \varphi_i^{(1)}$, $\chi = \theta^{(2)} - \theta^{(1)}$, *which corresponds to the null data. Then the Lagrange identity has the form (22).*

Based on the identity (22), we shall prove the uniqueness and continuous dependence results. We proceed first to obtain the uniqueness of the solution.

Theorem 2. *Assume that the conductivity tensor* k_{ij} *is positive definite in the sense that there exists a positive constant* k_0 *such that*

$$k_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i, \quad \forall \xi_i. \tag{23}$$

If ∂B_3 *is not empty or* $a(x) \neq 0$ *on* B, *then the mixed problem of thermoelastodynamics in the linear theory of micropolar bodies has at most one solution.*

Proof. In other words, we shall prove that

$$v_i(x,t) = 0, \quad \psi_i(x,t) = 0, \quad \chi(x,t) = 0, \quad \forall (x,t) \in B \times [0,T). \tag{24}$$

Because the solutions $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta_i^{(\alpha)})$, correspond to the same body force, body couple, and heat supply, it results that their differences correspond to the null body force, body couple, and heat supply. Thus (22) may be restated as follows

$$2\int_B [\varrho v_i \dot{v}_i + I_{ij}\psi_i\dot{\psi}_i] dV + \int_B \frac{1}{T_0} k_{ij} \left(\int_0^t \chi_{,i}(\xi) d\xi\right) \left(\int_0^t \chi_{,j}(\xi) d\xi\right) dV = 0,$$

and then, by integrating on $[0, s]$, $s \in [0, \frac{T}{2})$,

$$\int_B \varrho v_i(s) v_i(s) dV + \int_B I_{ij}\psi_i(s)\psi_j(s) dV +$$
$$+ \int_0^s \int_B \frac{1}{T_0} k_{ij} \left(\int_0^\tau \chi_{,i}(\xi) d\xi\right) \left(\int_0^\tau \chi_{,j}(\xi) d\xi\right) dV d\tau = 0$$

which proves that

$$v_i(x,t) = 0, \quad \psi_i(x,t) = 0, \quad \chi_{,i}(x,t) = 0 \text{ on } B \times [0, \frac{T}{2}). \tag{25}$$

If ∂B_3 is not empty, from (7) we deduce that (24) holds. If $a(x) \neq 0$, from the energy equation we get $\dot{\chi} = 0$. However, χ vanishes initially, so that (24) again hold true. If T is infinite, then the proof of Theorem 2 is complete. If T is finite, then we set

$$v_i(x, \frac{T}{2}) = \dot{v}_i(x, \frac{T}{2}) = \psi_i(x, \frac{T}{2}) = \dot{\psi}_i(x, \frac{T}{2}) = \chi(x, \frac{T}{2}) = 0,$$

and repeat the above procedure on $[\frac{T}{2}, \frac{T}{2} + \frac{T}{4}]$ in order to extend the conclusion (24) on $B \times [0, \frac{3T}{4})$, and so on. □

We are now ready to state and prove the continuous dependence theorem upon body force, body couple and heat supply on the compact subintervals of $[0, T)$ for the solutions of the initial-boundary value problems defined by (8), (6) and (7).

Theorem 3. *Let* $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\alpha)})$, $(\alpha = 1, 2)$ *be solutions of the problem defined by (8), (6) and (7) which correspond to the same boundary and initial data, but to the body forces, body couples and heat supplies* $(F_i^{(\alpha)}, M_i^{(\alpha)}, r^{(\alpha)})$, $(\alpha = 1, 2)$ *respectively, where*

$$F_i^{(2)} = F_i^{(1)} + G_i, \;\; M_i^{(2)} = M_i^{(1)} + L_i, \;\; r^{(2)} = r^{(1)} + P.$$

Then we have the following estimate

$$\begin{aligned} &\int_B [\varrho v_i(s) v_i(s) + I_{ij}\psi_i(s)\psi_j(s)] dV + \\ &+ \int_0^s \int_B \frac{1}{T_0} k_{ij} \left(\int_0^t \chi_{,i}(\xi) d\xi\right) \left(\int_0^t \chi_{,j}(\xi) d\xi\right) dV ds \leq \\ &\leq t_* M \left[\int_0^{t_*} \int_B \varrho G_i(t) G_i(t) dV dt\right]^{\frac{1}{2}} + \\ &+ t_* N \left[\int_0^{t_*} \int_B \frac{\varrho}{T_0} \left(\int_0^t P(\xi) d\xi\right)^2 dV dt\right]^{\frac{1}{2}} + \\ &+ t_* Q \left[\int_0^{t_*} \int_B \varrho L_i(t) L_i(t) dV dt\right]^{\frac{1}{2}}, \quad s \in [0, \frac{t_*}{2}], \end{aligned} \tag{26}$$

provided there exists $t_* \in [0,T)$ *such that*

$$\int_0^{t_*}\int_B \varrho G_i(t)G_i(t)dVdt \le M_1^2,\quad \int_0^{t_*}\int_B \varrho L_i(t)L_i(t)dVdt \le M_2^2,$$

$$(27)\int_0^{t_*}\int_B \frac{\varrho}{T_0}\left(\int_0^t P(\xi)d\xi\right)^2 dVdt \le M_3^2,\quad \int_0^{t_*}\int_B \varrho v_i(t)v_i(t)dVdt \le M^2,$$

$$\int_0^{t_*}\int_B \frac{\varrho}{T_0}\chi^2(\xi)dVdt \le N^2,\quad \int_0^{t_*}\int_B I_{ij}\psi_i(t)\psi_j(t)dVdt \le Q^2,$$

Proof. According to Schwarz's inequality, it follows

$$\int_0^t\int_B \varrho v_i(2t-s)G_i(s)dVds \le$$

$$\le \left[\int_0^t\int_B \varrho G_i(s)G_i(s)dVds\right]^{\frac{1}{2}}\left[\int_t^{2t}\int_B \varrho v_i(s)v_i(s)dvds\right]^{\frac{1}{2}} \le$$

$$\le M\left[\int_0^t\int_B \varrho G_i(s)G_i(s)dVds\right]^{\frac{1}{2}},$$

where, at last, we use the substitution $2t-s\to s$. We proceed analogously with the other integrals in (22), and then by integrating over $[0,s]$, $s\in[0,\frac{t}{2}]$ we obtain estimate (26), and the proof of Theorem 3 is thus complete. □

Estimate (26) will be used to obtain a continuous result upon initial data.

Theorem 4. *Let* $(u_i^{(1)}, v_i^{(1)}, \theta^{(1)})$, $(u_i^{(1)}+w_i, \varphi_i^{(1)}+\gamma_i, \theta^{(1)}+\sigma)$ *be two solutions of the problem (8), (6) and (7), which correspond to the same body force, body couple, heat supply and boundary data, but to the initial data* $(a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}, \theta_0^{(1)})$ *and* $(a_i^{(1)}+\alpha_i, b_i^{(1)}+\beta_i, c_i^{(1)}+\gamma_i, d_i^{(1)}+\delta_i, \theta_0^{(1)}+\varphi)$, *where the perturbations* $(\alpha_i,\beta_i,\gamma_i,\delta_i,\varphi)$ *obey the following restrictions*

$$\int_B \varrho(\alpha_i\alpha_i+\beta_i\beta_i)dV \le M_4^2,\quad \int_B \varrho(\gamma_i\gamma_i+\delta_i\delta_i)dV \le M_5^2,\quad \int_B \frac{T_0}{\varrho}\eta_0^2 dV \le M_6^2,$$

where we used the notation

$$\eta_0(x) = E_{ij}(x)[\alpha_{j,i}(x)+\varepsilon_{jik}\gamma_k(x)] + D_{ij}(x)\gamma_{j,i}(x) + \frac{a}{T_0}\varphi(x).$$

If we define

$$(28)\quad v_i(x,t)=\int_0^t\int_0^s w_i(x,t)d\tau ds,$$

$$\psi_i(x,t)=\int_0^t\int_0^s w_i(x,t)d\tau ds,\quad \chi(x,t)=\int_0^t\int_0^s \sigma(x,t)d\tau dt,$$

then the estimate

$$
\begin{aligned}
&\int_B [\varrho v_i(t) v_i(t) + I_{ij}\psi_i(t)\psi_j(t)]dV + \\
&+\int_0^t \int_B \frac{1}{T_0} K_{ij} \left(\int_0^s \chi_{,i}(\xi)d\xi\right)\left(\int_0^s \chi_{,j}(\xi)d\xi\right) dV ds \leq \\
(29) \qquad &\leq t_* M \left((t_* + \frac{t_*^2}{2})\int_B \varrho a_i a_i dV + (\frac{t_*^2}{2} + \frac{t_*^2}{3})\int_B \varrho b_i b_i dV\right)^{\frac{1}{2}} + \\
&+t_* Q \left((t_* + \frac{t_*^2}{2})\int_B \varrho c_i c_i dV + (\frac{t_*^2}{2} + \frac{t_*^2}{3})\int_B I_{ij} d_i d_i dV\right)^{\frac{1}{2}} + \\
&+N t_*^{\frac{7}{2}} (20)^{\frac{-1}{2}} \left(\int_B \frac{T_0}{\varrho}\eta_0^2 dV\right)^{\frac{1}{2}},
\end{aligned}
$$

holds, provided the perturbations (v_i, ψ_i, χ) *(from (28)) satisfy (27).*

Proof. Integrating by parts in (28), we deduce

$$
v_i(x,t) = \int_0^t (t-s) w_i(x,s)ds, \quad \psi_i(x,t) = \int_0^t (t-s)\omega_i(x,s)ds,
$$
$$
\chi(x,t) = \int_0^t (t-s)\sigma(x,s)ds.
$$

It is easy to prove that the functions (w_i, ω_i, σ) satisfy the equations of motion and energy with $F_i = M_i = r = 0$, and the initial conditions

$$
w_i(x,0) = \alpha_i(x), \quad \dot{w}_i(x,0) = \beta_i(x), \quad \omega_i(x,0) = \gamma_i(x),
$$
$$
\dot{\omega}_i(x,0) = \delta_i(x), \quad \sigma(x,0) = \varphi(x).
$$

A straightforward calculus proves that the functions (v_i, ψ_i, θ) defined in (28) satisfy the equations of motion and the energy equation, in which

$$
F_i(x,t) = \alpha_i(x) + t\beta_i(x), \quad M_i(x,t) = \gamma_i(x) + t\delta_i(x),
$$
$$
r(x,t) = T_0[E_{ij}(x)\alpha_{j,i}(x) + \varepsilon_{jik}E_{ij}(x)\gamma_k(x) + D_{ij}\gamma_{j,i}(x) + \tfrac{a}{T_0}\varphi(x)]t.
$$

With these specifications, estimate (29) follows from (27). □

Finally, we obtain a continuous dependence result of the solution of our problem, upon the thermoelastic coefficients, again as a consequence of Theorem 3.

Theorem 5. *Let* $\partial B_1^c = \partial B_2^c = \partial B_3^c = 0$ *and let* $(u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \theta^{(\alpha)})$ *be solutions of the problem defined by (8), (6) and (7) corresponding to the same boundary and initial data, same body force, body couple and heat supply, but to the thermoelastic coefficients*

$$(A_{ijmn}^{(1)}, B_{ijmn}^{(1)}, C_{ijmn}^{(1)}, E_{ij}^{(1)}, D_{ij}^{(1)}, k_{ij}^{(1)}, a^{(1)}),$$

$(A_{ijmn}^{(1)} + \mathcal{A}_{ijmn}, B_{ijmn}^{(1)} + \mathcal{B}_{ijmn}, C_{ijmn}^{(1)} + \mathcal{C}_{ijmn}, E_{ij}^{(1)} + \mathcal{E}_{ij}, D_{ij}^{(1)} + \mathcal{D}_{ij}, a^{(1)} + \alpha, k_{ij}^{(1)} + \kappa_{ij})$ *respectively. If the perturbations* (v_i, ψ_i, χ) *of solutions satisfy (27), then any solution of the problem defined by (8), (6) and (7), for which*

$$\int_B (u_{i,j}u_{i,j} + u_{i,jk}u_{i,jk} + \varphi_{i,j}\varphi_{i,j} + \theta_{,j}\theta_{,j} + \theta_{,jk}\theta_{,jk} + \dot{u}_{i,j}\dot{u}_{i,j} + \dot{\varphi}_{i,j}\dot{\varphi}_{i,j} + \theta^2)dVds \leq M_7^2 \tag{30}$$

depends continuously on the thermoelastic coefficients on $[0, \frac{t_*}{2}]$, *in*

$$\int_B [\varrho v_i(t)v_i(t) + I_{ij}\psi_i(t)\psi_j(t)]dV + \int_0^t \int_B \frac{1}{T_0} k_{ij} \left(\int_0^s \chi_{,i}(\xi)d\xi\right)\left(\int_0^s \chi_{,j}(\xi)d\xi\right) dVds.$$

Proof. In the usual way, we can prove that the perturbations (v_i, ψ_i, χ) of two solutions, verify the equations of motion and energy with the following body force, body couple, and heat supply

$$F_i = \mathcal{A}_{ijmn}\varepsilon_{mn}^{(2)} + \mathcal{B}_{ijmn}\gamma_{mn}^{(2)} - \mathcal{E}_{ij}\theta^{(2)}, \quad M_i = \mathcal{B}_{mnij}\varepsilon_{mn}^{(2)} + \mathcal{C}_{ijmn}\gamma_{mn}^{(2)} - \mathcal{D}_{ij}\theta^{(2)},$$

$$\frac{\varrho}{T_0} r = \mathcal{E}_{ij}\dot{\varepsilon}_{ij}^{(2)} + \mathcal{D}_{ij}\dot{\gamma}_{ij}^{(2)} + \frac{a}{T_0}\dot{\theta}^{(2)} - \frac{1}{T_0}[\kappa_{ij}\theta_{,j}^{(2)}]_{,i}.$$

Thus, the problem is analogous to that from Theorem 4. So, according to (29) and (26), we obtain the desired result. □

References

[1] Brun, L., Methodes energetiques dans les systemes evolutifs lineaires. Theoremes d'unicite, J. Mecanique, 8 (1969), 167-192.

[2] Weiner, J.H., A uniqueness theorem for the coupled thermoelasticity problem, Quart. Appl. Math., 15 (1957),993-1005.

[3] Ionescu-Cazimir, V., Problems of linear coupled thermoelasticity IV. Uniqueness theorem, Bull Acad. Polon, 12 (1964), 473-485.

[4] Green, A.E., Laws, N., On the entropy production inequality, Arch. Rational Mech. Anal., 45 (1972), 47-59.

[5] Rusu, G., On existence and uniqueness in thermoelasticity of materials with voids, Bull. Acad. Polon, 8 (1987), 339-346.

[6] Marin, M., The Lagrange identity method in thermoelasticity of bodies with microstructure, Int. J. Engng. Sci., 8 (1994), 1229-1240.

[7] Marin, M., Sur l'existence et l'unicite dans la thermoelasticite, C.R. Acad. Sci. Paris., t.321, Serie II b, 1995, 475-480.

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