Novi Sad J. Math.
Vol. 26, No. 2, 1996, 87–109

COMPATIBLE MAPPINGS OF TYPE (P) AND FIXED POINT THEOREMS IN METRIC SPACES AND PROBABILISTIC METRIC SPACES [1]

H. K. Pathak
Department of Mathematics, Kalayan Mahavidyalaya
Bhilai Nagar (M.P.) 490006, India
Y. J. Cho
Department of Mathematics, Gyeongsang National University
Chinju 660–701, Korea
S. S. Chang
Department of Mathematics, Sichuan University
Chengdu, Sichuan 610064, P. R. China
S. M. Kang
Department of mathematics, Gyeongsang National University
Chinju 660–701, Korea

Abstract

Some common fixed point theorems for compatible mappings of type (P) in metric spaces and probabilistic metric spaces are given. Also, we extend Caristi's fixed point theorem and Ekeland's variational principle in metric spaces to probabilistic metric spaces.

AMS Mathematics Subject Classification (1991): 47H10, 54H25
Key words and phrases: non-Archimedean Menger probabilistic metric spaces, compatible, comptible mappings of type (A) and compatible mappings of type (P), common fixed points

---

[1]The Present Studies were supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1994, Project No. BSRI-94-1405.

1. Introduction and Preliminaries

Since the formulation of Banach's contraction principle, a number of fixed point theorems for single-valued and multi-valued mappings in metric spaces and probabilistic metric spaces have been proved by many authors ([2], [3], [5], [7]-[10], [16], [22]-[24], [30], [34]-[37]). Every metric space is a probabilistic metric space and so we can use many results in probailistic metric spaces to prove some fixed point theorems metric spaces.

The purpose of this paper is, firstly, to prove some common fixed point theorems in metric spaces and probabilistic metric spaces and, secondly, to extend Caristi's fixed point theorem and Ekeland's variational principle in metric spaces to probabilistic metric spaces. From our main results, we can obtain many fixed point theorems for commuting, weakly commuting, compatible and compatible mappings of type (A) in metric spaces and probabilistic metric spaces ([2], [10], [14], [15], [19], [26], [27], [31], [33], [35]).

Let R denote the set of real numbers and R^+ the set of non-negative real numbers. The mapping $\mathcal{F} : R \to R^+$ is called a *distribution function* if it is a nondecreasing and left continuous function with $\inf \mathcal{F} = 0$ and $\sup \mathcal{F} = 1$. We will denote by $\mathcal{L}$ a set of all distribution functions.

Definition 1.1. *A* **probabilistic metric spaces** *(briefly, a PM-space) is a pair $(X, \mathcal{F})$ where X is a nonempty set and $\mathcal{F}$ is a mapping from $X \times X$ to $\mathcal{L}$. For $(u, v) \in X \times X$, the distribution function $\mathcal{F}(u, v)$ is denoted by $F_{u,v}$. The function $F_{u,v}$ is assumed to satisfy the following conditions:*

(PM-1) $F_{u,v}(x) = 1$ *for every* $x > 0$ *if and only if* $u = v$,

(PM-2) $F_{u,v}(0) = 0$ *for every* $u, v \in X$,

(PM-3) $F_{u,v}(x) = F_{v,u}(x)$ *for every* $u, v \in X$,

(PM-4) *If* $F_{u,v}(x) = 1$ *and* $F_{v,w}(y) = 1$, *then* $F_{u,w}(x + y) = 1$ *for every* $u, v, w \in X$.

Definition 1.2. *A mapping $\triangle : [0,1] \to [0,1]$ is called the* t-**norm** *if it satisfies the following conditions: For all $a, b, c, d \in [0,1]$*

(T-1) $\triangle(a, 1) = a$,

(T-2) $\triangle(a, b) = \triangle(b, a)$,

(T-3) $\triangle(c, d) \geq \triangle(a, b)$ *for* $c \geq a$ *and* $d \geq b$,

(T-4) $\triangle(\triangle(a, b), c) = \triangle(a, \triangle(b, c))$.

every $a \in [0, 1]$.

Definition 1.3. *A* **Menger space** *is a triple $(X, \mathcal{F}, \triangle)$, where $(X, \mathcal{F})$ is a*

PM-space, and $\triangle$ is a t-norm with the following condition:

(PM-5) $F_{u,v}(x+y) \geq \triangle(F_{u,v}(x), F_{v,w}(y))$
for every $u, v, w \in X$ and $x, y \in R^+$.

Definition 1.4. *A* **non-Archimedean Menger PM-space** *(N.A. Menger PM-space) is a triple $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a t-norm and the space $(X, \mathcal{F})$ satisfies the conditions (PM-1)~(PM-3) and (PM-6):*

(PM-6) $F_{u,w}(\max\{t_1, t_2\}) \geq \triangle(F_{u,v}(t_1), F_{v,w}(t_2))$
for all $u, v, w \in X$ and $t_1, t_2 \geq 0$.

The concept of neighbourhoods in PM-spaces was introduced by B. Schweizer and K. Sklar [28], [29]. If $u \in X$, $\epsilon > 0$ and $\lambda \in (0,1)$, then the (ϵ, λ)-neighbourhood of u is denoted by $U_u(\epsilon, \lambda) = \{v \in x : F_{u,v}(\epsilon) > 1 - \lambda\}$. If $(X, \mathcal{F}, \triangle)$ is a Menger space with the continuous t-norm $\triangle$, then the family $\{U_u(\epsilon, \lambda) : u \in X, \epsilon > 0, \lambda \in (0,1)\}$ of neighbourhoods induces a Hausdorff topology on X, which is denoted by (ϵ, λ)-topology.

Definition 1.5. *A PM-space $(X, \mathcal{F})$ is said to be of* **type** $(C)_{g,h}$ *if there exist elements $g, h \in Q$ such that $h(t) \leq g(t)$ for all $t \in [0,1]$ and*

$$g(F_{x,y}(t)) \leq h(F_{x,z}(t)) + h(F_{z,y}(t))$$

for all $x, y, z \in X$ and $t \geq 0$, where $Q = \{g : [0,1] \to [0, \infty]\}$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 1.6. *An N.A. Menger PM-space $(X, \mathcal{F}, \triangle)$ is said to be of* **type** $(D)_{g,h}$ *if there exist elements $g, h \in Q$ such that $h(t) \leq g(t)$ for all $t \in [0,1]$ and*

$$g(\triangle(s,t)) \leq h(s) + h(t)$$

for all $s, t \in [0,1]$.

Remark 1.1. ([11]) (1) If an N.A. PM-space $(X, \mathcal{F}, \triangle)$ is of type $(D)_{g,h}$, then $(X, \mathcal{F}, \triangle)$ is of type $(C)_{g,h}$.

(2) If $(X, \mathcal{F}, \triangle)$ is an N.A. PM-space and $\triangle \geq \triangle_m$, where $\triangle_m(s,t) = \max\{s+t-1, 0\}$, then $(X, \mathcal{F}, \triangle)$ is of type $(D)_{g,h}$ for $g, h \in Q$ defined by $g(t) = 1 - t$ and $h(t) = (1-t)^2$, respectively.

(3) If a PM-space $(X, \mathcal{F})$ is of type $(C)_{g,h}$, then it is metrizable, where the metric d on X is defined by

$$(*) \qquad d(x,y) = \int_0^1 h(F_{x,y}(t))dt$$

for all $x, y \in X$.

(4) If an N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is of type $(D)_{g,h}$, then it is metrizable, where the metric d on X is given by $(*)$. On the other hand, the (ϵ, λ)-topology coincides with the topology induced by the metric d defined by $(*)$.

(5) If $(X, \mathcal{F}, \Delta)$ is an N.A. Menger PM-space with the t-norm Δ such that $\Delta(s,t) \geq \Delta_m(s,t) = \max\{s+t-1, 0\}$ for $s, t \in [0,1]$, then the assertion (4) is also true.

2. Compatible Mappings of Type (P)

In this section we introduce the concept of compatible mappings of type (P) in metric space (X, d) and compare it with the compatible and compatible mappings of type (A). We also recall the following definitions and properties of compatible mappings and compatible mappings of type (A) ([26], [27]).

Definition 2.1. *Let $S, T : (X, d) \to (X, d)$ be mappings. S and T are said to be* **compatible** *if*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

Definition 2.2. *Let $S, T : (X, d) \to (X, d)$ be mappings. S and T are said to be compatible of type (A) if*

$$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(STx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 2.3. *Let $S, T : (X, d) \to (X, d)$ be mappings. S and T are said to be compatible of type (P) if*

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in x such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

The following propositions show that Definitions 2.1 and 2.2 are equivalent under some conditions:

Proposition 2.1. *Let $S, T : (X, d) \to (X, d)$ be continuous mappings. If S and T are compatible, then they are compatible of type (A).*

Proposition 2.2. *Let $S, T : (X, d) \to (X, d)$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible.*

The following is a direct consequence of Propositions 2.1 and 2.2:

Proposition 2.3. *Let $S, T : (X, d) \to (X, d)$ be continuous mappings. Then S and T are compatible if and only if they are compatible of type (A).*

Remark 2.1. In [27], we can find two examples that Proposition 2.3 is not true if S and T are not continuous on X.

We can also show that S and T are continuous, then S and T are compatible if and only if they are compatible of type (P) as follows:

Proposition 2.4. *Let $S, T : (X, d) \to (X, d)$ be continuous mappings. Then, S and T are compatible if and only if they are compatible of type (P).*

Proof Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = Tx_n = z$ for some $z \in X$. Since S and T are continuous,

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Sz,$$

and

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} TTx_n = Tz.$$

Suppose that S and T are compatible. Then we have

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0.$$

Now, since we have

$$\begin{aligned} d(SSx_n, TTx_n) &\leq d(SSx_n, STx_n) + d(STx_n, TTx_n) \\ &\leq d(SSx_n, STx_n) + d(STx_n, TSx_n) + d(TSx_n, TTx_n), \end{aligned}$$

it follows that $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$. Thus, S and T are compatible of type (P).

Conversely, suppose that S and T are compatible mappings of type (P), that is,

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0.$$

We then have

$$\begin{aligned} d(STx_n, TSx_n) &\leq d(STx_n, SSx_n) + d(SSx_n, TSx_n) \\ &\leq d(STx_n, SSx_n) + d(SSx_n, TTx_n) + d(TTx_n, TSx_n). \end{aligned}$$

Therefore, it follows that $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$. This completes the proof.□

Proposition 2.5. *Let $S, T : (X, d) \to (X, d)$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible of type (P).*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Suppose that S and T are compatible mappings of type (A). Assume, without loss of generality, that S is continuous. We then have

$$d(SSx_n, TTx_n) \leq d(SSx_n, STx_n) + d(STx_n, TTx_n),$$

and so, since S and T are compatible of type (A), we have

$$\lim_{n\to\infty} d(SSx_n, TSx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(STx_n, TTx_n) = 0.$$

Therefore, it follows that

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0.$$

This completes the proof.□

As a direct consequence of Propositions 2.3∼2.5, we have the following:

Proposition 2.6. *Let $S, T : (X, d) \to (X, d)$ be continuous mappings. Then*

(1) S and T are compatible if and only if they are compatible of type (P).

(2) S and T are compatible of type (A) if and only if they are compatible of type (P).

Next, we give several properties of compatible mappings of type (P) for one main theorems.

Proposition 2.7. *Let $S, T : (X, d) \to (X, d)$ be mappings. If S and T are compatible of type (P) and $Sz = Tz$ for some $z \in X$, then $SSz = STz = TSz = TTz$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = z$, $n = 1, 2, \cdots$, and $Sz = Tz$ for some $z \in X$. Then we have $Sx_n, Tx_n \to Sz$ as $n \to \infty$. Since S and T are compatible of type (P), we have

$$d(SSz, TTz) = \lim_{n \to \infty} d(SSx_n, TTx_n) = 0.$$

Therefore, $SSz = TTz$. But $Sz = Tz$ implies $SSz = STz = TSz = TTz$. This completes the proof.□

Proposition 2.8. *Let $S, T : (X, d) \to (X, d)$ be mappings. Let S and T are compatible mappings of type (P) and let $Sx_n, Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then we have the following:*

(1) $\lim_{n \to \infty} TTx_n = Sz$ if S is continuous at z,

(2) $\lim_{n \to \infty} SSx_n = Tz$ if T is continuous at z,

(3) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

Proof. (1) Suppose that S is continuous at z. Since we have $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$ for some $z \in X$, $SSx_n \to Sz$ as $n \to \infty$. Again, since S and T compatible of type (P), we have $\lim_{n \to \infty} d(TTx_n, SSx_n) = 0$ and so we have

$$d(TTx_n, Sz) \le d(TTx_n, SSx_n) + d(SSx_n, Sz),$$

it follows that $TTx_n \to Sz$ as $n \to \infty$.

(2) The proof of $\lim_{n\to\infty} SSx_n = Tz$ follows on the similar lines as argued in (1).

(3) Suppose that S and T are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by (1), $TTx_n \to Sz$ as $n \to \infty$. On the other hand, Since $Tx_n \to z$ as $n \to \infty$ and T is also continuous at z, $TTx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of the limit and by Proposition 2.7, $TSz = STz$. This completes the proof.□

3. Fixed Point Theorems in Metric Spaces

In this section, we give several fixed point theorems for compatible mappings of type (P) in metric spaces (X, d).

Let $\mathcal{G}$ be the family of all mappings $\phi : (R^+)^5 \to R^+$ such that ϕ is upper semicontinuous, non-decreasing in each coordinate variable, and for any $t > 0$,

$$\phi(t, t, 0, \alpha t, 0) \leq \beta t \quad \text{and} \quad \phi(t, t, 0, 0, \alpha t) \leq \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$.

$$\gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t$$

where $\gamma : R^+ \to R^+$ is a mapping and $a_1 + a_2 + a_3 = 4$.

First we have some lemmas for our main theorems:

Lemma 3.1. *([33]) For any $t > 0$, $\gamma(t) < t$, if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$ where γ^n denotes the n-times composition of γ.*

Let A, B, S and T be mappings from a metric space (X, d) into itself such that

(3.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(3.2) there exists $\phi \in \mathcal{G}$ such that

$$d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty))$$

for all $x, y \in X$.

Then, by (3.1), since $A(X) \subset T(X)$, for any point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$(3.3) \qquad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for $n = 0, 1, 2, \cdots$.

Lemma 3.2. $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, *where* $\{y_n\}$ *is the sequence in* X *defined by* (3.3).

Proof. Let $d_n = d(y_n, y_{n+1}) = 0, n = 0, 1, 2, \cdots$. Now, we shall prove that the sequence $\{d_n\}$ is non-increasing in R^+, that is, $d_{n+1} \le d_n$ for $n = 0, 1, 2, \cdots$. By (3.2), we have

$$\begin{aligned}(3.4) \quad d_{2n+1} &= d(y_{2n+1}, y_{2n+2}) \\ &= d(Ax_{2n+2}, Bx_{2n+1}) \\ &\le \phi(d(Ax_{2n+2}, Sx_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(Ax_{2n+2}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n+2}), \\ &\qquad d(Sx_{2n+2}, Tx_{2n+1})) \\ &= \phi(d((y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}), \\ &\qquad d(y_{2n+2}, y_{2n}), d(y_{2n+1}, y_{2n+1}), d(y_{2n+1}, y_{2n})) \\ &\le \phi(d_{2n+1}, d_{2n}, d_{2n+1} + d_{2n}, 0, d_{2n}).\end{aligned}$$

Suppose that $d_n < d_{n+1}$ for some n. Then, for some $\alpha < 2$, $d_n + d_{n+1} = \alpha d_{n+1}$. Since ϕ is non-decreasing in each coordinate variable and $\beta < 1$ for some $\alpha < 2$, by (3.4), we have

$$d_{2n+1} \le \phi(d_{2n+1}, d_{2n}, \alpha d_{2n+1}, 0, d_{2n+1}) \le \beta d_{2n+1} < d_{2n+1}.$$

Similary, we have

$$d_{2n+2} \le \phi(d_{2n+2}, d_{2n+2}, \alpha d_{2n+2}, 0, d_{2n+2}) \le \beta d_{2n+2} < d_{2n+2}.$$

Hence, for every $n = 0, 1, 2, \cdots$, $d_n \le \beta d_n < d_n$, which is a contradiction. Therefore, $\{d_{2n}\}$ is a non-increasing sequence in R^+. Now, again, by (3.2), we have

$$\begin{aligned} d_1 &= d(y_1, y_2) \\ &= d(Ax_1, Bx_2) \\ &\le \phi(d(Ax_2, Sx_2), d(Bx_1, Tx_1), d(Ax_2, Tx_1), \\ &\qquad d(Bx_1, Sx_2), d(Sx_2, Tx_1)) \\ &= \phi(d((y_2, y_1), d(y_1, y_0), d(y_2, y_0), d(y_1, y_1), d(y_1, y_0)) \\ &\le \phi(d_1, d_0, d_0 + d_1, 0, d_0) \\ &\le \phi(d_0, d_0, 2d_0, d_0, d_0) \\ &= \gamma(d_0). \end{aligned}$$

In general, we have $d_n \le \gamma^n(d_0)$ for $n = 0, 1, 2, \cdots$, which implies that, if $d_0 > 0$, then, by Lemma 3.1, we have

$$\lim_{n\to\infty} d_n \le \lim_{n\to\infty} \gamma^n(d_0) = 0.$$

Therefore, it follows that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

For $d_0 = 0$, since $\{d_n\}$ is non-increasing, we have clearly $\lim_{n\to\infty} d_n = 0$. This completes the proof.□

Lemma 3.3. *The sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X.*

Proof. By Lemma 3.2, it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in X. Then, there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \ge 2k$ such that

$$(3.5) \qquad d(y_{2m(k)}, y_{2n(k)}) > \epsilon.$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.5), that is,

$$(3.6) \qquad d(y_{2n(k)}, y_{2m(k)-2}) \le \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon.$$

Then for each even integer $2k$, we have

$$\begin{aligned}\epsilon &< d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).\end{aligned}$$

It follows from Lemma 3.2 and (3.6) that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{3.7}$$

By the triangle inequality,

$$\begin{aligned}|d(y_{2n(k)}, y_{2m(k)-1}) &- d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)-1}, y_{2m(k)}) \\ &\text{and} \\ |d(y_{2n(k)+1}, y_{2m(k)-1}) &- d(y_{2n(k)}, y_{2m(k)})| \\ &\leq d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).\end{aligned}$$

From Lemma 3.2 and (3.7), as $k \to \infty$, it follows that

$$d(y_{2n(k)+1}, y_{2m(k)-1}) \to \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)-1}) \to \epsilon. \tag{3.8}$$

Therefore, by (3.2) and (3.3), we have

$$\begin{aligned}d(y_{2n(k)}, y_{2m(k)}) &\leq d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)}) \\ &= d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1}) \\ &\leq d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(Ax_{2m(k)}, Sx_{2m(k)}), \\ &\quad d(Bx_{2n(k)+1}, Tx_{2n(k)+1}), d(Ax_{2m(k)}, Tx_{2n(k)+1}), \\ &\quad d(Bx_{2m(k)-1}, Sx_{2m(k)}), d(Sx_{2m(k)}, Tx_{2n(k)+1})) \\ &= d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(y_{2m(k)}, y_{2m(k)-1}), \\ &\quad d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)}, y_{2n(k)}), \\ &\quad d(y_{2n(k)+1}, y_{2m(k)-1}), d(y_{2m(k)-1}, y_{2n(k)})).\end{aligned} \tag{3.9}$$

Since ϕ is upper semicontinuous, as $k \to \infty$ in (3.9), by Lemma 3.2, (3.7) and (3.8), we have

$$\epsilon \leq \phi(0, 0, \epsilon, \epsilon, \epsilon) \leq \gamma(\epsilon) < \epsilon,$$

which is a contradiction. Therefore, the sequence $\{y_{2n}\}$ is a Cauchy sequence in X and so is $\{y_n\}$. This completes the proof.□

Now, we are ready to prove the main theorem:

Theorem 3.1. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions* (3.1), (3.2), *and*

(3.10) *one of A, B, S, and T is continuous,*

(3.11) *the pairs A, S and B, T are compatible of type (P).*

Then A, B, S, and T have a unique common fixed point z in X.

Proof. By Lemma 3.3, the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X and so, since (X, d) is complete, it converges to a point z in X. On the other hand, subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the point z.

Now, suppose that T is continuous. Since B and T are compatible of type (P), by Proposition 2.8, $BBx_{2n+1}, TBx_{2n+1} \to Tz$ as $n \to \infty$. Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (3.2), we have

$$\begin{aligned}(3.12)\quad d(Ax_{2n}, BBx_{2n+1}) \leq \phi(&d(Ax_{2n}, Sx_{2n}), d(BBx_{2n+1}, TBx_{2n+1}),\\ &d(Ax_{2n}, TBx_{2n+1}), d(BTx_{2n+1}, Sx_{2n}),\\ &d(Sx_{2n}, TBx_{2n+1})).\end{aligned}$$

Taking $n \to \infty$ in (3.12), since $\phi \in \mathcal{G}$, we have

$$\begin{aligned}d(z, Tz) &\leq \phi(0, 0, d(z, Tz), d(z, Tz), d(z, Tz))\\ &< \gamma(d(z, Tz))\\ &< d(z, Tz),\end{aligned}$$

which is a contradiction. Thus, we have $Tz = z$. Similarly, if we replace x by x_{2n} and y by z in (3.2), respectively, and taking $n \to \infty$, then we have $Bz = z$. Since $B(X) \subset S(X)$, there exists a point u in X such that $Bz = Su = z$. By using (3.2) again, we have

$$\begin{aligned}d(Au, z) &= d(Au, Bz)\\ &\leq \phi(d(Au, Su), d(Bz, Tz), d(Au, Tz), d(Bz, Su), d(Su, Tz))\\ &= \phi(d(Au, Su), 0, d(Au, z), 0, 0)\\ &< \gamma(d(Au, z))\\ &< d(Au, z),\end{aligned}$$

which is a contradiction and, so $Au = z$. Since A and S are compatible mappings of type (P) and $Au = Su = z$, by Proposition 2.7, $d(ASu, SSu) =$

0 and hence $Az = ASu = SSu = Sz$. Finally, by (3.2), we have again

$$\begin{aligned} d(Az, z) &= d(Az, Bz) \\ &\le \phi(d(Az, Sz), d(Bz, Tz), d(Az, Tz), d(Bz, Sz), d(Sz, Tz)) \\ &= \phi(d(Az, z), 0, d(Az, z), 0, 0) \\ &< \gamma(d(Az, z)) \\ &< d(Az, z), \end{aligned}$$

which implies that $Az = z$. Therefore, we have $Az = Bz = Tz = z$, that is, z is a common fixed point of the given mappings A, B, S, and T. The uniqueness of the common fixed point z follows easily from (3.2).

Similarly, we can also complete the proof when A or B or T is continuous. This completes the proof.□

By Theorem 3.1, we have the following:

Theorem 3.2. *Let $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of mappings from a complete metric space (X, d) into itself such that $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to the self-mappings A, B, S and T on X, respectively.*

Suppose that for $n = 1, 2, \cdots$, x_n is a common fixed point of A_n, B_n, S_n and T_n. Further, let self-mappings A, B, S and T on X satisfy the conditions (3.1), (3.2), (3.10) *and* (3.11). *If x is a common fixed point of A, B, S, T and $\sup\{d(x_n, x)\} < \infty$, then $x_n \to x$ as $n \to \infty$.*

Theorem 3.3. *Let $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of mappings from a complete metric space (X, d) into itself such that, for $n = 1, 2, 3, \cdots$,*

(3.13) *$A_n(X) \subset T_n(X)$ and $B_n(X) \subset S_n(X)$,*

(3.14) *one of A_n, B_n, S_n and T_n is continuous,*

(3.15) *the pairs A_n, S_n and B_n, T_n are compatible of type (P),*

(3.16) *there exists $\phi \in \mathcal{G}$ such that*

$$\begin{aligned} d(A_n x, B_n y) \le\ & \phi(d(A_n x, S_n x), d(B_n y, T_n y), d(A_n x, T_n y), \\ & d(B_n y, S_n y) d(S_n y, S_n y)) \end{aligned}$$

for all $x, y \in X$.

If $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ *and* $\{T_n\}$ *converges uniformly to self-mappings* A, B, S, *and* T *on* X, *respectively, then* A, B, S *and* T *satisfy the conditions* (3.1), (3.2), (3.10) *and* (3.11).

Further, the sequence $\{x_n\}$ *of unique common fixed points* x_n *of* A_n, B_n, S_n *and* T_n *converges to a unique common fixed point* x *of* A, B, S *and* T, *if* $\sup\{d(x_n, x)\} < \infty$.

Remark 3.1. Our main theorems extend and improve a number of fixed point theorems for commuting, weakly commuting, and compatible mappings, in metric spaces ([15], [26], [27], [31], [33]).

4. Fixed Point Theorems in PM-spaces

In [6] and [21], K. Caristi and I. Ekeland proved the following theorems, respectively:

Theorem 4.1. *Let* (X, d) *be a complete metric space and* T *be a mapping from* X *into itself. If there exists a lower semicontinuous function* $\Phi : X \to R^+$ *such that*

$$d(x, Tx) \leq \Phi(x) - \Phi(Tx)$$

for all $x \in X$, *then* T *has a fixed point.*

Theorem 4.2. *Let* (X, d) *be a complete metric space and* f *be a proper, bounded below and lower semicontinuous function from* X *into* R. *Then, for each* $\epsilon > 0$ *and* $u \in X$ *such that* $f(u) \leq \inf\{f(x) : x \in X\} + \epsilon$, *there exists a point* $v \in X$ *such that*

(4.1) $f(v) \leq f(u)$,

(4.2) $d(u, v) \leq 1$,

(4.3) $f(w) > f(v) - \epsilon d(v, w)$ *for all* $w \in X$, $w \neq v$.

Remark 4.1. (1) If $\Phi(x) = \frac{1}{k} d(x, Tx)$ for $0 \leq k < 1$, then from Theorem 4.1, we have Banach's contraction principle.

(2) In fact, in [21], I. Ekeland proved that Theorems 4.1 and 4.2 are equivalent. Also, he gave some applications of Theorem 4.2.

(3) In [12] and [14], S. S. Chang et al. proved that Theorems 4.1 and 4.2 are also equivalent in the setting of probabilistic metric spaces.

(4) The generalizations of Theorems 4.1 and 4.2 in different ways are given in [1], [4], [12], [13], [17], [18], [20], [25], [32].

In this section, we extend Caristi's fixed point theorem and Ekeland's variational principle in PM-spaces. Also, we prove some common fixed point theorems in PM-spaces by using the results from Section 2.

First, by using Theorem 4.1, we prove the following:

Theorem 4.3. *Let $(X, \mathcal{F})$ be a PM-space of type $(C)_{g,h}$ and (X, d) be a complete metric space, where the metric d on X is defined by $(*)$. If $\Phi : X \to R$ is a lower semicontinuous and bounded below function and the mapping $T : X \to X$ satisfies the following condition: for all $x \in X$ and $t \geq 0$,*

$$g(F_{x,Tx}(t)) \leq \Phi(x) - \Phi(Tx), \tag{4.4}$$

then T has a fixed point in X.

Proof. From (4.4), we have

$$\begin{aligned} d(x, Tx) &= \int_0^1 h(F_{x,Tx}(t))dt \leq \int_0^1 g(F_{x,Tx}(t))dt \\ &\leq \int_0^1 (\Phi(x) - \Phi(Tx))dt = \Phi(x) - \Phi(Tx) \end{aligned}$$

and thus, by Theorem 4.1, T has a fixed point in X.□

Corollary 4.1. *Let $(X, \mathcal{F})$ be a PM-space of type $(C)_{g,h}$ and (X, d) be a complete metric space, where the metric d on X is defined by $(*)$, and the function $\Phi(x,t) : X \times R^+ \to R^+$ be integrable in t. If the function $\psi(x) = \int_0^1 \phi(x,t)dt$ is lower semicontinuous and bounded below and the mapping $T : X \to X$ satisfies the following condition:*

for all $x \in X$ and $t > 0$,

$$g(F_{x,Tx}(t)) \leq \phi(x,t) - \phi(Tx,t), \tag{4.5}$$

then T has a fixed point in X.

Proof. From (4.5), we have

$$\begin{aligned} d(x,Tx) &= \int_0^1 h(F_{x,Tx}(t))dt \le \int_0^1 g(F_{x,Tx}(t))dt \\ &\le \int_0^1 (\phi(x,t) - \phi(Tx,t))dt \\ &= \int_0^1 \phi(x,t)dt - \int_0^1 \phi(Tx,t)dt \\ &= \psi(x) - \psi(Tx) \end{aligned}$$

Therefore, by Theorem 4.3, T has a fixed point in X.□

Theorem 4.4. *Let $(X,\mathcal{F})$ be a PM-space of type $(C)_{g,h}$ and (X,d) be a complete metric space, where the metric d on X is defined by $(*)$. If the function $\Phi : X \to R$ is a proper, lower semicontinuous and bounded below, and T is a multi-valued mapping from X into 2^X such that for each $x \in X$, there exists a point $fx \in Tx$ such that $f : X \to X$ is a function satisfying the following condition: for all $x \in X$ and $t \ge 0$,*

$$g(F_{x,fx}(t)) \le \Phi(x) - \Phi(fx), \tag{4.6}$$

then f and T have a common fixed point in X.

Proof. Since Φ is proper, there exists a point $u \in X$ such that $\Phi(x) < +\infty$ and so let

$$A = \{x \in X : g(F_{x,u}(t)) \le \Phi(u) - \Phi(x) \text{ for every } t > 0\}.$$

Then A is a nonempty closed set in X. Since $g(F_{x,fx}(t)) \le \Phi(x) - \Phi(fx)$, for each $x \in X$, $fx \in A$ and so we have

$$\Phi(fx) + g(F_{x,fx}(t)) \le \Phi(x) \le \Phi(u) - g(F_{x,u}(t)).$$

Thus we have

$$\begin{aligned} g(F_{u,fx}(t)) &\le h(F_{u,x}(t)) + h(F_{x,fx}(t)) \\ &\le g(F_{u,x}(t)) + g(F_{x,fx}(t)) \\ &\le \Phi(u) - \Phi(x) + \Phi(x) - \Phi(fx) \\ &= \Phi(u) - \Phi(fx). \end{aligned}$$

Therefore, by Theorem 4.3, the function $f : A \to A$ has a fixed point in A, say x_0, and so $x_0 = fx_0 \in Tx_0$, that is, the point x_0 is a common fixed point of f and T. This completes the proof.□

By Theorem 4.4, we have Ekeland's variational principle in PM-spaces.

Theorem 4.5. *Let $(X, \mathcal{F})$ be a PM-space of type $(C)_{g,h}$ and (X, d) be a complete metric space, where the metric d on X is defined by $(*)$. If the function $\Phi : X \to R$ is proper, lower semicontinuous and bounded below and, for each $\epsilon > 0$, there exists a point $u \in X$ such that $\Phi(u) \leq \inf\{\Phi(x) : x \in X\} + \epsilon$, then there exists a point $v \in X$ such that*

(4.7) $\Phi(v) \leq \Phi(u)$,

(4.8) $g(F_{u,v}(t)) \leq 1$,

(4.9) *For all $x \in X$, there exists $t \geq 0$ such that $\Phi(v) - \Phi(u) \leq \epsilon g(F_{u,x}(t))$.*

Proof. Let $\epsilon > 0$ and let a point $u \in X$ such that $\Phi(u) \leq \inf\{\Phi(u) : x \in X\} + \epsilon$. Letting $A = \{x \in X : \Phi(x) \leq \Phi(u) - \epsilon g(F_{u,x}(t)\}$, then A is a nonempty closed set in X and so, since (X, d) is complete, A is complete. For each $x \in A$, let $Sx = \{y \in X : \Phi(y) \leq \Phi(x) - \epsilon g(F_{x,y}(t)), x \neq y, \ t \geq 0\}$ and define

$$Tx = \begin{cases} x & \text{if } Sx \text{ is empty} \\ Sx & \text{if } Sx \text{ is nonempty.} \end{cases}$$

Then T is a multi-valued mapping from A into 2^A. Since $Tx = x \in A$ if $Sx = \emptyset$ and $Tx = Sx$ if $Sx \neq \emptyset$, we have, for each $y \in Tx = Sx$,

$$\Phi(y) \leq \Phi(x) - \epsilon g(F_{x,y}(t))$$

and

$$\begin{aligned} \epsilon g(F_{u,y}(t)) &\leq \epsilon h(F_{u,x}(t)) + h(F_{x,y}(t)) \\ &\leq \epsilon g(F_{u,x}(t)) + \epsilon g(F_{x,y}(t)) \\ &\leq \Phi(u) - \Phi(x) + \Phi(x) - \Phi(y) \\ &= \Phi(u) - \Phi(y), \end{aligned}$$

which implies $y \in A$, and so we have $Tx = Sx$ in A. Assume that T has no fixed point in A. Then for each $x \in A$ and $y \in Tx = Sx$, we have

$$\epsilon g(F_{x,y}(t)) \leq \Phi(x) - \Phi(y)$$

and

$$g(Fx, y(t)) \leq \frac{1}{\epsilon}\Phi(x) - \frac{1}{\epsilon}\Phi(y).$$

Thus, by Theorem 4.4, T has a fixed point v in A, which is a contradiction. Therefore, $Sv = \emptyset$, that is, for each $x \in X$, $x \neq v$, $\Phi(x) \geq \Phi(v) - \epsilon g(F_{v,x}(t))$. Since $v \in A$, $\Phi(v) \leq \Phi(u) - \epsilon g(F_{u,v}(t))$ and so $\Phi(v) \leq \Phi(u)$.

On the other hand, we have

$$\begin{aligned} \epsilon g(F_{u,v}(t)) &\leq \Phi(u) - \Phi(v) \\ &\leq \Phi(u) - \inf\{\Phi(x) : x \in X\} \\ &\leq \epsilon \end{aligned}$$

and so $g(F_{u,v}(t)) \geq 1$. This completes the proof.□

Next, by using Theorem **??**, we prove common fixed point theorems in PM-spaces. We introduce some definitions and properties of compatible mappings of type (P) in PM-spaces.

Definition 4.1. *Let $(X, \mathcal{F}, \triangle)$ be an N.A. Menger PM-space of type $(D)_{g,h}$ and A, S be mappings from X into itself. A and S are said to be g-***compatible** *if*

$$\lim_{n\to\infty} g(F_{ASx_n,SSx_n}(t)) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 4.2. *Let $(X, \mathcal{F}, \triangle)$ be an N. A. Menger PM-space of type $(D)_{g,h}$ and A, S be mappings from X into itself. A and S are said to be g-***compatible of type** *(A) if*

$$\lim_{n\to\infty} g(F_{AAx_n,SSx_n}(t)) = 0, \quad \lim_{n\to\infty} g(F_{SAx_n,AAx_n}(t)) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 4.3. *Let $(X, \mathcal{F}, \triangle)$ be an N.A. Menger PM-space of type $(D)_{g,h}$ and A, S be mappings from X into itself. A and S are said to be g-***compatible of type** *(P) if*

$$\lim_{n\to\infty} g(F_{AAx_n,SSx_n}(t)) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Remark 4.2. (1) The mappings A and S are h-compatible, h-compatible of type (A) and h-compatible of type (P), whenever they are g-compatible, g-compatible of type (A) and g-compatible of type (P), respectively.

(2) In fact, since $(X, \mathcal{F}, \Delta)$ is a N.A. Menger PM-space of type $(D)_{g,h}$ and it is metrizable by the metric d defined by $(*)$, Definitions 2.1 and 4.1, 2.2 and 4.2, 2.3 and 4.3 are equivalent each other, respectively.

(3) By using Definitions 4.1~4.3, we can obtain the same properties, that is, Propositions 2.1~2.8 between compatible mappings, compatible mappings of type (A) and compatible mappings of type (P) in PM-spaces.

Theorem 4.6. *Let $(X, \mathcal{F}, \Delta)$ be a τ-complete N.A. Menger PM-space with the t-norm Δ such that $\Delta(s,t) \geq \Delta_m(s,t) = \max\{s+t-1, 0\}$, $s,t \in [0,1]$. Let A, B, S and T be mappings from X into itself such that*

(4.10) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

(4.11) *one of A, B, S and T is τ-continuous,*

(4.12) *the pairs A, S and B, T are g-compatible mappings of type (P),*

(4.13) *there exists a $\phi \in \mathcal{G}$ such that*

$$\begin{aligned}\int_0^1 F_{Ax,By}(t)dt \quad &\geq \quad 1 - \phi\Big\{\int_0^1 (1 - F_{Ax,Sx}(t))^2 dt, \\ &\qquad \int_0^1 (1 - F_{By,Ty}(t))^2 dt, \int_0^1 (1 - F_{Ax,Ty}(t))^2 dt \\ &\qquad \int_0^1 (1 - F_{Ax,Sx}(t))^2 dt, \int_0^1 (1 - F_{Ax,Sx}(t))^2 dt\Big\}\end{aligned}$$

for all $x, y \in X$ and $t \geq 0$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Since $(X, \mathcal{F}, \Delta)$ is an N.A. Menger PM-space with the t-norm such that $\Delta(s,t) \geq \Delta_m(s,t) = \max\{s+t-1, 0\}$, $s,t \in [0,1]$, by Remark 1.1 (5), it is metrizable by the metric d defined by $(*)$. Thus, if we define $g(t) = 1-t$

and $h(t) = (1-t)^2$, from (4.13), we have

$$\begin{aligned} d(Ax, By) &= \int_0^1 h(F_{Ax,Sx}(t))dt = \int_0^1 (1 - F_{Ax,By}(t))^2 dt \\ &= \int_0^1 (1 - F_{Ax,By}(t))dt < 1 - \int_0^1 F_{Ax,Sx}(t)dt \\ &\leq \phi\Big\{ \int_0^1 (1 - F_{Ax,Sx}(t))^2 dt, \int_0^1 (1 - F_{By,Ty}(t))^2 dt, \\ &\qquad \int_0^1 (1 - F_{Ax,Ty}(t))^2 dt, \int_0^1 (1 - F_{Ax,Sx}(t))^2 dt, \\ &\qquad \int_0^1 (1 - F_{Ax,Sx}(t))^2 dt \Big\}, \end{aligned}$$

i.e.,

$$d(Ax, By) < \phi(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty))$$

for all $x, y \in X$. Therefore, by Theorem 3.1, A, B, S and T have a unique common fixed point in X. This completes the proof.□

As an immediate consequence of Theorem 4.6, we have the following:

Corollary 4.2. *Let $(X, \mathcal{F}, \Delta)$ be as in Theorem 4.6 Let A, B, S and T be mappings from X into itself satisfying the conditions* (4.10), (4.11), (4.12) *and*

(4.14) *there exists a number $c \in (0,1)$ such that*

$$\begin{aligned} \int_0^1 F_{Ax,By}(t)dt &\geq 1 - c \cdot \max\Big\{ \int_0^1 (1 - F_{Ax,Sx}(t))^2 dt, \\ &\qquad \int_0^1 (1 - F_{By,Ty}(t))^2 dt, \int_0^1 (1 - F_{Ax,Ty}(t))^2 dt, \\ &\qquad \int_0^1 (1 - F_{By,Sx}(t))^2 dt, \int_0^1 (1 - F_{Sx,Ty}(t))^2 dt \Big\} \end{aligned}$$

for all $x, y \in X$ and $t \geq 0$.

Then A, B, S and T have a unique common fixed point in X.

Acknowledgement. The authors are extremely grateful to Prof. O. Hadžić for her valuable suggestions on this paper.

References

[1] Bhakta, P.C., Basu, T., Some fixed point theorems on metric spaces, J. Indian Math. Soc. 45(1981), 399–404.

[2] Bharucha-Reid, A.T., Fixed point theorems in probabilistic analysis, Bull. Amer. Math. Soc. 82(1976), 641–657.

[3] Bocsan, Gh., On some fixed point theorems in probabilistic metric spaces, Math. Balkanica 4(1974), 67–70.

[4] Bollenbacher, A., Hicks, T.L., A fixed point theorems revisited, Proc. Amer. Math. Soc. 102(1988), 898–900.

[5] Cain Jr., G.L., Kasriel, R.H., Fixed and periodic points of local contraction mappings of probabilistic metric spaces, Math. Systems Theory 9(1976), 289–297.

[6] Caristi, J., Fixed point theorems for mappings satisfying inwardness condition, Train. Amer. Math. Soc. 215(1976), 241–251.

[7] Chang, S.S., On some fixed point theorems in probabilistic metric spaces and its applications, Z. Wahr. verw. Gebiete 63(1983), 463–474.

[8] Chang, S.S., On the theory of probabilistic metric spaces with applications, Acta Math. Sinica, New Series 1(4)(1985), 366-377.

[9] Chang, S.S., Probabilistic metric spaces and fixed point theorems for mappings, J. Math. Research Expos. 3(1985), 23–28.

[10] Chang, S.S., Fixed point theorems for single-valued and multi-valued mappings in non-Archimedean Menger probabilistic metric spaces, Math. Japonica 35(5)(1990), 875–885.

[11] Chang, S.S., Xiang, S.W., Topological structure and metrization problem of probabilistic metric spaces and application, J. Qufu Normal Univ. 16(3) (1990), 1–8.

[12] Chang, S.S., Chen, Y.Q., Guo, J.L., Ekeland's variational principal and Caristi's fixed point theorem in probabilistic metric spaces, Acta Math. Appl. Sinica 7(3)(1991), 217–229.

[13] Chang, S.S., Cho, Y.J., Kim, J.K., Ekeland's variational principle and Caristi's coincidence theorem for set-valued mappings in probabilistic metric spaces, submitted.

[14] Cho, Y.J., Murthy, P.P., Stojakovic, M., Compatible mapping of type (A) and commom fixed point in Menger spaces, Comm. of Korean Math. Soc. 7(2) (1992), 325–339.

[15] Cho, Y.J., Park, K.S., Chang, S.S., Fixed point theorems in metric spaces and probabilistic metric spaces, to appear in IJMMS.

[16] Ćirić, Lj., On fixed points of generalized contractions of probabilistic metric spaces, Publ. Inst. Math. (Beograd) 18(32) (1975), 71–78.

[17] Ćirić, Lj., Periodic and fixed point theorems in a quasi-metric space, J. Austral Math. Soc., Series A 54 (1993), 80–85.

[18] Dien, N.H., Fixed Point and Fuzzy mappings, Ph. D. Dissertation, Institute of Mathematics, Sofia, Bulgaria (1986).

[19] Dien, N.H., On common fixed points of weakly commuting contractive mappings, Hanoi Institute of Mathematics, preprint 92/30 (1992).

[20] Dien, N.H., Some remarks on common fixed point theorems, J. Math. Anal. Appl. 189(1994), 76–90.

[21] Ekeland, I., Nonconvex minimization problems, Bull. Amer. Math. Soc. 1(1979), 443–475.

[22] Hadžić, O., A fixed point theorem in Menger spaces, Publ. Inst. Math. (Beograd) 20(40) (1979), 107–112.

[23] Hadžić, O., Some theorems on the fixed points in probabilistic metric and random normed spaces, Boll. Un. Mat. Ital. 13(5) 18(1981), 1-11.

[24] Hicks, T.L., Fixed point theory in probabilistic metric spaces, Review of Research, Fac. Sci. Math. Series, Univ. of Novi Sad 13(1983), 63-72.

[25] Jung, J.S., Cho, Y.J., Kim, J.K., Minimization theorems and fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 61(1994), 199–207.

[26] Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 4 (1986), 771–779.

[27] Jungck, G., Murthy, P.P., Cho, Y.J., Compatible mappings of type (A) and common fixed point theorems, Math. Japon. 38(2) (1993), 381–390.

[28] Schweizer, B., Sklar, A., Statistical metric spaces, Pacific J. Math. 10(1960), 313-334.

[29] Schweizer, B., Sklar, A., Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Math. (1983).

[30] Sehgal, V., Rharucha-Reid, A.T., Fixed points of contraction mappings of probabilistic metric spaces, Math. Systems Theory 6 (1972), 92–102.

[31] Sessa, S., On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. (Beograd) 46 (1982), 146–153.

[32] Siegel, J., A new proof of Caristi's fixed point theorem, Proc. Amer. Math. Soc. 66(1977), 54–56.

[33] Singh, S.P., Meade, B.A., On common fixed point theorems, Bull. Austral. Math. Soc. 16(1977), 49–53.

[34] Stojakovic, M., Fixed point theorem in probabilistic metric spaces, Kobe J. Math. 2(1985), 1–9.

[35] Stojakovic, M., Common fixed point theorems in complete metric and probabilistic metric spaces, Bull. Austral. Math. Soc. 36(1987), 73–78.

[36] Stojakovic, M., A common fixed point theorems in probabilistic metric spaces and its applications, Glasnik Mat. 23(43) (1988), 203–211.

[37] Tan, N.X., Generalized probabilistic metric spaces and fixed point theorems, Math. Nachr. 129(1986), 205–218.

Received by the editors July 14, 1994.