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A PARAMETRIZATION OF DIGITAL PARABOLOIDS BT LEAST SQUARE FITS

Joviša Žunić
Faculty of Engineering, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

We prove that the digital paraboloid segments and their least squares paraboloid fits are in one-to-one correspondence, which gives a simple representation of a digital paraboloid segment by its base description and coefficients of the least squares paraboloid fit. This leads to a first known constant space representation of digital paraboloid segments.

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1. Introduction

Consider a paraboloid ρ in the three-dimensional Euclidean space with the equation $z = Ax^2 + By^2 + Cxy + Dx + Ey + F$. The paraboloid ρ will be digitized using the digitization scheme in which the first digital points (points with integer coordinates, often referred to as pixels) below a given paraboloid are taken. Obviously, it is equivalent to translating the paraboloid by -0.5 in the vertical direction and rounding.

So, the associated set of digital points for the paraboloid ρ, called a digital paraboloid, is defined as

$$P(\rho) = \{(i, j, \lfloor Ai^2 + Bj^2 + Cij + Di + Ej + F \rfloor), i \text{ and } j \text{ are integers}\},$$

where $\lfloor u \rfloor$ is the greatest integer not bigger than u. In general, we will be dealing with finite subsets of $P(\rho)$, or more precisely, with the digital paraboloid segments which are obtained by digitizing parts of the paraboloids whose projections on the xy-plane are bounded regions, called bases of digital paraboloid segments. So, if the paraboloid ρ is digitized and if a region Q in the xy-plane is given, then the digital paraboloid segment $P(\rho, Q)$ (more precisely, digital paraboloid segment with the base Q) is defined as:

$$P(\rho,Q) = \{(i,j,\lfloor Ai^2 + Bj^2 + Cij + Di + Ej + F \rfloor),(i,j) \in Q \text{ where } i \text{ and } j \text{ are integers}\}.$$

It is natural, for practical reasons, that the bases of digital paraboloid segments are assumed to be squares. If

$$Q = Q(p,q,r,s) = \{(i,j), p \leq i < q, r \leq j < s, \ i,j \text{ are integers}, q-p = s-r\},$$

then the digital paraboloid seqment $P(\rho, Q)$ will be denoted by $P(\rho, p, q, r, s)$. For convenience and without loss of generality, we can assume that p and r are equal to zero ($p = r = 0$), while q and s are equal to an integer, let say m. Under the previous assumptions, the digital paraboloid segments $P(\rho, p, q, r, s) = P(\rho, 0, m, 0, m)$ will be denoted by $P_m(\rho)$.

The major contribution of this paper is to give a constant space representation for digital paraboloid segments which have the constant space representation of their bases. The representation consists of the base representation plus six coefficients of the least squares paraboloid fit corresponding to the observed digital paraboloid segment. For example, the representation of digital paraboloid segments requires ten numbers.

The idea of using the least squares fitting techniques for representations of digital objects is not new. Namely, Melter and Rosenfeld (1989) introduced the concept of a noisy straight line segment, based on least squaress line fitting and defined in terms of bounds on correlation coefficients, and showed that it is a generalization of a digital straight line. They posed the following question: If a continuous line is digitized and least squaress are applied to the points of the digital line segment, can the original line be recovered? They gave a partial answer, for some special case. Melter, Stojmenović and Žunić (1993) answered positively the question for any line segment. More precisely, they proved that the least squares line fit uniquely determines the digital line on a segment. Thus any digital line segment can be uniquely coded by four numbers (x_1, n, b_0, b_1), where x_1 and n are its x-coordinate of the left endpoint and the number of digital points, respectively, while b_0 and b_1 are the coefficients of the least squares line fit $Y = b_0 + b_1X$ for the given digital line segment. It matches digital line segments with their least squares line fits. Therefore they obtained a new representation of digital line segments by four numbers. This representation is an alternative to the well known representation of digital lines by adjacent pairs given by Lindenbaum and Koplowitz (1991) and to the one suggested by Dorst and Smeulders (1984).

While constant space representations for digital lines exist in the literature, no such representation currently is known for digital paraboloid segments. The representation given in this paper is the first one.

2. Preliminaries

Suppose that we are given a finite set T of pixels in the three-dimensional Euclidian space (denoted as R^3). The least squaress paraboloid fit for T is a paraboloid which minimizes the sum of the squares of the vertical distances to all points in T. The method for determining such paraboloids is well-known from statistics, (see e.g., Burr (1974)).

If T is given by $\{(x_i, y_i, z_i), i = 1, 2, \ldots, t\}$ and if the equation of its least squaress paraboloid fit is $z = ax^2 + by^2 + cxy + dx + ey + f$, then the function $F(a, b, c, d, e, f) = \sum_{i=1}^{n}(ax_i^2 + by_i^2 + cx_iy_i + dx_i + ey_i + f - z_i)^2$ should be minimized.

By solving the equations $\frac{\partial F}{\partial a} = 0$, $\frac{\partial F}{\partial b} = 0$, $\frac{\partial F}{\partial c} = 0$, $\frac{\partial F}{\partial d} = 0$, $\frac{\partial F}{\partial e} = 0$, $\frac{\partial F}{\partial f} = 0$, , i.e.,

$$a\sum_{i=1}^{t} x_i^4 + b\sum_{i=1}^{t} x_i^2y_i^2 + c\sum_{i=1}^{t} x_i^3y_i + d\sum_{i=1}^{t} x_i^3 + e\sum_{i=1}^{t} x_i^2y_i + f\sum_{i=1}^{t} x_i^2 = \sum_{i=1}^{t} x_i^2z_i$$

$$a\sum_{i=1}^{t} x_i^2y_i^2 + b\sum_{i=1}^{t} y_i^4 + c\sum_{i=1}^{t} x_iy_i^3 + d\sum_{i=1}^{t} x_iy_i^2 + e\sum_{i=1}^{t} y_i^3 + f\sum_{i=1}^{t} y_i^2 = \sum_{i=1}^{t} y_i^2z_i$$

$$(1)\quad a\sum_{i=1}^{t} x_i^3y_i + b\sum_{i=1}^{t} x_iy_i^3 + c\sum_{i=1}^{t} x_i^2y_i^2 + d\sum_{i=1}^{t} x_i^2y_i + e\sum_{i=1}^{t} x_iy_i^2 + f\sum_{i=1}^{t} x_iy_i^2 = \sum_{i=1}^{t} x_iy_iz_i$$

$$a\sum_{i=1}^{t} x_i^3 + b\sum_{i=1}^{t} x_i y_i^2 + c\sum_{i=1}^{t} x_i^2 y_i + d\sum_{i=1}^{t} x_i^2 + e\sum_{i=1}^{t} x_i y_i + f\sum_{i=1}^{t} x_i = \sum_{i=1}^{t} x_i z_i$$
$$a\sum_{i=1}^{t} x_i^2 y_i + b\sum_{i=1}^{t} y_i^3 + c\sum_{i=1}^{t} x_i y_i^2 + d\sum_{i=1}^{t} x_i y_i + e\sum_{i=1}^{t} y_i^2 + f\sum_{i=1}^{t} y_i = \sum_{i=1}^{t} y_i z_i$$
$$a\sum_{i=1}^{t} x_i^2 + b\sum_{i=1}^{t} y_i^2 + c\sum_{i=1}^{t} x_i y_i + d\sum_{i=1}^{t} x_i + e\sum_{i=1}^{t} y_i + f\sum_{i=1}^{t} 1 = \sum_{i=1}^{t} z_i$$

one can obtain the coefficients a, b, c, d, e and f of the least squares paraboloid fit. The unique solution exists whenever the determinant of the system (1) is nonzero (for example, if all pixels belong to a parabola $z = \alpha x^2 + \beta x + \gamma$, then the solution is not unique). If T is a digital paraboloid segment $P_m(\rho)$, then T consists of m^2 digital points $(i, j, z(i,j))$, satisfying $0 \le i < m$, $0 \le j < m$ and $z(i,j) = \lfloor Ai^2 + Bj^2 + Cij + Di + Ej + F \rfloor$, and the system (1) becomes

$$a \cdot S_{40} + b \cdot S_{22} + c \cdot S_{31} + d \cdot S_{30} + e \cdot S_{21} + f \cdot S_{20} = \sum_{(i,j,z(i,j)) \in P_m(\rho)} i^2 z(i,j)$$
$$a \cdot S_{22} + b \cdot S_{40} + c \cdot S_{31} + d \cdot S_{21} + e \cdot S_{30} + f \cdot S_{20} = \sum_{(i,j,z(i,j)) \in P_m(\rho)} j^2 z(i,j)$$
$$a \cdot S_{31} + b \cdot S_{31} + c \cdot S_{22} + d \cdot S_{21} + e \cdot S_{21} + f \cdot S_{11} = \sum_{(i,j,z(i,j)) \in P_m(\rho)} ij z(i,j)$$
$$a \cdot S_{30} + b \cdot S_{21} + c \cdot S_{21} + d \cdot S_{20} + e \cdot S_{11} + f \cdot S_{10} = \sum_{(i,j,z(i,j)) \in P_m(\rho)} i z(i,j)$$
$$a \cdot S_{21} + b \cdot S_{30} + c \cdot S_{21} + d \cdot S_{11} + e \cdot S_{20} + f \cdot S_{10} = \sum_{(i,j,z(i,j)) \in P_m(\rho)} j z(i,j)$$
$$a \cdot S_{20} + b \cdot S_{20} + c \cdot S_{11} + d \cdot S_{10} + e \cdot S_{10} + f \cdot S_{00} = \sum_{(i,j,z(i,j)) \in P_m(\rho)} z(i,j),$$

where the coefficients $S_{ij} = S_{ji}$ for $i + j \le 4$ and $i, j = 0, 1, 2, 3, 4$ are defined as follows:

$$\begin{aligned} S_{40} &= \sum_{i=1}^{t} x_i^4 = \sum_{i=1}^{t} y_i^4 = \sum_{(i,j,z(i,j)) \in P_m(\rho)} i^4 = \sum_{(i,j,z(i,j)) \in P_m(\rho)} j^4 = \\ &= m \cdot (0^4 + 1^4 + \ldots + (m-1)^4) = \\ &= \frac{m^2(m-1)(2m-1)(3m^2-3m-1)}{30} \quad ; \\ S_{30} &= \sum_{i=1}^{t} x_i^3 = \sum_{i=1}^{t} y_i^3 = \sum_{(i,j,z(i,j)) \in P_m(\rho)} i^3 = \sum_{(i,j,z(i,j)) \in P_m(\rho)} j^3 = \end{aligned}$$

$$
\begin{aligned}
&= m\cdot(0^3+1^3+\ldots+(m-1)^3) = \frac{m^3(m-1)^2}{4} \quad ; \\
S_{31} &= \sum_{i=1}^{t} x_i^3 y_i = \sum_{i=1}^{t} x_i y_i^3 = \sum_{(i,j,z(i,j))\in P_m(\rho)} i^3 j = \sum_{(i,j,z(i,j))\in P_m(\rho)} i j^3 = \\
&= (0+1+\ldots+(m-1))\cdot(0^3+1^3+\ldots+(m-1)^3) = \\
&= \frac{m^3(m-1)^3}{8} \quad ; \\
S_{20} &= \sum_{i=1}^{t} x_i^2 = \sum_{i=1}^{t} y_i^2 = \sum_{(i,j,z(i,j))\in P_m(\rho)} i^2 = \\
&= m\cdot(0^2+1^2+\ldots+(m-1)^2) = \frac{m^2(m-1)(2m-1)}{6} \quad ; \\
S_{21} &= \sum_{i=1}^{t} x_i^2 y_i = \sum_{i=1}^{t} x_i y_i^2 = \sum_{(i,j,z(i,j))\in P_m(\rho)} i^2 j = \sum_{(i,j,z(i,j))\in P_m(\rho)} i j^2 = \\
&= (0+1+\ldots+(m-1))\cdot(0^2+1^2+\ldots+(m-1)^2) = \\
&= \frac{m^2(m-1)^2(2m-1)}{12} \quad ; \\
S_{22} &= \sum_{i=1}^{t} x_i^2 y_i^2 = \sum_{(i,j,z(i,j))\in P_m(\rho)} i^2 j^2 = \\
&= (0^2+1^2+\ldots+(m-1)^2)\cdot(0^2+1^2+\ldots(m-1)^2) = \\
&= \frac{m^2(m-1)^2(2m-1)^2}{36} \quad ; \\
S_{10} &= \sum_{i=1}^{t} x_i = \sum_{i=1}^{t} y_i = \sum_{(i,j,z(i,j))\in P_m(\rho)} i = \sum_{(i,j,z(i,j))\in P_m(\rho)} j = \\
&= m\cdot(0+1+\ldots+(m-1)) = \frac{m^2(m-1)}{2} \\
S_{11} &= \sum_{i=1}^{t} x_i y_i = \sum_{(i,j,z(i,j))\in P_m(\rho)} ij = \\
&= (0+1+\ldots+(m-1))\cdot(0+1+\ldots+(m-1)) = \frac{m^2(m-1)^2}{4} \quad ; \\
S_{00} &= \sum_{i=1}^{t} 1 = \sum_{(i,j,z(i,j))\in P_m(\rho)} 1 = m^2.
\end{aligned}
$$

For the above identities see, e.g., Krechmar (1974).

The previous system has the determinant equal to $\frac{m^{12}(m^2-1)^6(m^2-4)^2}{671846400}$, and so, for $m > 2$ the coefficients a, b, c, d, e, and f are uniquely determined. These coefficients give the least squares paraboloid fit $L_m(\rho)$ for a given digital paraboloid segment $P_m(\rho)$.

As a consequence of the previous observation, we have the following lemma.

Lemma 1. *If the digital paraboloid segment $P_m(\rho)$ is given, then its least squares paraboloid fit $L_m(\rho)$ is uniquely determined whenever $m > 2$.*

In the rest of the paper, the condition $m > 2$ will not be mentioned but assumed.

3. One-to-one correspondence between digital paraboloid segments and least squares paraboloid fits

In the previous section we showed that if a digital paraboloid segment $P_m(\rho)$ is given, then its least squares paraboloid fit, denoted as $L_m(\rho)$, can be determined uniquely.

A key question is whether there exist two different digital paraboloid segments with the same corresponding least squares paraboloid fits. The answer is no, and this implies that the digital paraboloid segments and their least squares paraboloid fits are in one-to-one correspondence. This is the main result of the paper.

Theorem 1. *Let $P_m(\rho)$ and $P_m(\rho')$ be two digital paraboloid segments and let $L_m(\rho)$ and $L_m(\rho')$ be their corresponding least squares paraboloid fits. Then $P_m(\rho) = P_m(\rho')$ is equivalent to $L_m(\rho) = L_m(\rho')$.*

Proof. Let the paraboloids ρ and ρ' be given by equations $z = Ax^2 + By^2 + Cxy + Dx + Ey + F$ and $z = A'x^2 + B'y^2 + C'xy + D'x + E'y + F'$, respectively. Then their corresponding digital paraboloid segments are

$$P_m(\rho) = \{(i, j, \lfloor Ai^2 + Bj^2 + Cij + Di + Ej + F \rfloor), 0 \le i < m, 0 \le j < m\} =$$

$$= \{(i,j,z(i,j)), 0 \le i < m, 0 \le j < m\},$$

and

$$P_m(\rho') = \{(i,j,\lfloor A'i^2+B'j^2+C'ij+D'i+E'j+F'\rfloor), 0 \le i < m, 0 \le j < m\} =$$

$$= \{(i,j,z'(i,j)), 0 \le i < m, 0 \le j < m\}.$$

Also, let the paraboloids $L_m(\rho)$ and $L_m(\rho')$ have the equations $z = ax^2 + by^2 + cxy + dx + ey + f$ and $z = a'x^2 + b'y^2 + c'xy + d'x + e'y + f'$, respectively.

The implication

$$P_m(\rho) = P_m(\rho') \Rightarrow L_m(\rho) = L_m(\rho')$$

follows from Lemma 1.

The opposite direction will be proved by a contradiction.

Suppose that $P_m(\rho)$ and $P_m(\rho')$ are two different digital paraboloid segments with the same associated least squares paraboloid fits $L_m(\rho)$ and $L_m(\rho')$, i.e.,

$$a = a' \quad \text{and} \quad b = b' \quad \text{and} \quad c = c' \quad \text{and} \quad d = d' \quad \text{and} \quad e = e' \quad \text{and} \quad f = f'.$$

Since a, b and c, as well as a', b' and c' are the solutions of system (2), it follows that

(2) $$\sum_{(i,j,z(i,j)) \in P_m(\rho)} z(i,j) = \sum_{(i,j,z'(i,j)) \in P_m(\rho')} z'(i,j) \quad ,$$

(3) $$\sum_{(i,j,z(i,j)) \in P_m(\rho)} iz(i,j) = \sum_{(i,j,z'(i,j)) \in P_m(\rho')} iz'(i,j) \quad ,$$

(4) $$\sum_{(i,j,z(i,j)) \in P_m(\rho)} jz(i,j) = \sum_{(i,j,z'(i,j)) \in P_m(\rho')} jz'(i,j) \quad ,$$

(5) $$\sum_{(i,j,z(i,j)) \in P_m(\rho)} ijz(i,j) = \sum_{(i,j,z'(i,j)) \in P_m(\rho')} ijz'(i,j) \quad ,$$

(6) $$\sum_{(i,j,z(i,j)) \in P_m(\rho)} i^2z(i,j) = \sum_{(i,j,z'(i,j)) \in P_m(\rho')} i^2z'(i,j) \quad ,$$

$$(7) \qquad \sum_{(i,j,z(i,j))\in P_m(\rho)} j^2 z(i,j) = \sum_{(i,j,z'(i,j))\in P_m(\rho')} j^2 z'(i,j) \quad ,$$

are satisfied.

Without loss of generality, we may assume that $z(i,j)$ and $z'(i,j)$ are strictly positive integers (for $0 \le i < m$ and $0 \le j < m$), otherwise, both paraboloids ρ and ρ' can be translated by a positive integer in the vertical direction, until all z-coordinates become positive. This process will increase all quantities (2)-(7) for both paraboloids by an equal amount.

Let S denote the set of all digital points (x,y,z) lying below the paraboloid ρ and above the xy-plane, satisfying $0 \le x < m$ and $0 \le y < m$, while S' denotes the set of all digital points (x,y,z) lying below the paraboloid ρ' and above the xy-plane, also satisfying $0 \le x < m$ and $0 \le y < m$ (x, y and z are integers).

In order to make a contradiction we start by interpretations of equalities (2)-(7).

- (2) implies that the number of digital points belonging to S and the number of digital points belonging to S' is the same. Moreover, we can write that the cardinality of the set $S \backslash S'$ and the cardinality of the set $S' \backslash S$ is the same and different from zero, i.e., $\#(S\backslash S') = \#(S'\backslash S) \neq 0$ (the inequality follows because $P_m(\rho) \neq P_m(\rho')$).

$$(8) \qquad \sum_{(x,y,z)\in S\backslash S'} 1 = \sum_{(x,y,z)\in S'\backslash S} 1.$$

- (3) implies that the sum of x-coordinates of all digital points from S coincides with the sum of all digital points from S'. Namely, suppose that each digital point (x,y,z) receives "weight" x. The sum of weights of digital points from S is $\sum_{(x,y,z)\in S} x$ (there are $z(i,j)$ points with the weight i, for fixed i and j). Similary, the sum of weights of digital points from S' is equal to $\sum_{(x,y,z)\in S'} x$. Further, after removing common points from both sides, we have

$$(9) \qquad \sum_{(x,y,z)\in S\backslash S'} x = \sum_{(x,y,z)\in S'\backslash S} x.$$

- Analogously to the previous item one can derive (from (4), (5), (6) and (7)),

(10) $$\sum_{(x,y,z)\in S\setminus S'} y = \sum_{(x,y,z)\in S'\setminus S} y \quad ,$$

(11) $$\sum_{(x,y,z)\in S\setminus S'} xy = \sum_{(x,y,z)\in S'\setminus S} xy \quad ,$$

(12) $$\sum_{(x,y,z)\in S\setminus S'} x^2 = \sum_{(x,y,z)\in S'\setminus S} x^2 \quad ,$$

(13) $$\sum_{(x,y,z)\in S\setminus S'} y^2 = \sum_{(x,y,z)\in S'\setminus S} y^2 \quad .$$

The projection of the intersection of the paraboloids ρ and ρ' on the xy-plane is the conic curve $(A-A')x^2+(B-B')y^2+(C-C')xy+(D-D')x+(E-E')y=F-F'$. This conic curve separates the projections of the points from $S\setminus S'$ and $S'\setminus S$ on xy-plane. Without loss of generality, suppose that the digital points from $S\setminus S'$ satisfy the inequality $(A-A')x^2+(B-B')y^2+(C-C')xy+(D-D')x+(E-E')y<F-F'$, while those from $S'\setminus S$, satisfy the opposite inequality $(A-A')x^2+(B-B')y^2+(C-C')xy+(D-D')x+(E-E')y>F-F'$.

Consider the expression

$$\begin{aligned}
S &= (A-A')\cdot \sum_{(x,y,z)\in S\setminus S'} x^2 + (B-B')\cdot \sum_{(x,y,z)\in S'\setminus S} y^2 + (C-C')\cdot \sum_{(x,y,z)\in S\setminus S'} xy + \\
&\quad +(D-D')\cdot \sum_{(x,y,z)\in S'\setminus S} x + (E-E')\cdot \sum_{(x,y,z)\in S\setminus S'} y = \\
&= \sum_{(x,y,z)\in S\setminus S'} (A-A')x^2 + (B-B')y^2 + (C-C')xy + \\
&\quad +(D-D')x + (E-E')y < \\
&< \sum_{(x,y,z)\in S\setminus S'} F'-F = (F-F')\cdot \#(S\setminus S').
\end{aligned}$$

On the oher hand, by using (8)-(13),

$$S = (A-A')\cdot\sum_{(x,y,z)\in S\setminus S'} x^2 + (B-B')\cdot\sum_{(x,y,z)\in S'\setminus S} y^2 + (C-C')\cdot\sum_{(x,y,z)\in S\setminus S'} xy +$$

$$+(D-D')\cdot\sum_{(x,y,z)\in S'\setminus S} x + (E-E')\cdot\sum_{(x,y,z)\in S\setminus S'} y =$$

$$= (A-A')\cdot\sum_{(x,y,z)\in S'\setminus S} x^2 + (B-B')\cdot\sum_{(x,y,z)\in S\setminus S'} y^2 + (C-C')\cdot\sum_{(x,y,z)\in S'\setminus S} xy +$$

$$+(D-D')\cdot\sum_{(x,y,z)\in S\setminus S'} x + (E-E')\cdot\sum_{(x,y,z)\in S'\setminus S} y =$$

$$= \sum_{(x,y,z)\in S'\setminus S} (A-A')x^2 + (B-B')y^2 + (C-C')xy +$$

$$+(D-D')x + (E-E')y >$$

$$> \sum_{(x,y,z)\in S\setminus S'} F' - F = (F-F')\cdot\#(S'\setminus S),$$

which leads to the contradiction

$$(F-F')\cdot\#(S\setminus S') < S < (F-F')\cdot\#(S'\setminus S),$$

because $\#(S\setminus S')$ is equal to $\#(S'\setminus S)$. □

4. Conclusion

In this paper the least squares fitting technique is applied for digital paraboloid segments. It is shown that digital paraboloid segments and their least squares paraboloid fits are in one-to-one correspondence if the base for digital paraboloid segments is fixed. This result enables the first known constant space representation for digital paraboloid segments with bases which can be represented with a finite number of parameters. For such a representation it is enough to have the parameters for the base representation plus six parameters which are the coefficients of their least squares paraboloid fit.

The situation where the bases are assumed to be squares in the xy-plane is studied in detail. It is easy to extend the result to the cases when the bases are of some other shape representable by a constant number of parameters.

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