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DISTRIBUTION'S PARALLELISM AND GEODESICS IN THE $F(3,\varepsilon)$-STRUCTURE LAGRANGIAN MANIFOLD

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Abstract

If an almost product structure P on the tangent space $T(E) = T_V(E) \oplus T_H(E)$ of Lagrangian $2n$ dimensional manifold E is defined, and if $f_v(3,\varepsilon)$-structure on $T_V(E)$ is defined, then $f_h(3,\varepsilon)$-structure on $T_H(E)$ are defined in the natural way. We can define $F(3,\varepsilon)$-structure on $T(E)$.

In the Lagrangian $F(3,\varepsilon)$-manifold we have studied two linear connections, defined in [4] in terms of an arbitrary connection, distribution's parallelism and geodesic curve.

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1. Introduction

Let $\mathcal{M}$ be an n-dimensional and E $2n$-dimensional differentiable manifold and let $\eta = (E, \pi, \mathcal{M})$ be vector bundles and $\pi E = \mathcal{M}$. The differential structures (U, ϕ, R^{2n}) are vector charts of the vector bundles η. Hence the canonical coordinates on $\pi^{-1}(U)$ are $(x^1, \dots, x^n, y^1, \dots, y^n) = (x^i, y^a)$, $i = 1, 2, \dots, n$; $a = 1, \dots, n$. The transformation maps on E are $x^{i'} =$

$x^{i'}(x^1, x^2, \dots, x^n)$, $y^{a'} = M_a^{a'}(x^1, \dots, x^n)y^a$ and $\operatorname{rank}\left[\frac{\partial x^{i'}}{\partial x^i}\right] = \operatorname{rank}\left[\frac{\partial y^{a'}}{\partial y^a}\right] = \operatorname{rank} M_a^{a'} = n$.

The inverse transformations are

$$x^i = x^i(x^{1'}, x^{2'}, \dots, x^{n'}),\ y^a = M_{a'}^a(x^{i'}, \dots, x^{n'})y^{a'},\ \text{where } M_{a'}^a M_b^{a'} = \delta_b^a.$$

The local natural bases of the tangent space $T(E)$ are $\{\partial_i, \partial_a\}$

$$\partial_a = \frac{\partial}{\partial y^a} = M_a^{a'}(x^i)\partial_{a'},\quad \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\partial_{i'} + (\partial_i M_b^{a'}(x^i))y^b\partial_{a'}.$$

The nonlinear connection on E is the distribution $N : u \in E \to N_u \subset T_u(E)$ which is supplementary to the distribution V,

$$T_u(E) = N_u \oplus V_u, \quad \forall_u \in E. \tag{1.1}$$

They are locally determined by $\delta_i = \partial_i - N_i^a\partial_a$. The local basis adapted to decompositions in (1.1) is $\{\delta_i, \partial_a\}$.

It is easy to prove [3] that on $\{\delta_i, \partial_a\}$

$$\delta_{i'} = \delta_i \frac{\partial x^i}{\partial x^{i'}},\quad \partial_{a'} = \frac{\partial y^a}{\partial y^{a'}}\partial_a.$$

The subspace of $T(E)$ spanned by $\{\delta_i\}$ will be denoted by $T_H(E)$ and the subspace spaned by $\{\partial_a\}$ will be denoted by $T_V(E)$, $T(E) = T_H(E) \oplus T_V(E)$, $\dim T_H(E) = n = \dim T_V(E)$.

Definition 1.1. *If the Riemannian metrical structure on $T(E)$ is given by $G = g_{ij}(x^i, y^a)dx^i \otimes dx^j + g_{ab}(x^i, y^a)\delta y^a \otimes \delta y^b$, where $g_{ij}(x^i, y^a) = g_{ij}(x^i)$, $g_{ab} = \frac{1}{2}\partial_a\partial_b L(x^i, y^a)$ and $L(x^i, y^a)$ is a Lagrange function, then such a space we call Lagrangian space.*

Let $X \in T(E)$, then $X = X^i\delta_i + \bar{X}^a\partial_a$ and the automorphism $P : \mathcal{X}(T(E)) \to \mathcal{X}(T(E))$ defined by $PX = \bar{X}^i\delta_i + X^a\partial_a$ is the natural almost product structure on $T(E)$ i.e, $P^2 = I$. If we denote by v and h the projection morphism of $T(E)$ to $T_V(E)$ and $T_H(E)$ respectively, we have $P \circ h = v \circ P$.

2. $f(3,\varepsilon)$-structures

Definition 2.1. *We call Lagrange vertical $f_v(3,\varepsilon)$-structure of rank r on $T_V(E)$ a non-null tensor field f_v of type (1,1) and of class C^∞ such that $f_v^3+\varepsilon f_v=0$, $\varepsilon=\pm 1$, and rank $f_v=r$, where r is constant everywhere.*

Definition 2.2. *We call Lagrange horizontal $f_h(3,\varepsilon)$-structure on $T_H(E)$ a non-null tensor field f_h on $T_H(E)$ of type (1,1) of class C^∞ satisfying $f_h^3+\varepsilon f_h=0$, $\varepsilon=\pm 1$, rank $f_h=r$, where r is constant everywhere.*

An $F(3,\varepsilon)$-structure on $T(E)$ is a non-null tensor field F of type $\binom{11}{11}$ such that $F^3+\varepsilon F=0$, $\varepsilon=\pm 1$, rank $F=2r=$const.

For our study it is very convenient to consider f_v or f_h as a morphism of vector bundles [1], [2]

$$f_v:\mathcal{X}(T_V(E))\to\mathcal{X}(T_V(E)),\quad f_h:\mathcal{X}(T_H(E))\to\mathcal{X}(T_H(E)).$$

Let f_v be a Lagrange vertical $f_v(3,\varepsilon)$-structure of rank r. We define the morphisms $\mathbf{l}_v=-\varepsilon f_v^2$ and $\mathbf{m}_v=\varepsilon f_v^2+I_{T_V(E)}$, where $I_{T_V(E)}$ denotes the identity morphism on $T_V(E)$.

It is clear that $\mathbf{l}_v+\mathbf{m}_v=I$. Also we have $\mathbf{l}_v\mathbf{m}_v=\mathbf{m}_v\mathbf{l}_v=-f_v^4-\varepsilon f_v^2=-f_v(f_v^3+\varepsilon f_v)=0$, $\mathbf{m}_v^2=\mathbf{m}_v$, $\mathbf{l}_v^2=\mathbf{l}_v$.

Hence the morphisms $\mathbf{l}_v$, $\mathbf{m}_v$ applied to the $\mathcal{X}(T_V(E))$ are complementary projection morphisms, then there exist complementary distributions L_v and M_v corresponding to the projection morphisms $\mathbf{l}_v$ and $\mathbf{m}_v$, respectively, such that dim $L_v=r$ and dim $M_v=n-r$. It is easy to see that

$$\mathbf{l}_v f_v=f_v\mathbf{l}_v=f_v,\ \mathbf{m}_v f_v=f_v\mathbf{m}_v=0,\tag{2.1}$$
$$f_v^2\mathbf{l}_v=\mathbf{l}_v f_v^2=-I,\ f_v^2\mathbf{m}_v=0,$$

Proposition 2.1. *If a Lagrange $f_v(3,\varepsilon)$-structure of rank r is defined on $T_V(E)$, then the horizontal $f_h(3,\varepsilon)$-structure of rank r is defined on $T_H(E)$ by the natural almost product structure of $T(E)$.*

Proof. If we put

$$f_hX=Pf_vPX,\quad \forall X\in T_H(E)\tag{2.2}$$

it is easy to see that $f_h^3X=Pf_v^3PX$ and $f_h^3+\varepsilon f_h=0$, and rank $f_h=r$.

Proposition 2.2. *If a Lagrange $f_v(3,\varepsilon)$-structure of rank r is defined on $T_V(E)$, then an $F(3,\varepsilon)$-structure is defined on $T(E)$ by the natural almost product structure of $T(E)$.*

Proof. We put

$$F = f_h h + f_v v, \tag{2.3}$$

where f_h, is defined by (2.2), and h, v are the projection morphisms of $T(E)$ to $T_H(E)$ and $T_V(E)$. Then it is easy to check that $F^2 = f_h^2 h + f_v^2 v$, $F^3 = f_h^3 h + f_v^3 v$. Thus $F^3 + \varepsilon F = 0$. It is clear that rank $F = 2r$.

If $\mathbf{l}_h$, $\mathbf{m}_h$ are complementary projection morphisms of the horizontal $f_h(3,\varepsilon)$-structure, which is defined by the natural almost product structure of $T(E)$, we have

$$\begin{aligned}
&\mathbf{l}_h X = -\varepsilon f_h^2 X = -\varepsilon P f_v^2 P X = P\mathbf{l}_v P X, \forall X \in T_H(E)\\
&\mathbf{m}_h X = (\varepsilon f_h^2 + I_{T_H(E)})X = \varepsilon P f_v^2 P X + P I_{T_V(E)} P X = P\mathbf{m}_v P X,\\
&\mathbf{m}_h X = P\mathbf{m}_v P X,\ \forall X \in T_H(E).
\end{aligned}$$

If $\mathbf{l}$, $\mathbf{m}$ are complementary projection morphism of the $F(3,\varepsilon)$-structure on $T(E)$, then we have

$$\mathbf{l} = -\varepsilon F^2 = -\varepsilon f_h^2 h - \varepsilon f_v^2 v = \mathbf{l}_h h + \mathbf{l}_v v \tag{2.4}$$

$$\mathbf{m} = \varepsilon F^2 + I_{T(E)} = \varepsilon f_h^2 h + \varepsilon f_v^2 v + I_{T_H(E)} h + I_{T_V(E)} v = \mathbf{m}_h h + \mathbf{m}_v v.$$

Thus, if there is given a Lagrange $f_v(3,\varepsilon)$-structure on $T_V(E)$ of rank r, then there exist complementary distributions L_h, M_h of $T_H(E)$, corresponding to the morphisms $\mathbf{l}_h$, $\mathbf{m}_h$ such that $L_h = PL_v$, $M_h = PM_v$. Thus we have the decompositions

$$T(E) = T_H(E) \oplus T_V(E) = PL_v \oplus PM_v \oplus L_v \oplus M_v. \tag{2.5}$$

If L and M denote complementary distributions corresponding to the morphisms $\mathbf{l}$ and $\mathbf{m}$ respectively, then from (2.4) and (2.5) we have $L = PL_v \oplus L_v$, $M = PM_v \oplus M_v$.

Let $\bar{g}_v$ be a pseudo-Riemannian metric tensor, which is symmetric, bilinear and non-degenerate on $T_V(E)$

$$\bar{g}_v : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E)) \to \mathcal{F}(T(E)).$$

(For examples $\bar{g}_v$ can be the vertical part of a Lagrange metric structure).

The mapping $a_v : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E)) \to \mathcal{F}(T(E))$ which is defined by

$$a_v(X,Y) = \frac{1}{2}[\bar{g}(\mathbf{l}_v X, \mathbf{l}_v Y) + \bar{g}(\mathbf{m}_v X, \mathbf{m}_v Y)] \;\; \forall X,Y \in \mathcal{X}(T_V(E))$$

is a pseudo-Riemannian structure on $T(E)$ such that $a_v(X,Y) = 0$, $\forall X \in \mathcal{X}(L_v)$, $Y \in \mathcal{X}(M_v)$.

If a Lagrange $f_v(3,\varepsilon)$-structure of rank r is defined on $T_V(E)$, then there exist a pseudo-Riemannian structure g_v of $T_V(E)$ with respect to which the complementary distributions L_v and M_v are orthogonal and f_v is an isometry on $T_V(E)$, [3].

$$g_v(X,Y) = \frac{1}{2}[a_v(X,Y) + a_v(f_v X, f_v Y)].$$

We can define a metric structure g_h on $T_H(E)$: $g_h(X,Y) = g_v(PX,PY)$, $\forall X,Y \in \mathcal{X}(T_H(E))$. Using (2.5) the distributions L_h, M_h are orthogonal with respect to g_h and the horizontal $f_h(3,\varepsilon)$-structure which is define by $f_h X = P f_v P X$, $\forall X \in \mathcal{X}(T_H(E))$ is an isometry on $T_H(E)$ with respect to g_h.

We can also define a metric tensor G on $T(E)$.

$$G(X,Y) = g_h(X,Y)h + g_v(X,Y)v. \tag{2.6}$$

The distributions L and M are orthogonal with respect to G and the $F(3,\varepsilon)$-structure which is defined by $FX = f_h h + f_v v$, $X \in T(E)$ is an isometry on $T(E)$ with respect to G.

3. Distribution's parallelism

Let E^{2n} be an $F(3,\varepsilon)$-structure Lagrangian manifold as in Chapter 2.

The two operators [4] $\bar{\nabla}$ and $\tilde{\nabla}$, defined in terms of an arbitrary connection ∇ in E^{2n} and $\mathbf{l}$, $\mathbf{m}$

$$\bar{\nabla}_X Y = \mathbf{l}\nabla_X(\mathbf{l}Y) + \mathbf{m}\nabla_X(\mathbf{m}Y) \tag{3.1}$$

and

$$\tilde{\nabla}_X Y = \mathbf{l}\nabla_{\mathbf{l}X}(\mathbf{l}Y) + \mathbf{m}\nabla_{\mathbf{m}X}(\mathbf{m}Y) + \mathbf{l}[\mathbf{m}X, lY] + \mathbf{m}[\mathbf{l}X, \mathbf{m}Y] \tag{3.2}$$

are linear connections on E^{2n}.

Definition 3.1. *The distribution D is ∇-parallel if for $X \in D$ and $Y \in T(E)$ the vector field $\nabla_Y X \in D$, or equivalently ∇_Y, is a transformation of D for each $Y \in T(E)$.*

Definition 3.2. *The distribution D is ∇-half parallel if for $X \in D$ and $Y \in T(E)$ the vector field $(\Delta F)(X,Y) \in D$ where*

$$(\Delta F)(X,Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y(FX). \tag{3.3}$$

Applying Definition 3.1 to the distributions M and L we have the following theorems:

Theorem 3.1. *In an $F(3,\varepsilon)$-manifold the distributions L and M are $\bar{\nabla}$-parallel as well as $\tilde{\nabla}$-parallel.*

Proof. Since $\mathbf{ml} = 0$, we have by (3.1), (3.2)

$$\mathbf{m}\bar{\nabla}_X Y = 0,\ \mathbf{m}\tilde{\nabla}_X Y = 0,\ X \in T(E),\ Y \in L.$$

Consequently, L is $\bar{\nabla}$-parallel as well as $\tilde{\nabla}$-parallel. The same can be shown for the distribution M (and for L_v, M_v and L_h, M_h too, with respect to $\mathbf{l}_v$, $\mathbf{m}_v$ and $\mathbf{l}_h$, $\mathbf{m}_h$, respectively).

Theorem 3.2. *In an $F(3,\varepsilon)$-manifold the distributions L and M are ∇-parallel if and only if the connections $\bar{\nabla}$ and $\tilde{\nabla}$ are equal.*

Proof. If L and M are ∇-parallel, then

$$\mathbf{m}\nabla_X(\mathbf{l}Y) = 0,\ \mathbf{l}\nabla_X(\mathbf{m}Y) = 0 \text{ for } X, Y \in T(E).$$

Therefore, since $\mathbf{l} + \mathbf{m} = I$,

$$\nabla_X(\mathbf{l}Y) = \mathbf{l}\nabla_X(\mathbf{l}Y),\ \nabla_X(\mathbf{m}Y) = \mathbf{m}\nabla_X(\mathbf{m}Y).$$

Since by (3.1), ∇ and $\bar{\nabla}$ are equal. In the same way, the converse can be shown.

Theorem 3.3. *In an $F(3,\varepsilon)$-manifold the distribution L is $\bar{\nabla}$-half parallel if $\mathbf{m}\nabla_{FX}(\mathbf{m}Y)=0$, for $X\in L$ and $Y\in T(E)$.*

Proof. Since $\mathbf{m}F=0$, according to (3.3)

$$\mathbf{m}(\Delta F)(X,Y)=-\mathbf{m}\bar{\nabla}_{FX}+\mathbf{m}\bar{\nabla}_{Y}(FX)\text{ for }X\in L\text{ and }Y\in T(E).$$

Thus, by (3.1), $\mathbf{m}(\Delta F)(X,Y)=-\mathbf{m}\nabla_{FX}(\mathbf{m}Y)$ which proves the theorem.

Theorem 3.4. *In an $F(3,\varepsilon)$-manifold the distribution M is $\bar{\nabla}$-half parallel if $F\nabla_X(\mathrm{l}Y)=0$, for $X\in M$ and $Y\in T(E)$.*

Proof. According to (3.2) since $\mathrm{l}F=F\mathrm{l}$,

$$\mathrm{l}(\Delta F)(X,Y)=F\bar{\nabla}_XY-F\bar{\nabla}_YX-\mathrm{l}\bar{\nabla}_{FX}Y+\mathrm{l}\bar{\nabla}_Y(FX)$$

$$\text{for }X\in M,\ Y\in T(E).$$

Thus, by (3.1), (2.4) we have $\mathrm{l}(\Delta F)(X,Y)=F\nabla_X(\mathrm{l}Y)$, which proves the theorem.

Theorem 3.5. *In an $F(3,\varepsilon)$-manifold the distribution L is $\tilde{\nabla}$-half parallel if the vector field $[\mathbf{m}Y,FX]\in L$ for $X\in L$ and $Y\in T(E)$.*

Proof. Taking account of the relation (3.3), and since $\mathbf{m}F=0$, we have

$$\mathbf{m}(\Delta F)(X,Y)=-\mathbf{m}\tilde{\nabla}_{FX}Y+\mathbf{m}\tilde{\nabla}_Y(FX)\text{ for }X\in L,\ Y\in T(E).$$

Thus by (3.2) (2.4), we have $\mathbf{m}(\Delta F)(X,Y)=-\mathbf{m}[FX,\mathbf{m}Y]$ which proves the theorem.

4. Geodesic in E^{2n}

Let c be a curve in E^{2n}, Z a tangent vector field, and ∇ an arbitrary connection in E^{2n}.

Definition 4.1. *The curve c is ∇-geodesic if $\nabla_ZZ=0$ along c.*

Applying the above definition onto the connection $\bar{\nabla}$ and $\tilde{\nabla}$ we have the following results.

Theorem 4.1. *The curve c is* $\bar{\nabla}$*-geodesic if the vector field* $(\nabla_Z Z - \nabla_Z(\mathbf{m}Z)) \in M$ *and* $\nabla_Z(\mathbf{m}Z) \in L$.

Proof. By (3.1) we have $\bar{\nabla}_Z Z = \mathbf{l}\nabla_Z(\mathbf{l}Z) + \mathbf{m}\nabla_Z(\mathbf{m}Z)$, and since $\mathbf{l} + \mathbf{m} = I$,

$$\bar{\nabla}_Z Z = \mathbf{l}(\nabla_Z Z - \mathbf{l}\nabla_Z(\mathbf{m}Z)) + \mathbf{m}\nabla_Z(\mathbf{m}Z)$$

which proves the theorem, as far as $\mathbf{l}^2 = \mathbf{l}$.

Similarly, by (3.2) and (3.1) we have

Theorem 4.2. *The curve c is* $\tilde{\nabla}$*-geodesic if the vector fields*

$$\nabla_{\mathbf{l}Z}(\mathbf{l}Z) + [\mathbf{m}Z, \mathbf{l}Z] \in M \text{ and } \nabla_{\mathbf{m}Z}(\mathbf{m}Z) + [\mathbf{l}Z, \mathbf{m}Z] \in L.$$

$$\tilde{\nabla}_Z Z = \mathbf{l}\nabla_{\mathbf{l}Z}(\mathbf{l}Z) + \mathbf{m}\nabla_{\mathbf{m}Z}(\mathbf{m}Z) + \mathbf{l}[\mathbf{m}Z, \mathbf{l}Z) + \mathbf{m}[\mathbf{l}Z, \mathbf{m}Z].$$

Theorem 4.3. *The (1,1) tensor fields* $\mathbf{l}$ *and* $\mathbf{m}$ *are* $\bar{\nabla}$*-covariantly constant as well as* $\tilde{\nabla}$*-covariantly constant.*

Proof. We have $\bar{\nabla}_X(\mathbf{l}Y) = (\bar{\nabla}_X\mathbf{l})Y + \mathbf{l}(\bar{\nabla}_X Y)$.

Thus, by (3.1) $(\bar{\nabla}_X\mathbf{l})Y = 0$, $Y \in T(E)$. Similarly, $\bar{\nabla}_X\mathbf{m} = 0$, $\tilde{\nabla}_X\mathbf{l} = 0$, $\tilde{\nabla}_X\mathbf{m} = 0$.

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