NOVI SAD J. MATH.
VOL. 27, NO. 1, 1997, 83-92

ON SPECIAL ELEMENTS OF BISEMILATTICES

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Abstract

We define certain special elements of bisemilattices, such as distributive, neutral, absorptive, and so on. After giving some properties of these elements we consider two kinds of ideals and filters of bisemilattices and their connection with special elements. We prove structure theorems for bisemilattices as generalizations of the well known structure theorems for lattices. Finally, we give an application related to bisemilattice identities satisfied on a bisemilattice and a consequence of this result for lattices are given.

AMS Mathematics Subject Classification (1991): 06A12, 06A10, 06B99
Key words and phrases: bisemilattices, special elements

1. Introduction

Bisemilattices were introduced first by J.Plonka in [6] under the name of quasi-lattices. Padmanabhan in [5] called it bisemilattices. Bisemilattices were investigated by many authors (see [1], [4-8], [10], and the references there in).

A bisemilattice $\mathcal{A} = (A, +, \circ)$ is an algebra of the type $(2,2)$, where $(A, +)$ and $(A, \circ)$ are semilattices. Therefore, a lattice is a bisemilattice with

[1]Work supported by Science Fund of Serbia.

absorption laws. Various classes of bisemilattices have been investigated, such as distributive bisemilattices, bisemilattices with a distributive, and with an absorptive law, with generalized absorptive laws, Birkhoff systems (bisemilattices which satisfy $x + (y \circ x) = (x + y) \circ x$), and so on.

There are two kinds of orderings corresponding to a bisemilattice $(A, +, \circ)$ [1]:

$$x \leq_+ y \text{ if and only if } x + y = y$$

and

$$x \leq_\circ y \text{ if and only if } x \circ y = x.$$

The notion of a filter in a distributive bisemilattice was defined by Balbes in [1] in the following way: a filter F in $(A, +, \circ)$ is a nonempty subset of A which satisfies:

$$\text{if } x \leq_\circ y, \text{ then } x \in F \text{ implies } y \in F$$

and

$$\text{if } x, y \in F \text{ then } x \circ y \in F.$$

An ideal was defined dually (i.e. where $\circ$ and $\leq_\circ$ are replaced with $+$ and $\leq_+$, respectively).

We shall use here only the principal ideals and filters. The following definitions and notations will be used:
$x \uparrow_\circ$ is a $\circ$- filter generated by x and defined by:

$$x \uparrow_\circ = \{y \mid x \leq_\circ y\} = [x)_\circ$$

$x \downarrow_\circ$ is a $\circ$- ideal generated by x and defined by:

$$x \downarrow_\circ = \{y \mid y \leq_\circ x\}$$

$x \uparrow_+$ is a $+$- filter generated by x and defined by:

$$x \uparrow_+ = \{y \mid x \leq_+ y\} = (x]_+$$

$x \downarrow_+$ is a $+$- ideal generated by x and defined by:

$$x \downarrow_+ = \{y \mid y \leq_+ x\}.$$

If we consider distributive bisemilattices, all kinds of principal ideals and filters are sub-bisemilattices of the bisemilattice, which is not true for bisemilattices in general. Filters and ideals are only subsets of the underlying set.

In the paper [8], A. Romanowska and J.D.H. Smith defined a **pointed bisemilattice** $(A, +, \circ, 0)$ as a bisemilattice with an element 0, which satisfies the following property:

$$0 \leq_\circ x, \text{ and } 0 \leq_+ x, \text{ for every } x \in A.$$

We will call the element 0 the **bottom element**, and the element 1 with the dual property the **top element**, if it exists.

2. On a class of special elements of bisemilattices

In this part, certain special elements are defined for bisemilattices. They have their analogues in lattices, but do not have analogous properties.

Let $(A, +, \circ)$ be a bisemilattice.
Element $a \in A$ is **distributive** if and only if for every $x, y \in A$

$$a + (x \circ y) = (a + x) \circ (a + y).$$

If an element satisfies the dual law, then it is called **codistributive**.
Element $a \in A$ is **costandard** if and only if for every $x, y \in A$

$$x + (a \circ y) = (x + a) \circ (x + y).$$

If an element satisfies the dual law, then it is called **standard**.
Element $a \in A$ is **neutral** if and only if for every $x, y \in A$

$$(x \circ a) + (a \circ y) + (x \circ y) = (x + a) \circ (a + y) \circ (x + y).$$

Element $a \in A$ is **cancelable** if and only if for every $x, y \in A$

$$\text{if } a + x = a + y \text{ and } a \circ x = a \circ y \text{ then } x = y.$$

If $\mathcal{A} = (A, +, \circ, 0, 1)$ is a bisemilattice with bottom and top elements, then an element $y \in A$ is a **complement** of $x \in A$ if

$$x \circ y = 0 \text{ and } x + y = 1.$$

Lemma 1. *If $a \in A$ is a costandard (standard) element of a bisemilattice $\mathcal{A} = (A, +, \circ, 0, 1)$ having a complement, then this complement is unique.*

Proof. Let y and z be two complements for a. Then $y = y + 0 = y + (a \circ z) = (y + a) \circ (y + z) = 1 \circ (y + z) = (z + a) \circ (y + z) = z + (a \circ z) = z + 0 = z$. □

Lemma 2. *If $a \in A$ is a cancelable element of a bisemilattice $\mathcal{A} = (A, +, \circ, 0, 1)$ having a complement, then this complement is unique.*

Proof. Straightforward. □

Lemma 3. *If a is a codistributive element of a bisemilattice $\mathcal{A} = (A, +, \circ)$ then $a \uparrow_\circ$ and $a \downarrow_\circ$ are sub-bisemilattices of $\mathcal{A}$.*

Proof. Let $y, z \in a \uparrow_\circ$. Then $a \leq_\circ y$ and $a \leq_\circ z$, i.e. $a \circ y = a$ and $a \circ z = a$. Obviously, $y \circ z \in a \uparrow_\circ$. Since a is a codistributive element, $a = a \circ y + a \circ z = a \circ (y + z)$, and $y + z \in a \uparrow_\circ$. The proof for $a \downarrow_\circ$ is similar. □

The dual statement is also satisfied:

Lemma 4. *If a is a distributive element of a bisemilattice $\mathcal{A} = (A, +, \circ)$ then $a \uparrow_+$ and $a \downarrow_+$ are sub-bisemilattices of $\mathcal{A}$.* □

Proposition 1. *Let $\mathcal{A} = (A, +, \circ)$ be a bisemilattice and $a \in A$. Then the following conditions are equivalent:*

(i) a is a distributive element of $\mathcal{A}$;

(ii) the filter $a \uparrow_+$ is a sub-bisemilattice of $\mathcal{A}$ and the mapping $f : A \rightarrow a \uparrow_+$ defined by $f(x) = a + x$, is a bisemilattice homomorphism.

Proof. $(i) \longrightarrow (ii)$
The first part follows from the previous lemma. Since $f(x \circ y) = a + (x \circ y) = (a + x) \circ (a + y) = f(x) \circ f(y)$ and $f(x + y) = a + x + y = f(x) + f(y)$, f is a homomorphism.
$(ii) \longrightarrow (i)$ Since f is a homomorphism, for every $x, y \in A$, $a + (x \circ y) = f(x \circ y) = f(x) \circ f(y) = (a + x) \circ (a + y)$. □
The dual proposition is also satisfied:

Proposition 2. *If $\mathcal{A} = (A, +, \circ)$ is a bisemilattice and $a \in A$, then the following conditions are equivalent:*

(i) a is a codistributive element of $\mathcal{A}$;

(ii) the ideal $a\downarrow_\circ$ is a sub-bisemilattice of $\mathcal{A}$ and the mapping $f : A \to a\downarrow_\circ$ defined by $f(x) = a \circ x$, is a bisemilattice homomorphism. □

There are propositions which are valid for lattices, but the analogous propositions are not valid for bisemilattices. For instance, if an element of a lattice is neutral or even standard, it is cancelable as well, which is not the case for bisemilattices.

Example 1.

Let $\mathcal{A} = (A, +, \circ)$ be a bisemilattice, such that $A = \{a, b, c\}$ and operations $+$ and $\circ$ are defined with: $a + b = b$, $a + c = a$, $b + c = b$, and $x \circ y = b$, for $x \neq y$. Then, the element b is distributive, codistributive, standard, costandard, neutral, but it is not cancelable, because:

$$b + a = b = b + c \text{ and } b \circ a = b = b \circ c, \text{ but } a \neq c.$$

3. Absorptive elements of bisemilattices

In this part we define a class of special elements which do not have their analogues in lattices, that is, various types of absorptive elements. Clearly, when bisemilattice $\mathcal{A} = (A, +, \circ)$ is a lattice, then all elements from A are absorptive.

Let $\mathcal{A} = (A, +, \circ)$ be a bisemilattice, and $a \in A$, then
a is $+-$**absorptive** if for every $x \in A$:

$$a + (x \circ a) = a;$$

a is $\circ-$**absorptive** if for every $x \in A$:

$$a \circ (x + a) = a;$$

a is $+-$**coabsorptive** if for every $x \in A$:

$$x + (a \circ x) = x;$$

a is $\circ-$**coabsorptive** if for every $x \in A$:

$$x \circ (a + x) = x.$$

If an element a is $+-$ and $\circ-$ absorptive then it is called **absorptive**. If an element a is $+-$ and $\circ-$ coabsorptive then it is called **coabsorptive**.

Lemma 5. *Let $\mathcal{A} = (A, +, \circ)$ be a bisemilattice, and $a \in A$. The following is satisfied:*
(i) if a is $\circ-$coabsorptive and standard, then it is cancelable;
(ii) if a is $+-$coabsorptive and costandard, then it is cancelable;
(iii) if a is $\circ-$absorptive and has at least one of the following properties: distributivity, codistributivity, standardness, costandardness, then it is $+-$ absorptive as well, i.e. absorptive;
(iv) if a is standard and absorptive, then it is distributive;
(v) if a is costandard and absorptive, then it is codistributive;
(vi) if a is absorptive, then from $x \in a \uparrow_+$ and $y \in a \downarrow_\circ$ it follows that $y \leq_+ x$ and $y \leq_\circ x$.

Proof.
(i) If $a + x = a + y$ and $a \circ x = a \circ y$, then
$x = x \circ (a+x) = x \circ (a+y) = (x \circ a) + (x \circ y) = (y \circ a) + (x \circ y) = y \circ (a+x) = y \circ (a + y) = y$.
(ii) Dual to (i).
(iii) If a is, for instance, distributive, then we have,

if $a \circ (a + x) = a$, then: $a + (a \circ x) = (a + a) \circ (a + x) = a \circ (a + x) = a$.
(iv) $(a+x) \circ (a+y) = ((a+x) \circ a) + ((a+x) \circ y) = a + (a \circ y) + (x \circ y) = a + (x \circ y)$.
(v) Dual to (iv).
(vi) Let $a \leq_+ x$ and $y \leq_\circ a$. Since $a + x = x$ and $a \circ y = y$, consequently $x + y = (a+x) + (a \circ y) = a + x = x$, and $x \circ y = (a+x) \circ (a \circ y) = a \circ y = y$. □

4. Structure theorems for bisemilattices

Theorem 1. *If $\mathcal{A} = (A, +, \circ)$ is a bisemilattice and $a \in A$ then the following conditions are equivalent:*
(i) a is distributive, codistributive and cancelable;
(ii) $a \uparrow_+$ and $a \downarrow_\circ$ are sub-bisemilattices of $\mathcal{A}$ and the mapping $g : A \to a \uparrow_+ \times a \downarrow_\circ$ defined by $g(x) = (a + x, a \circ x)$ is an embedding.

Proof. $(i) \longrightarrow (ii)$ The first part follows from Lemmas 3 and 4, and hence $a \uparrow_+ \times a \downarrow_\circ$ is also a bisemilattice. Since a is distributive and codistributive, $g(x+y) = (a+(x+y), a\circ(x+y)) = ((a+x)+(a+y), (a\circ x)+(a\circ y)) = g(x)+g(y)$ and $g(x\circ y) = (a+(x\circ y), a\circ(x\circ y)) = ((a+x)\circ(a+y), (a\circ x)\circ(a\circ y)) = g(x)\circ g(y)$. Obviously, since a is cancelable, g is an injection. $(ii) \longrightarrow (i)$ Since g is an injection, a is cancelable. Since g is a homomorphism, following the proof for $(i) \rightarrow (ii)$, it is simple to prove that a is distributive and codistributive. □

Corollary 1. *If $\mathcal{A} = (A, +, \circ, 0, 1)$ is a bisemilattice with top and bottom elements and $a \in A$, then the following conditions are equivalent:*
(i) a is distributive, codistributive, cancelable, absorptive and has a complement;
(ii) $a \uparrow_+$ and $a \downarrow_\circ$ are sub-bisemilattices of $\mathcal{A}$ and the mapping $g : A \rightarrow a \uparrow_+ \times a \downarrow_\circ$ defined by $g(x) = (a+x, a\circ x)$ is an isomorphism.

Proof. $(i) \longrightarrow (ii)$ Considering the previous theorem, the only thing left to prove is that g is a surjection. Let b be a complement of the element a (which is unique, due to the cancelability), and let (x,y) be an arbitrary element from $a \uparrow_+ \times a \downarrow_\circ$. We show that $g(x\circ(y+b)) = (x,y)$. $(x\circ(y+b))+a =$

$(x+a)\circ(y+b+a) = (x+a)\circ(y+1) = (x+a)\circ 1 = x+a = x$, and
$(x\circ(y+b))\circ a = x\circ((y\circ a)+(b\circ a)) = x\circ((y\circ a)+0) = x\circ y\circ a = (a+x)\circ a\circ y = a\circ y = y$.
$(ii) \longrightarrow (i)$ Distributivity, codistributivity and cancelability follow from the previous theorem.

There exists a unique element $b \in A$ such that $g(b) = (b+a, b\circ a) = (1,0)$. Since $g(0) = (a,0)$, $g(1) = (1,a)$, $g(a) = (a,a)$, $(1,0)$ belongs to $a \uparrow_+ \times a \downarrow_\circ$, and

$$(a,a)+(1,0) = (1,a) \text{ and } (a,a)\circ(1,0) = (a,0),$$

the element b is a complement of a.

For $x \in A$, and b a complement of a, $a\circ(x+a) = 1\circ a\circ(x+a) = (b+a)\circ(x+a)\circ a = ((b\circ x)+a)\circ a = (b\circ x\circ a)+a = 0\circ x+a = 0+a = a$, and by Lemma 5(iii), a is absorptive. □

Theorem 2. *If $\mathcal{A} = (A, +, \circ, 0, 1)$ is a bisemilattice with top and bottom elements then there is, up to the isomorphism, one-to-one correspondence between all decompositions of $\mathcal{A}$ into a direct product of two bisemilattices* *and all elements of $\mathcal{A}$ which are distributive, codistributive, cancelable, absorptive and have a complement.*

Proof.

By the previous corollary, to every distributive, codistributive, cancelable, absorptive element a, having a complement, corresponds the decomposition of $\mathcal{A}$ into bisemilattices $a \uparrow_+$ and $a \downarrow_\circ$.

Conversely, if $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, for bisemilattices $\mathcal{B}$ and $\mathcal{C}$, then $\mathcal{B}$ and $\mathcal{C}$ both have top and bottom elements. Indeed, if $1 = (b_1, c_1) \in B \times C$, then for all $(x, y) \in B \times C$,
$(x, y) + (b_1, c_1) = (b_1, c_1)$ and $(x, y) \circ (b_1, c_1) = (x, y)$. Since $x + b_1 = b_1$ and $x \circ b_1 = x$, for all $x \in B$, b_1 is the top element of the bisemilattice $\mathcal{B}$. In the same way, c_1 is the top element of $\mathcal{C}$, and considering the element (b_0, c_0), such that $0 = (b_0, c_0)$, we conclude that b_0 and c_0 are bottom elements of $\mathcal{B}$ and $\mathcal{C}$, respectively.

Further, we consider the elements a and a' from A, such that $a = (b_0, c_1)$, and $a' = (b_1, c_0)$. It is not difficult to prove that the element a is distributive, codistributive, absorptive, cancelable, and have a complement a', and also, that $\mathcal{B}$ is isomorphic to $a \uparrow_+$, i.e. to $(b_0, c_1) \uparrow_+$, and $\mathcal{C}$ is isomorphic to $a \downarrow_\circ$, i.e. to $(b_0, c_1) \downarrow_\circ$.

This completes the proof. □

Example 2.

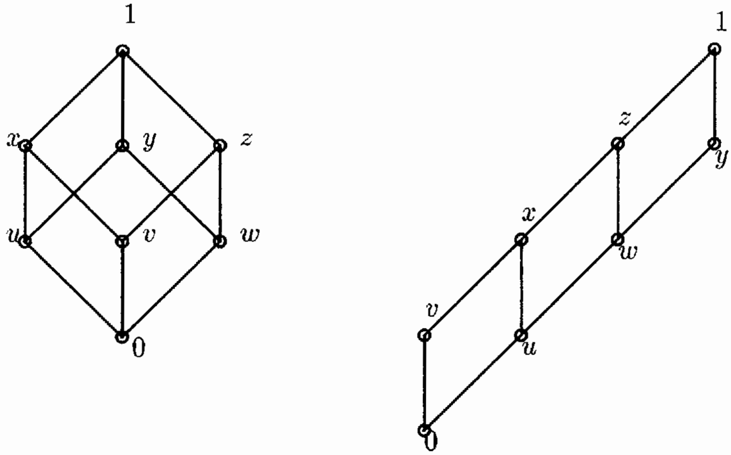
Let $\mathcal{A} = (A, +, \circ, 0, 1)$ be a bisemilattice with top and bottom elements, where $A = \{0, 1, x, y, z, u, v, w\}$ (Fig. 1).

The element v is distributive, codistributive, cancelable, absorptive and has the complement y, i.e. satisfies the conditions of Corollary 2. The bisemilattice $\mathcal{A}$ is isomorphic with the direct product of $v \uparrow_+$ and $v \downarrow_\circ$.

Remark. Naturally, a picture of a bisemilattice consists of two Hasse diagrams, one for each ordering. We use the convention that if $x \leq_+ y$ and $z \leq_\circ t$, we draw x below y, and z below t in the corresponding diagrams.

Corollary 2. *If a is a distributive, codistributive and cancelable element of the bisemilattice $\mathcal{A} = (A, +, \circ)$ then an arbitrary bisemilattice identity is satisfied on $\mathcal{A}$ if and only if it is satisfied on $a \uparrow_+$ and $a \downarrow_\circ$.*

Proof. Straightforward, by Theorem 1. □



$(A, +)$ $(A, \circ)$

Fig. 1

5. Consequences for lattices

Since a bisemilattice whose all elements are absorptive is a lattice, and all ideals and filters in a lattice are sublattices, we obtain the following statements which are consequences of Theorems 1-2 and Corollaries 1-2.

Theorem 3. ([3]) *If $\mathcal{L} = (L, \wedge, \vee)$ is a lattice and $a \in A$, then the following conditions are equivalent:*
(i) a is neutral;
(ii) the mapping $g : A \to a \uparrow \times a \downarrow$ defined by $g(x) = (a \vee x, a \wedge x)$ is an embedding. □

Theorem 4. ([3]) *If $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is a lattice with top and bottom elements and $a \in A$, then the following conditions are equivalent:*
(i) a is in the center of $\mathcal{L}$;
(ii) the mapping $g : A \to a \uparrow \times a \downarrow$ defined by $g(x) = (a \vee x, a \wedge x)$ is an isomorphism. □

Theorem 5. ([3]) *If $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is a lattice with top and bottom elements then there is, up to the isomorphism, one-to-one correspondence* *between all decompositions of $\mathcal{L}$ into a direct product of two lattices and all elements belonging to the center of $\mathcal{L}$.* □

Theorem 6. ([9]) *If a is a neutral element of the lattice $\mathcal{L} = (L, \wedge, \vee)$, then an arbitrary lattice identity is satisfied on $\mathcal{L}$ if and only if it is satisfied on $a \uparrow$ and $a \downarrow$.* □

References

[1] Balbes, R., A representation theorem for distributive quasi-lattices, Fund. Math. 68 (1970), 207-214.

[2] Davey, B.A., Priestly, H.A., Introduction to lattices and order, Cambridge University Press, 1990.

[3] Grätzer, G., General Lattice Theory, Akademie-Verlag Berlin, 1978.

[4] Nieminen, J., Ideals in distributive quasilattice, Studia Univ. Babes-Bolyai Math., 22 (1977), 6-11.

[5] Padmanabhan, R., Regular identities in lattices, Trans. Amer. Math. Soc. 158 (1971), 179-188.

[6] Plonka, J., On distributive quasi-lattices, Fund. Math. 60 (1967), 191-200.

[7] Romanowska, A. , On bisemilattices with one distributive law, Algebra Universalis 10 (1980), 36-47.

[8] Romanowska, A., Smith, J.D.H., Bisemilattices of subsemilattices, Journal of Algebra 70 (1981), 78-88.

[9] Šešelja, B., Tepavčević, A., Special elements of the lattice and lattice identities, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20,2, (1990), 21-29.

[10] Ušan, J., Tepavčević, A., On one class of bisemilattices, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 19,2, (1989), 93-104.

Received by the editors August 29, 1995.