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DIFFERENT ADAPTED BASES AND METRICAL GENERALIZED CONNECTIONS

Irena Čomić
Faculty of Engineering, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

In the tangent space of the vector bundle $\xi = (E, \pi, M)$, $\dim M = n$, $\dim E = n + m$, using two different nonlinear connections $N_i^a(x, y)$ and $\overline{N}_i^a = N_i^a + T_i^a$ (T_i^a is a tensor field), two adapted bases $B = \{\delta_i, \partial_a\}$ and $\overline{B} = \{\overline{\delta}_i, \partial_a\}$ are introduced. The relations between the components of the generalized linear connection ∇, the metric tensor and the torsion tensor expressed in these two bases are given.

It is proved that the distinguished d–connection in the basis B will be d–connection in the basis $\overline{B}$ iff T_i^a is h– and v–parallel tensor field. The metrical connection ∇ in the basis B will be also metrical in the basis $\overline{B}$.

The coefficients of the metrical generalized connections, when $T_H(E)$ is not orthogonal to $T_V(E)$, are obtained. They are functions of the given metric tensor, nonlinear connection and arbitrary torsion tensor. When the metrical generalized connection is compatible with co–symplectic, almost Hermitian, conformal, almost complex, almost product etc. structures, then, some restrictions for the torsion tensor are obtained, but their exact form is an open problem. For the special case, for the Finsler bundle and d–connection most of these problems are solved ([1]-[9], [13], [14]).

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1. The Geometry of Vector Bundles

Let $\xi = (E, \pi, M)$ be a C^∞ vector bundle with $dimM = n, dimE = n + m$. In some local chart the point $u \in E$ has the coordinates

$$(x^1, \ldots, x^n, y^1, \ldots, y^m) = (x^i, y^a) = (x, y)$$

$$i, j, k, l = 1, \ldots, n \qquad a, b, c, d, e, f = 1, \ldots, m.$$

The allowable coordinate transformations $(x, y) \to (x', y')$ are given by

$$(1.1) \quad \begin{array}{ll} \text{(a)}\ x^{i'} = x^{i'}(x^1, \ldots, x^n) & \text{rank}[\partial x^{i'}/\partial x^i] = n \\ \text{(b)}\ y^{a'} = M_a^{a'}(x) y^a & \text{rank}[M_a^{a'}] = m, \end{array}$$

so the inverse transformation $(x', y') \to (x, y)$ exists, and is determined by

$$(1.2) \quad \begin{array}{ll} \text{(a)}\ x^i = x^i(x^{1'}, \ldots, x^{n'}) & \text{(c)}\ M_a^{a'} M_{b'}^a = \delta_{b'}^{a'} \\ \text{(b)}\ y^a = M_{a'}^a(x') y^{a'} & \text{(d)}\ M_b^{a'} M_{a'}^a = \delta_b^a. \end{array}$$

The tangent space $T(E)$ is spanned by $\{\partial_i, \partial_a\}$, where

$$(1.3) \quad \partial_i = \partial/\partial x^i \qquad \partial_a = \partial/\partial y^a.$$

They have the following law of transformation:

$$(1.4) \quad \begin{array}{ll} \text{(a)}\ \partial_a = M_a^{a'}(x)\partial_{a'} & \partial_{a'} = M_{a'}^a(x')\partial_a \\ \text{(b)}\ \partial_i = (\partial_i x^{i'})\partial_{i'} + (\partial_i M_b^{a'}(x)) y^b \partial_{a'}. & \end{array}$$

If M is paracompact, then there exists a family of functions $N_i^a(x, y)$, with the following law of transformation:

$$(1.5) \quad N_i^a(x, y) = N_{i'}^{a'}(x', y')(\partial_i x^{i'}) M_{a'}^a(x') - (\partial_{i'} M_{b'}^a(x')) y^{b'} (\partial_i x^{i'}).$$

These functions $N_i^a(x, y)$ are called the coefficients of nonlinear connection. With them, we can transform the basis $\{\partial_i, \partial_a\}$, whose vectors under the coordinate transformation (1.1) or (1.2) are not transforming as vectors, into the basis $B = \{\delta_i, \partial_a\}$, whose vectors have this property. δ_i is defined by

(1.6) $\delta_i = \partial_i - N_i^a \partial_a.$

It is easy to prove from (1.4) and (1.5) that

(1.7) $\delta_{i'} = \delta_i(\partial_{i'} x^i).$

Any vector field X in $T(E)$ can be represented in the basis B in the following form

(1.8) $X = X^i \delta_i + X^a \partial_a, \quad X^i = X^{i'}(\partial_{i'} x^i), \quad X^a = X^{a'} M_{a'}^a.$

We call $X^a \partial_a$ vertical vector field and $X^i \delta_i$ horizontal vector field of $T(E)$, respectively. We shall denote the subspace of $T(E)$ spanned by $\{\delta_i\}$ by $T_H(E)$ and the subspace spanned by $\{\partial_a\}$ by $T_V(E)$. So we have

$$T(E) = T_H(E) \oplus T_V(E) \quad \dim T_H(E) = n \quad \dim T_V(E) = m.$$

Let us consider the dual tangent space of E, the space $T^*(E)$. The natural basis in $T^*(E)$ is

$$\{dx^1, \ldots, dx^n, dy^1, \ldots, dy^m\} = \{dx^i, dy^a\}.$$

From (1.1) we have

(1.9) (a) $dx^{i'} = (\partial_i x^{i'}) dx^i$ (b) $dy^{a'} = (\partial_i M_a^{a'}) y^a dx^i + M_a^{a'} dy^a.$

It is obvious that dy^a $a = 1, \ldots, m$ are not transforming as tensors, so we introduce a new basis

$$B^* = \{dx^i, \delta y^a\} \text{ of } T^*(E), \text{ where}$$

(1.10) $\delta y^a = dy^a + N_i^a dx^i$.

By the coordinate transformation (1.1) and (1.2) the bases B^* and $B^{*'} = \{dx^{i'}, \delta y^{a'}\}$ are related by (1.9a) and

(1.11) (a) $\delta y^a = M_{a'}^a(x')\delta y^{a'}$ (b) $\delta y^{a'} = M_a^{a'}(x)\delta y^a$.

If B^* and $B^{*'}$ are two bases of $T^*(E)$ related by (1.9a) and (1.11), then any 1–form $w \in T^*(E)$ satisfies the relations

(1.12) $w = w_i dx^i + w_a \delta y^a = w_{i'} dx^{i'} + w_{a'} \delta y^{a'}$,

where

(1.13) $w_{i'} = w_i(\partial_{i'} x^i)$ $w_{a'} = w_a M_{a'}^a$.

Let us consider $T^*(E) \otimes T^*(E)$. In this space, a metric tensor G will be given with respect to the basis B^*, by

(1.14) $G = g_{ij} dx^i \otimes dx^j + g_{ib} dx^i \otimes \delta y^b + g_{aj} \delta y^a \otimes dx^j + g_{ab} \delta y^a \otimes \delta y^b$.

$T_H(E)$ is orthogonal to $T_V(E)$ with respect to the metric G iff $g_{ib} = 0$ and $g_{aj} = 0$ for $\forall i, j = 1, \ldots, n$ and $\forall a, b = 1, \ldots, m$.

With respect to the coordinate transformations (1.1) and (1.2) the coordinates of the metric tensor G are transforming in the following way

(1.15)
$$g_{i'j'} = g_{ij}(\partial_{i'} x^i)(\partial_{j'} x^j) \qquad g_{i'b'} = g_{ib}(\partial_{i'} x^i) M_{b'}^b$$
$$g_{a'j'} = g_{aj} M_{a'}^a (\partial_{j'} x^j) \qquad g_{a'b'} = g_{ab} M_{a'}^a M_{b'}^b.$$

We shall define the covariant coordinates of vector $X = X^i \delta_i + X^a \partial_a$ by

(1.16) $X_i = g_{ij} X^j + g_{ib} X^b$ $X_a = g_{aj} X^j + g_{ab} X^b$.

Definition 1.1. *The generalized connection* $\nabla : T(E) \otimes T(E) \to T(E)$ $(\nabla : (X, Y) \to \nabla_X Y)$ *or equivalently* $\nabla_X : T(E) \to T(E)$ $(\nabla_X : Y \to \nabla_X Y)$ *is a linear connection for which*

$$(1.17)\quad \begin{array}{ll} \nabla_{\delta_i}\delta_j = F_{j\ i}^{\ k}\delta_k + F_{j\ i}^{\ c}\partial_c & \nabla_{\delta_j}\partial_a = F_{a\ j}^{\ k}\delta_k + F_{a\ j}^{\ c}\partial_c \\ \nabla_{\partial_a}\delta_j = C_{j\ a}^{\ k}\delta_k + C_{j\ a}^{\ c}\partial_c & \nabla_{\partial_a}\partial_b = C_{b\ a}^{\ k}\delta_k + C_{b\ a}^{\ c}\partial_c. \end{array}$$

The linearity means that for $\forall X, Y \in T(E)$

$$(1.18)\quad X = X^i\delta_i + X^a\partial_a \quad , \quad Y = Y^j\delta_j + Y^b\partial_b$$

we have

$$(1.19)\quad \begin{aligned} \nabla_X Y \ = \ & X^i(\delta_i Y^j)\delta_j + X^i Y^j \nabla_{\delta_i}\delta_j + X^i(\delta_i Y^b)\partial_b + X^i Y^b \nabla_{\delta_i}\partial_b + \\ & X^a(\partial_a Y^j)\delta_j + X^a Y^j \nabla_{\partial_a}\delta_j + X^a(\partial_a Y^b)\partial_b + X^a Y^b \nabla_{\partial_a}\partial_b. \end{aligned}$$

From (1.17)–(1.19) follows

Proposition 1.1. *For linear connection ∇ defined by (1.16) and (1.19) and vector fields X and Y given by (1.18) we have*

$$(1.20)\quad \nabla_X Y = (Y^j_{|i}X^i + Y^j|_a X^a)\delta_j + (Y^b_{|i}X^i + Y^b|_a X^a)\partial_b,$$

where

$$(1.21)\quad \begin{array}{ll} (a)\ Y^x_{|i} = \delta_i Y^x + F_{k\ i}^{\ x}Y^k + F_{b\ i}^{\ x}Y^b, & \\ (b)\ Y^x|_a = \partial_a Y^x + C_{k\ a}^{\ x}Y^k + C_{b\ a}^{\ x}Y^b, & x = j \ \text{or}\ x = b. \end{array}$$

Proposition 1.2. *If (x, y) and (x', y') are two coordinate systems connected by the transformation laws (1.1) and (1.2), then*

$$(1.22)\quad \nabla_{X'}Y' = \nabla_X Y$$

if and only if

$$(1.23)\quad \begin{array}{ll} (a)\ Y^j_{|i} = Y^{j'}_{|i'}(\partial_i x^{i'})(\partial_{j'}x^j) & (b)\ Y^j|_a = Y^{j'}|_{a'}(\partial_{j'}x^j)M^{a'}_a \\ (c)\ Y^b_{|i} = Y^{b'}_{|i'}(\partial_i x^{i'})M^b_{b'} & (d)\ Y^b|_a = Y^{b'}|_{a'}M^{a'}_a M^b_{b'} \end{array}$$

or equivalently

$$(1.24)\quad (a)\quad F_{k\ i}^{\ j} = F_{k'\ i'}^{\ j'}(\partial_k x^{k'})(\partial_{j'}x^j)(\partial_i x^{i'}) + (\partial_k\partial_i x^{j'})(\partial_{j'}x^j)$$

$$\begin{array}{ll}
(b) & F_{c\,i}^{\,b} = F_{c'\,i'}^{\,b'}(\partial_i x^{i'})M_{b'}^{b}M_c^{c'} + (\partial_i M_c^{b'})M_{b'}^{b} \\
(c) & F_{b\,i}^{\,j} = F_{b'\,i'}^{\,j'}M_b^{b'}(\partial_{j'}x^j)(\partial_i x^{i'}) \\
(d) & F_{k\,i}^{\,b} = F_{k'\,i'}^{\,b'}(\partial_k x^{k'})(\partial_i x^{i'})M_{b'}^{b} \\
(e) & C_{k\,a}^{\,j} = C_{k'\,a'}^{\,j'}(\partial_{j'}x^j)(\partial_k x^{k'})M_a^{a'} \\
(f) & C_{b\,a}^{\,j} = C_{b'\,a'}^{\,j'}(\partial_{j'}x^j)M_b^{b'}M_a^{a'} \\
(g) & C_{k\,a}^{\,b} = C_{k'\,a'}^{\,b'}(\partial_k x^{k'})M_{b'}^{b}M_a^{a'} \\
(h) & C_{c\,a}^{\,b} = C_{c'\,a'}^{\,b'}M_{b'}^{b}M_c^{c'}M_a^{a'}
\end{array}$$

From (1.24b) and (1.16) follows that

$$F_{c\,i}^{\,b} = F_{c'\,i'}^{\,b'}(\partial_i x^{i'})M_{b'}^{b} + (\partial_i M_c^{b'})y^c M_{b'}^{b}.$$

From (1.2d) follows

$$(\partial_i M_c^{b'})y^c M_{b'}^{b} = -(\partial_{i'} M_{b'}^{b})u^{b'}(\partial_i x^{i'}),$$

so the comparation with (1.5) shows that $F_{c\,i}^{\,b}y^c$ has the same law of transformation as N_i^b, and can take

$$N_i^b = F_{c\,i}^{\,b}y^c.$$

Proposition 1.3. *The torsion tensor*

(1.25) $T(X,Y) = \nabla_X Y - \nabla_Y X - [XY]$

for the connection ∇ *has the form*

$$(1.26)\quad \begin{aligned} T(X,Y) = {} & (T_{j\,i}^{\,k}Y^jX^i + T_{b\,i}^{\,k}Y^bX^i + T_{j\,a}^{\,k}Y^jX^a + T_{b\,a}^{\,k}Y^bX^a)\delta_k + \\ & (T_{j\,i}^{\,c}Y^jX^i + T_{b\,i}^{\,c}Y^bX^i + T_{j\,a}^{\,c}Y^jX^a + T_{b\,a}^{\,c}Y^bX^a)\partial_c, \end{aligned}$$

where

$$(1.27)\quad \begin{array}{ll}
(a)\ T_{j\,i}^{\,k} = F_{j\,i}^{\,k} - F_{i\,j}^{\,k} & (e)\ T_{j\,i}^{\,c} = F_{j\,i}^{\,c} - F_{i\,j}^{\,c} + \delta_i N_j^c - \delta_j N_i^c \\
(b)\ T_{b\,i}^{\,k} = F_{b\,i}^{\,k} - C_{i\,b}^{\,k} & (f)\ T_{b\,i}^{\,c} = F_{b\,i}^{\,c} - C_{i\,b}^{\,c} - \partial_b N_i^c \\
(c)\ T_{j\,a}^{\,k} = C_{j\,a}^{\,k} - F_{a\,j}^{\,k} & (g)\ T_{j\,a}^{\,c} = C_{j\,a}^{\,c} - F_{a\,j}^{\,c} + \partial_a N_j^c \\
(d)\ T_{b\,a}^{\,k} = C_{b\,a}^{\,k} - C_{a\,b}^{\,k} & (h)\ T_{b\,a}^{\,c} = C_{b\,a}^{\,c} - C_{a\,b}^{\,c}
\end{array}$$

2. The Adapted Bases Formed with Different Nonlinear Connections

Let us consider the nonlinear connection $\overline{N}_i^a(x,y)$ which, under coordinate transformation (1.1) and (1.2), transforms in the following way

$$(2.1)\quad \overline{N}_i^a(x,y) = \overline{N}_{i'}^{a'}(x',y')(\partial_i x^{i'})M_{a'}^a(x') - (\partial_{i'} M_{b'}^a(x'))y^{b'}(\partial_i x^{i'})$$

We shall use the notation

$$(2.2)\quad T_i^a = N_i^a - \overline{N}_i^a.$$

From (1.5) and (2.1) we obtain

$$(2.3)\quad T_i^a = T_{i'}^{a'}(\partial_i x^{i'})M_{a'}^a(x'),$$

which shows that the difference of two nonlinear connection with respect to the coordinate transformation (1.1) and (1.2) is transforming as tensor.

We denote the adapted basis formed with $\overline{N}_i^a$ by $\overline{B} = \{\overline{\delta}_i, \partial_a\}$ where

$$(2.4)\quad \overline{\delta}_i = \partial_i - \overline{N}_i^a(x,y)\partial_a$$

From (1.6) and (2.4) it follows that

$$(2.5)\quad \overline{\delta}_i = \delta_i + T_i^a\partial_a$$

The dual basis of $\overline{B}$ is $\overline{B}^* = \{dx^i, \overline{\delta}y^a\}$, where

$$(2.6)\quad \overline{\delta}y^a = dy^a + \overline{N}_i^a dx^i.$$

From (1.10) and (2.6) follows

$$(2.7)\quad \delta y^a = \overline{\delta}y^a + T_i^a dx^i.$$

The vector field $X \in T(E)$ and the one form field $w \in T^*(E)$ given by (1.8) and (1.12) respectively, are in the basis $\overline{B}$ given by

(2.8) $X = \overline{X}^i\overline{\delta}_i + \overline{X}^a\partial_a \qquad w = \overline{w}_i dx^i + \overline{w}_a\overline{\delta}y^a.$

Their coordinates in the bases B and $\overline{B}$ are connected by

$$(2.9) \qquad \begin{aligned} X^i &= \overline{X}^i \\ X^a &= \overline{X}^a + \overline{X}^i T_i^a \end{aligned} \qquad\qquad \begin{aligned} \overline{w}_i &= w_i + w_a T_i^a \\ \overline{w}_a &= w_a \end{aligned}$$

(2.9) is the consequence of (2.4), (2.7), (1.8), (1.12) and (2.8). The metric tensor G given by (1.14), in the basis $\overline{B}^*$ has the form

$$(2.10) \quad G = \overline{g}_{ij}dx^i \otimes dx^j + \overline{g}_{ib}dx^i \otimes \overline{\delta}y^b + \overline{g}_{aj}\overline{\delta}y^a \otimes dx^j + \overline{g}_{ab}\overline{\delta}y^a \otimes \overline{\delta}y^b$$

The components of G in the bases B^* and $\overline{B}^*$ are connected by

$$(2.11) \qquad \begin{aligned} \overline{g}_{ij} &= g_{ij} + g_{ib}T_j^b + g_{aj}T_i^a + g_{ab}T_i^aT_j^b \\ \overline{g}_{ib} &= g_{ib} + g_{ab}T_i^a \\ \overline{g}_{aj} &= g_{aj} + g_{ab}T_j^b \\ \overline{g}_{ab} &= g_{ab} \end{aligned}$$

(2.11) follows from (1.14), (2.6) and (2.10).

If $T_H(E)$ spanned by $\{\delta_i\}$ is orthogonal to $T_V(E)$ spanned by $\{\partial_a\}$, i.e. $g_{ib} = 0$, then the components of the metric tensor G in the base $\overline{B}^*$ are given by

$$(2.11)' \qquad \begin{aligned} \overline{g}_{ij} &= g_{ij} + g_{ab}T_i^aT_j^b \\ \overline{g}_{ib} &= g_{ab}T_i^a \\ \overline{g}_{aj} &= g_{ab}T_j^b \\ \overline{g}_{ab} &= g_{ab} \end{aligned}$$

The linear connection $\nabla : (X,Y) \to \nabla_X Y$ defined by Definition 1.1. or (1.17) in the basis $\overline{B}$ is expressed in the following way:

$$(2.12) \qquad \begin{aligned} \nabla_{\overline{\delta}_i}\overline{\delta}_j &= \overline{F}_{j\ i}^{\ k}\overline{\delta}_k + \overline{F}_{j\ i}^{\ c}\partial_c \\ \nabla_{\partial_a}\overline{\delta}_j &= \overline{C}_{j\ a}^{\ h}\overline{\delta}_h + \overline{C}_{j\ a}^{\ c}\partial_c \end{aligned} \qquad\qquad \begin{aligned} \nabla_{\overline{\delta}_j}\partial_a &= \overline{F}_{a\ j}^{\ k}\overline{\delta}_k + \overline{F}_{a\ j}^{\ c}\partial_c \\ \nabla_{\partial_a}\partial_b &= \overline{C}_{b\ a}^{\ k}\overline{\delta}_k + \overline{C}_{b\ a}^{\ c}\partial_c. \end{aligned}$$

Lemma 2.1. *The coefficients of the same linear connection $\nabla : T(E) \otimes T(E) \to T(E)$ expressed in the bases B and $\overline{B}$ are connected by*

$$
\begin{aligned}
&(2.13) \quad (a) \quad \overline{F}_{j\,i}^{\,k} = F_{j\,i}^{\,k} + F_{b\,i}^{\,k}T_j^b + C_{j\,a}^{\,k}T_i^a + C_{b\,a}^{\,k}T_i^aT_j^b \\
&(b) \quad \overline{F}_{j\,i}^{\,c} = (F_{j\,i}^{\,c} - F_{b\,i}^{\,k}T_j^bT_k^c) + T_i^a(C_{j\,a}^{\,c} - C_{b\,a}^{\,k}T_j^bT_k^c) + T_{j|i}^c + T_j^c|_aT_i^a \\
&(c) \quad \overline{F}_{a\,j}^{\,k} = F_{a\,j}^{\,k} + C_{a\,b}^{\,k}T_j^b \\
&(d) \quad \overline{F}_{a\,j}^{\,c} = F_{a\,j}^{\,c} - F_{a\,j}^{\,k}T_k^c + (C_{a\,b}^{\,c} - C_{a\,b}^{\,k}T_k^c)T_j^b \\
&(e) \quad \overline{C}_{j\,a}^{\,k} = C_{j\,a}^{\,k} + C_{b\,a}^{\,k}T_j^b \\
&(f) \quad \overline{C}_{j\,a}^{\,c} = C_{j\,a}^{\,c} - C_{b\,a}^{\,k}T_j^bT_k^c + T_j^c|_a \\
&(g) \quad \overline{C}_{b\,a}^{\,k} = C_{b\,a}^{\,k} \\
&(h) \quad \overline{C}_{b\,a}^{\,c} = C_{b\,a}^{\,c} - C_{b\,a}^{\,k}T_k^c,
\end{aligned}
$$

where

$$
\begin{aligned}
T_{j|i}^c &= \delta_iT_j^c - F_{j\,i}^{\,k}T_k^c + F_{b\,i}^{\,c}T_j^b \\
T_j^c|_a &= \partial_aT_j^c - C_{j\,a}^{\,k}T_k^c + C_{b\,a}^{\,c}T_j^b.
\end{aligned}
$$

(2.13) follows from (2.12), (2.5), (1.17), and (1.19). From (2.13) and (1.24) there follows that $\overline{F}_{j\,i}^{\,h}$ and $\overline{F}_{a\,j}^{\,c}$ are transforming as usual connection coefficients, the others as tensors.

We can assert that (1.24) is valid if all letters F and C in them are overlined.

Theorem 2.1. *The Miron's d–connection ∇ in the adapted basis $B = \{\delta_i, \partial_a\}$ in general case will not be d–connection in the adapted basis $\overline{B} = \{\overline{\delta}_i, \partial_a\}$.*

Proof. As is known, the d–connection ∇ is the linear connection with the property

$$
\begin{aligned}
\nabla &: T(E) \otimes T_H(E) \to T_H(E) \\
\nabla &: T(E) \otimes T_V(E) \to T_V(E),
\end{aligned}
$$

or expressed in coordinates:

$$(2.14)\qquad \nabla_{\delta_i}\delta_j = F_{j\ i}^{\ k}\delta_k \qquad \nabla_{\delta_j}\partial_a = F_{a\ j}^{\ c}\partial_c$$
$$\nabla_{\partial_a}\delta_j = C_{j\ a}^{\ k}\delta_k \qquad \nabla_{\partial_a}\partial_b = C_{b\ a}^{\ c}\partial_c$$

Comparing (1.17) with (2.14) it is obvious that if we put

$$(2.15)\quad F_{j\ i}^{\ e} = 0 \qquad F_{a\ j}^{\ k} = 0 \qquad C_{j\ a}^{\ c} = 0 \qquad C_{b\ a}^{\ k} = 0$$

in (1.17) then the d connection defined by (2.14) is obtained.

By substituting (2.15) into (2.13) we obtain that the components of the d–connection in the basis $\overline{B}$ are given by

$$(2.16)\qquad \begin{array}{ll}
(a) & \overline{F}_{j\ i}^{\ k} = F_{j\ i}^{\ k} + C_{j\ a}^{\ k}T_i^a \\
(b) & \overline{F}_{j\ i}^{\ c} = T_{j|i}^c + T_j^c|_a T_i^a \\
(c) & \overline{F}_{a\ j}^{\ h} = 0 \\
(d) & \overline{F}_{a\ j}^{\ c} = F_{a\ j}^{\ c} + C_{a\ b}^{\ c}T_j^b \\
(e) & \overline{C}_{j\ a}^{\ k} = C_{j\ a}^{\ k} \\
(f) & \overline{C}_{j\ a}^{\ c} = T_j^c|_a \\
(g) & \overline{C}_{b\ a}^{\ k} = 0 \\
(h) & \overline{C}_{b\ a}^{\ c} = C_{b\ a}^{\ c}
\end{array}$$

From (2.16) and (2.12) we can see that the d–connection in the basis $\overline{B} = \{\overline{\delta}_i, \partial_a\}$ has the property

$$\nabla : T(E)\otimes \overline{T}_H(E) \not\to \overline{T}_H(E)$$
$$\nabla : T(E)\otimes T_V(E) \to T_V(E),$$

where $\overline{T}_H(E)$ is spanned by $\{\overline{\delta}_i\}$.

Theorem 2.2. *The necessary and sufficient conditions that the d–connection ∇ in the basis $B = \{\delta_i, \partial_a\}$ be also a d–connection in the basis $\overline{B} = \{\overline{\delta}_i, \partial_a\}$ are*

$$(2.17)\qquad T_{j|i}^c = 0 \qquad\qquad T_j^c|_a = 0,$$

i.e. the tensor field T_j^c in the basis B with respect to the d–connection ∇ should be h–parallel and v–parallel.

Proof. From (2.12), (2.16), and (2.17) it follows that

$$(2.18)\qquad \nabla_{\bar\delta_i}\bar\delta_j = \overline{F}_{j\,i}^{\,k}\bar\delta_k \qquad\qquad \nabla_{\bar\delta_j}\partial_a = \overline{F}_{a\,j}^{\,c}\partial_c$$
$$\nabla_{\partial_a}\bar\delta_j = \overline{C}_{j\,a}^{\,k}\bar\delta_k \qquad\qquad \nabla_{\partial_a}\partial_b = \overline{C}_{b\,a}^{\,c}\partial_c,$$

where the connection coefficients are determined by (2.16a, d, e, h). The converse is obvious.

Theorem 2.3. *The components of the same torsion tensor $T(X,Y)$ defined by (1.25) of the generalized connection ∇ in the bases B and $\overline{B}$ are connected by*

$$(2.19)\qquad (a)\quad \overline{T}_{j\,i}^{\,k} = T_{j\,i}^{\,k} + T_{b\,i}^{\,k}T_j^b + T_{j\,a}^{\,k}T_i^a + T_{b\,a}^{\,k}T_j^bT_i^a$$
$$(b)\quad \overline{T}_{b\,i}^{\,k} = T_{b\,i}^{\,k} + T_{b\,a}^{\,k}T_i^a$$
$$(c)\quad \overline{T}_{j\,a}^{\,k} = T_{j\,a}^{\,k} + T_{b\,a}^{\,k}T_j^b \qquad (d)\quad \overline{T}_{b\,a}^{\,k} = T_{b\,a}^{\,k}$$
$$(e)\quad \overline{T}_{j\,i}^{\,c} = T_{j\,i}^{\,c} - T_{j\,i}^{\,k}T_k^c - T_{b\,i}^{\,k}T_j^bT_k^c + T_{b\,j}^{\,k}T_i^bT_k^c$$
$$-T_{b\,a}^{\,k}T_j^bT_i^aT_k^c + T_{a\,i}^{\,c}T_j^a - T_{a\,j}^{\,c}T_i^a + T_{b\,a}^{\,c}T_j^bT_i^a$$
$$(f)\quad \overline{T}_{b\,i}^{\,c} = T_{b\,i}^{\,c} + T_{b\,a}^{\,c}T_i^a - T_{b\,i}^{\,k}T_k^c - T_{b\,a}^{\,k}T_i^aT_k^c$$
$$(g)\quad \overline{T}_{j\,a}^{\,c} = T_{j\,a}^{\,c} + T_{b\,a}^{\,c}T_j^b - T_{j\,a}^{\,k}T_k^c - T_{b\,a}^{\,k}T_j^bT_k^c$$
$$(h)\quad \overline{T}_{b\,a}^{\,c} = T_{b\,a}^{\,c} - T_{b\,a}^{\,k}T_k^c.$$

The proof follows from (1.25), (2.5) and (2.9).

Theorem 2.4. *The torsion tensor $T(X,Y)$ for the Miron's d-connection defined by (2.14) satisfies (1.25), (1.26), but their components are given by*

$$(2.20)\qquad (a)\ T_{j\,i}^{\,k} = F_{j\,i}^{\,k} - F_{i\,j}^{\,k} \qquad\qquad (e)\ T_{j\,i}^{\,c} = \delta_iN_j^c - \delta_jN_i^c$$
$$(b)\ T_{b\,i}^{\,k} = -C_{i\,b}^{\,k} \qquad\qquad (f)\ T_{b\,i}^{\,c} = F_{b\,i}^{\,c} - \partial_bN_i^c$$
$$(c)\ T_{j\,a}^{\,k} = C_{j\,a}^{\,k} \qquad\qquad (g)\ T_{j\,a}^{\,c} = \partial_aN_j^c - F_{a\,j}^{\,c}$$
$$(d)\ T_{b\,a}^{\,k} = 0 \qquad\qquad (h)\ T_{b\,a}^{\,c} = C_{b\,a}^{\,c} - C_{a\,b}^{\,c}.$$

The proof follows from (1.27) and (2.15).

Theorem 2.5. *The components of the same torsion tensor $T(X,Y)$ defined by (1.25) of Miron's d-connection in the bases B and $\overline{B}$ are connected by*

$$(2.21)\qquad (a)\quad \overline{T}_{j\,i}^{\,k} = T_{j\,i}^{\,k} + T_{b\,i}^{\,k}T_j^b + T_{j\,a}^{\,k}T_i^a$$

$$
\begin{aligned}
&(b) \quad \overline{T}_{b\,i}^{\,k} = T_{b\,i}^{\,k} \qquad (c)\ \overline{T}_{j\,a}^{\,k} = T_{j\,a}^{\,k} \\
&(d) \quad \overline{T}_{b\,a}^{\,k} = 0 = T_{b\,a}^{\,k} \\
&(e) \quad \overline{T}_{j\,i}^{\,c} = T_{j\,i}^{\,c} - T_{j\,i}^{\,k} T_k^c - T_{b\,i}^{\,k} T_j^b T_k^c + T_{b\,j}^{\,k} T_i^b T_k^c \\
&\qquad + T_{a\,i}^{\,c} T_j^a - T_{a\,j}^{\,c} T_i^a + T_{b\,a}^{\,c} T_j^b T_i^a \\
&(f) \quad \overline{T}_{b\,i}^{\,c} = T_{b\,i}^{\,c} + T_{b\,a}^{\,c} T_i^a - T_{b\,i}^{\,k} T_k^c \\
&(g) \quad \overline{T}_{j\,a}^{\,c} = T_{j\,a}^{\,c} + T_{b\,a}^{\,c} T_j^b - T_{j\,a}^{\,k} T_k^c \\
&(h) \quad \overline{T}_{b\,a}^{\,c} = T_{b\,a}^{\,c}.
\end{aligned}
$$

where the components of $T(X,Y)$ in the basis B are given by (2.20).

3. The Covariant Derivatives of the Metric Tensor

In $T^*(E) \otimes T^*(E)$ the metric tensor G expressed in the basis B^* is given by (1.14) and in the basis $\overline{B}^*$ by (2.10). The connections between the coordinates of the same metric tensor G in different bases are given by (2.11). The linear connection ∇ in the basis B is defined by (1.17) and in the basis $\overline{B}$ by (2.12). The realtion between the coefficients of the same connection ∇ expressed in different bases B and $\overline{B}$ are given by (2.13). The metrical connection has the property that h– and v–covariant derivatives of all coordinates of the metric tensor are equal to zero. The question is: when will the metrical connection in the basis B^* be metrical in the basis $\overline{B}^*$? To get the answer to this question it is necessary to obtain the relations between the covariant derivatives of the metrical tensor in these two bases.

Theorem 3.1. *The relations between the covariant derivatives of the same metric tensor G, with respect to the same generalized connection ∇ but in the different bases B^* and $\overline{B}^*$ are given by*

$$
\begin{aligned}
(3.1) \quad &(a) \quad \overline{g}_{ij\overline{|}k} = g_{ij|k} + g_{aj|k} T_i^a + g_{ib|k} T_j^b + g_{ab|k} T_i^a T_j^b + \\
&\qquad (g_{ij}|_c + g_{aj}|_c T_i^a + g_{ib}|_c T_j^b + g_{ab}|_c T_i^a T_j^b) T_k^c \\
&(b) \quad \overline{g}_{ib\overline{|}k} = g_{ib|k} + g_{ab|k} T_i^a + g_{ib}|_c T_k^c + g_{ab}|_c T_i^a T_k^c \\
&(c) \quad \overline{g}_{aj\overline{|}k} = g_{aj|k} + g_{ab|k} T_j^b + g_{aj}|_c T_k^c + g_{ab}|_c T_j^b T_k^c \\
&(d) \quad \overline{g}_{ab\overline{|}k} = g_{ab|k} + g_{ab}|_c T_k^c \\
&(e) \quad \overline{g}_{ij}\overline{|}_c = g_{ij}|_c + g_{aj}|_c T_i^a + g_{ib}|_c T_j^b + g_{ab}|_c T_i^a T_j^b
\end{aligned}
$$

$$\begin{array}{ll} (f) & \bar{g}_{ib}\bar{|}_c = g_{ib}|_c + g_{ab}|_c T_i^a \\ (g) & \bar{g}_{aj}\bar{|}_c = g_{aj}|_c + g_{ab}|_c T_j^b \\ (h) & \bar{g}_{ab}\bar{|}_c = g_{ab}|_c \end{array}$$

where

$$(3.2) \quad \begin{array}{l} (a)\ g_{xy|k} = \delta_k g_{xy} - F_{x\ k}^{\ h} g_{hy} - F_{x\ k}^{\ d} g_{dy} - F_{y\ k}^{\ h} g_{xh} - F_{y\ k}^{\ d} g_{xd} \\ (b)\ g_{xy}|_c = \partial_c g_{xy} - C_{x\ c}^{\ h} g_{hy} - C_{x\ c}^{\ d} g_{dy} - C_{y\ c}^{\ h} g_{xh} - C_{y\ c}^{\ d} g_{xd} \end{array}$$

$$x \in \{i, a\}, \qquad y \in \{j, b\}.$$

$\bar{g}_{xy\bar{|}k}$ is obtained from (3.2a) if in its right-hand side δ, g and F are overlined in all places where they appear.

$\bar{g}_{xy}\bar{|}_c$ is obtained from (3.2b) if in its right-hand side g and C are overlined in all places where they appear.

The proof of (3.1) follows from (2.11) and (2.13). From (3.1) we get

Theorem 3.2. *If ∇ is a metric connection with respect to the basis B^*, then it will be a metric connection also in the basis $\overline{B}^*$.*

The statement follows from (3.1a) and (3.1a').

If $T_H(E)$ spanned by $\{\delta_i\}$ is orthogonal to $T_V(E)$ spanned by $\{\partial_a\}$, i.e. $g_{ib} = 0, g_{aj} = 0$, then (3.2) becomes

$$(3.2)' \quad \begin{array}{ll} \text{(a)} & g_{ij|k} = \delta_k g_{ij} - F_{i\ k}^{\ h} g_{hj} - F_{j\ k}^{\ h} g_{ih} \\ \text{(b)} & g_{ib|k} = -F_{i\ k}^{\ d} g_{db} - F_{b\ k}^{\ h} g_{ih} \\ \text{(c)} & g_{aj|k} = -F_{a\ k}^{\ h} g_{hj} - F_{j\ k}^{\ d} g_{ad} \\ \text{(d)} & g_{ab|k} = \delta_k g_{ab} - F_{a\ k}^{\ d} g_{db} - F_{b\ k}^{\ d} g_{ad} \\ \text{(e)} & g_{ij}|_c = \partial_c g_{ij} - C_{i\ c}^{\ h} g_{hj} - C_{j\ c}^{\ h} g_{ih} \\ \text{(f)} & g_{ib}|_c = -C_{i\ c}^{\ d} g_{db} - C_{b\ c}^{\ h} g_{ih} \\ \text{(g)} & g_{aj}|_c = -C_{a\ c}^{\ h} g_{hj} - C_{j\ c}^{\ d} g_{ad} \\ \text{(h)} & g_{ab}|_c = \partial_c g_{ab} - C_{a\ c}^{\ d} g_{db} - C_{b\ c}^{\ d} g_{ad}. \end{array}$$

The coefficients of the generalized metrical connection are obtained from the conditions $g_{xy|k} = 0, g_{xy}|_c = 0, \forall x \in \{i, a\}, y\forall \in \{j, b\}$.

The following notations will be used

$$(3.3)\quad \begin{aligned} &(a)\quad F_{xyk} = F_{x\ k}^{\ h} g_{hy} + F_{x\ k}^{\ d} g_{dy} \\ &(b)\quad C_{xyc} = C_{x\ c}^{\ h} g_{hy} + C_{x\ c}^{\ d} g_{dy} \\ &(c)\quad L_{xyz} = \begin{cases} F_{xyk} & \text{for} \quad z = k \\ C_{xyc} & \text{for} \quad z = c \end{cases} \\ &(d)\quad \nabla_z = \begin{cases} \delta_k & \text{for} \quad z = k \\ \partial_c & \text{for} \quad z = c \end{cases} \\ &(e)\quad g_{xy|z} = \begin{cases} g_{xy|k} & \text{for} \quad z = k \\ g_{xy}|_c & \text{for} \quad z = c. \end{cases} \end{aligned}$$

Using the above notation, (3.2) can be written in the shorter form

$$(3.4)\quad g_{xy|z} = \nabla_z g_{xy} - L_{xyz} - L_{yxz}.$$

From (3.4) it follows that

$$(3.5)\quad \begin{aligned} 2L_{zxy} = {} & \nabla_z g_{xy} + \nabla_y g_{zx} - \nabla_x g_{yz} + \\ & (L_{zxy} - L_{yxz}) + (L_{yzx} - L_{xzy}) + (L_{zyx} - L_{xyz}) \end{aligned}$$

For $z = a, x = b, y = k$ (3.5) gives

$$(3.5a)\quad \begin{aligned} 2F_{abk} = {} & \delta_k g_{ab} + \partial_a g_{bk} - \partial_b g_{ka} + \\ & (F_{abk} - C_{kba}) + (C_{kab} - F_{bak}) + (C_{akb} - C_{bka}) \end{aligned}$$

The expressions in the last three brackets in (3.5) and (3.5a) are the functions of the torsion tensor and the nonlinear connection. The original coefficients can be obtained from (3.6)

$$(3.6)\quad \begin{bmatrix} L_{z\ y}^{\ i} \\ L_{z\ y}^{\ a} \end{bmatrix} = \begin{bmatrix} g^{ij} & g^{ib} \\ g^{aj} & g^{ab} \end{bmatrix} \begin{bmatrix} L_{zjy} \\ L_{zby} \end{bmatrix}$$

where the square matrix on right-hand side in (3.6) is the inverse matrix of the metric tensor.

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