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SOME PROBABILISTIC GENERALIZATIONS OF THE SUBMEASURE CONCEPT

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Abstract

According to the study of L. Drewnowski [3] on a set ring with the topology induced by numerical submeasures, this paper is concerned with the relation between the study of probabilistic submeasures (probabilistic generalization of the submeasure notion) and that of some random structures on set rings.

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1. Introduction

In the recent years non-additive set functions have obtained an important place in the mathematical investigations with many applications. Let us mention some of these classes: submeasures, $D-$submeasures, semimeasures, fuzzy measures, k - triangular set functions, etc.

Important problems of measure theory can be developed taking into consideration the topological point of view. These problems have led to a topological study of the set functions that are not necessarily additive, study inaugurated by Orlicz, developed and systematized by Drewnowski in a series of papers [3]. Thus we naturally come across the study of topological set rings with the topology induced by submeasures.

In numerous instances in which the measure theory is applied, the association of a single number as the (sub)measure of a set is rather an over-idealization. Probabilistic submeasures render the concept of submeasure as a probabilistic rather than a deterministic one.

In analogy with the case of positive submeasures it is but natural to topologically develop the study of random submeasures, as well as of random structures on set rings.

2. Preliminaries

Let $(\mathcal{S}, \Delta, \cap)$ be a ring of subsets of a fixed set S, with respect to the operations Δ (symmetric difference) and $\cap$ (intersection). In particular, $\mathcal{S}$ can be the ring $\mathcal{P}(S)$ of all subsets of S.

The ring $\mathcal{S}$ is said to be a topological ring of sets if on $\mathcal{S}$ a topology τ is given such that the set operations $(A, B) \to A \cap B$ and $(A, B) \to A \Delta B$ from $\mathcal{S} \times \mathcal{S}$ (endowed with a product topology) are also continuous. The general properties of topological rings remain valid in the special case of topological rings of sets. The topology τ, consistent with the ring structure of $(\mathcal{S}, \Delta, \cap)$, will be shortly called an r - topology on $\mathcal{S}$.

If the topology τ on $\mathcal{S}$ is such that for each neighborhood $\mathcal{U}$ of $\emptyset$ there is a neighborhood $\mathcal{V}$ of $\emptyset$ with the property that $B \in \mathcal{U}$, whenever $B \in \mathcal{S}$ and $B \subset A \in \mathcal{V}$, then τ is said to be a monotone (or a Fréchet-Nikodym) topology. And if $\mathcal{U}$ is a base of the neighborhoods of $\emptyset$, it is said to be a normal base of neighborhoods if for every $U \in \mathcal{U}$ one has $B \in \mathcal{U}$ provided $B \in \mathcal{S}$ and $B \subset A$ for some $A \in \mathcal{U}$.

A mapping $\eta : \mathcal{S} \to [0, \infty]$ is said to be a submeasure, [3] if:

(i) $\eta(\emptyset) = 0$

(ii) $\eta(A) \leq \eta(B)$ if $A, B \in \mathcal{S}$ and $A \subset B$

(iii) $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever $A, B \in \mathcal{S}$.

A submeasure η is said to be finite if $\eta(A) < \infty$ for every $A \in \mathcal{S}$.

For a submeasure η on $\mathcal{S}$, the classes $\mathcal{U}(\epsilon) = \{A \in \mathcal{S}, \eta(A) \leq \epsilon\}, \epsilon > 0$ constitute a normal base of neighborhoods of $\emptyset$ for a Fréchet-Nikodym topology denoted $\tau(\eta)$ and called the Fréchet-Nikodym topology generated by η on $\mathcal{S}$. This topology $\tau(\eta)$ is semimetrizable for example by the semimetric defined by: $d(A, B) = \eta(A \Delta B), A, B \in \mathcal{S}$ (if $\eta(A) < \infty$, for each $A \in \mathcal{S}$)

3. Probabilistic generalizations of the submeasure concept

Let D_+ be the family of all distribution functions F such that $F(0) = 0$ (recall that F is left continuous, nondecreasing and $\sup_{x>0} F(x) = 1$). If $a \geq 0$, then H_a defined by:

$$H_a(x) = \begin{cases} 0, & x \leq a \\ 1, & x > 1 \end{cases}$$

is an element of D_+. Moreover $F \leq H_0$ for every F in D_+.

Let T be a t - norm, i.e. an operation on $[0, 1]$ such that $([0, 1]), T)$ is an order semigroup with unit. Let $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

(θ_0) $$\theta(x, 0) = x$$

(θ_1) $$\theta(x, y) = \theta(y, x)$$

(θ_2) $$\theta(x, y) \leq \theta(x, z) \text{ if } y \leq z$$

(θ_3) $$\theta(x, \theta(y, z)) = \theta(\theta(x, y), z).$$

Such a function is called an operation (shortly Op), [10]. Suppose that θ is continuous and strictly increasing in each place. Then from [10] we have that there exists $f : [0, \infty) \to [0, \infty)$ which is strictly increasing, bicontinuous and such that: $\theta(x, y) = f^{-1}(f(x) + f(y))$. The function f is called the additive generator of θ. We note the following simple and useful examples:

$$\theta_k = (x, y) = (x^k + y^k)^{1/k} \quad \text{where} \quad k \in [0, \infty).$$

Here $f_k(x) = x^k$ is the additive generator of θ_k.

Also $\theta_\infty(x, y) = \lim_{k\to\infty} \theta_k(x, y) = \max\{x, y\}$ is an operation, which has no additive generator. The Op-s θ_1 (which is the classical Sum) and θ_∞ will be of principal interest in what follows.

Definition 3.1. *Let $\mathcal{S}$ be a ring of subsets of a fixed set S and a mapping $\gamma : \mathcal{S} \to D_+$ ($\gamma(A)$ will be denoted by γ_A) such that:*

(m_1) $$A = \emptyset \Leftrightarrow \gamma_A(x) = H_0(x), x > 0$$

(m_2) $$A \subset B \Rightarrow \gamma_A(x) \geq \gamma_B(x), x > 0$$

(m_3) $$\gamma_A(x) = 1, \gamma_B(y) = 1 \Rightarrow \gamma_{A\cup B}(\theta(x,y)) = 1, A, B \in \mathcal{S}.$$

The mapping γ is said to be a θ - probabilistic submeasure and the pair $(\mathcal{S}, \gamma)$ is called ring with θ - probabilistic submeasure.

Definition 3.2. *The mapping γ which verifies the axioms $(m_1), (m_2)$, and $(\theta m_3')$, where:*

$(\theta m_3')$ $$\gamma_{A\cup B}(\theta(x,y)) \geq T(\gamma_A(x), \gamma_B(y)), x, y > 0, A, B \in \mathcal{S}$$

where T is a fixed t - norm, is called θ - Menger submeasure.

The triplet $(\mathcal{S}, \gamma, T)$ is named a θ - Menger ring. For $\theta = \theta_1 =$ Sum the mapping γ is called probabilistic submeasure (Menger submeasure, respectively) and for $\theta = \theta_\infty =$ max we have the so called non Archimedean probabilistic submeasure (non-Archimedean Menger submeasure, respectively).

Remark 3.3. It is easy to see that every θ_a - probabilistic submeasure is θ_b - probabilistic submeasure if $\theta_a \leq \theta_b$.

Remark 3.4. Every ring with a numerical submeasure may be regarded as a ring with a θ - Menger submeasure of a special kind.

Let $(\mathcal{S}, \eta)$ be a ring with a numerical submeasure η. Then, a θ - Menger submeasure γ on $\mathcal{S}$ can be defined with respect to the t- norm $T(A, B) = \min\{a, b\}$, and $\theta =$ Sum, by $\gamma_A(x) = H_0(x - \eta(A))$. The number $\gamma_A(x)$ is interpreted as the probability that the submeasure $\eta(A)$ of A is less than x.

Theorem 3.5. *Let $(\mathcal{S},\eta)$ be a ring with a numerical submeasure η. Then a $\theta-$ Menger submeasure γ on $\mathcal{S}$ can be defined with respect to the t - norm:*

$$T(a,b) = \begin{cases} a, & b=1 \\ b, & a=1 \\ 0, & a\neq 1,\ b\neq 1 \end{cases}$$

and θ arbitrary, by

$$\gamma_A(x) = \begin{cases} 1-\dfrac{\eta(A)}{x+\eta(A)}, & x>0 \\ \\ 0, & x\leq 0 \end{cases}$$

Proof. It is easy to verify $(m_1),(m_2),(\theta m_3)$. For $(\theta m_3')$ we have: If $\eta(A)\neq 0$ and $\eta(B)\neq 0$ it results that $T(\gamma_A(x),\gamma_B(y))=0,\ x,y>0$ and $\gamma_{A\cup B}(\theta(x,y))\geq T(\gamma_A(x),\gamma_B(y))$. If $\eta(B)=0$ then $\eta(A\cup B)=\eta(A)$ and:

$$T(\gamma_A(x),\gamma_B(y))=\gamma_A(x)=1-\frac{\eta(A)}{x+\eta(A)}$$

Since $\eta(A\cup B)=\eta(A)$, and $\theta(x,y)\geq x$, we have:

$$\gamma_{A\cup B}(\theta(x,y))=1-\frac{\eta(A\cup B)}{\theta(x,y)+\eta(A\cup B)}=1-\frac{\eta(A)}{\theta(x,y)+\eta(A)}\geq$$

$$\geq 1-\frac{\eta(A)}{x+\eta(A)}=T(\gamma_A(x),\gamma_B(y))$$

Definition 3.6. *The mapping γ which verifies the axioms $(m_1),(m_2)$ and (Hm_3), where*

(Hm_3) $\forall\epsilon>0,\exists\delta>0$ *such that* $1-\gamma_A(\delta)<\delta, 1-\gamma_B(\delta)<\delta\Rightarrow 1-\gamma_{A\cup B}(\epsilon)<\epsilon,\quad A,B\in\mathcal{S}$

is said to be a probabilistic H - submeasure and the pair $(\mathcal{S},\gamma)$ is called a ring with probabilistic H - submeasure.

We consider below a generalization of (Hm_3), formulated in terms of "additive generators" of t - norms. (Hm_3) can be formulated by using the function $f_m(t) = 1 - t$, which is the generator of $T_m = \max(\text{Sum} - 1, 0)$. The notion and the notations are the same as in [11].

Let $f : [0,1] \to [0,\infty)$ be continuous, strictly decreasing and such that $f(1) = 0$. If $f(0) = b$, then the mapping $f^{(-1)} : [0,\infty) \to [0,1]$ given by:

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x), & x \leq b \\ 0, & x \geq b \end{cases}$$

is the pseudo-inverse of f. We note that $f^{(-1)} \circ f(x) = x$, and

$$f \circ f^{(-1)}(y) = \begin{cases} y, & y < b, \\ b, & y \geq b. \end{cases}$$

It is obvious that $f^{(-1)}$ is decreasing on $[0,b]$ and continuous.

Definition 3.7. *Let $\mathcal{S}$ be a ring of subsets of a fixed set S and a mapping $\gamma : \mathcal{S} \to D_+$ which verifies the axioms $(m_1), (m_2)$ and (fm_3) where:*

(fm_3) $\forall \epsilon > 0, \exists \delta > 0$ *such that* $f \circ \gamma_A(\delta) < \delta,\ f \circ \gamma_B(\delta) < \delta \Rightarrow f \circ \gamma_{A \cup B}(\epsilon) < \epsilon$

is called a probabilistic f - submeasure and the pair $(\mathcal{S}, \gamma)$ will be called a ring with probabilistic f - submeasure.

Remark 3.8. If (fm_3) holds for δ then it clearly holds for $\delta_1 < \delta$. If $f = f_m$ then (fm_3) is exactly (Hm_3).

Theorem 3.9. *Every θ - Menger submeasure for which* $\sup_{x<1} T(x,x) < 1$ *defines a probabilistic f - submeasure for any f.*

Proof. Let $\epsilon > 0$ be given. Evidently, we can suppose that $\epsilon < f(0)$, since f is decreasing. New let δ be such that $T(f^{-1}(\delta), f^{-1}(\delta)) > f^{-1}(\epsilon)$ and $\theta(\delta,\delta) < \epsilon$. Suppose that $f \circ \gamma_A(\delta) < \delta$ and $f \circ \gamma_B(\delta) < \delta$. This implies that $\gamma_A(\delta) \geq f^{-1}(\delta)$ and $\gamma_B(\delta) \geq f^{-1}(\delta)$. Then, by $(\theta m_3')$ we have that:

$$\gamma_{A\cup B}(\epsilon) \geq \gamma_{A\cup B}(\theta(\delta,\delta)) \geq T(\gamma_A(\delta), \gamma_B(\delta)) \geq T(f^{-1}(\delta), f^{-1}(\delta)) > f^{-1}(\epsilon))$$

and so $f \circ \gamma_{A\cup B}(\epsilon) < \epsilon$, that is (fm_3) holds.

Remark 3.10. If $(\mathcal{S}, \gamma, T)$ is a θ - Menger ring and T is Archimedean with the generator f, then $(\theta m_3')$ is equivalent to (fm_3'), where

(fm_3') $$f \circ \gamma_{A\cup B}(\theta(x,y)) \leq f \circ \gamma_A(x) + f \circ \gamma_B(y)$$

and (fm_3) easily follows in this case.

4. The topological set ring defined by generalized probabilistic submeasures

Theorem 4.1. *Let γ be a θ - Menger submeasure on the ring of set $\mathcal{S}$, and for $\epsilon, \lambda > 0$ let $\mathcal{U}(\epsilon,\lambda) = \{A \in \mathcal{S}, \gamma_A(\epsilon) > 1-\lambda\}$. If the $t-$ norm T is continuous and $\inf_{x>0}\theta(x,x) = 0$, then*

(i) the family $\mathbf{U} = \{\mathcal{U}(\epsilon,\lambda) : \epsilon > 0, \lambda > 0\}$ is a normal base of neighborhoods of $\emptyset$ for a Fréchet - Nikodym topology $\tau(\gamma)$ and $(\mathcal{S}, \Delta, \cap, \tau(\gamma))$ is a topological ring.

(ii) the map $\mathcal{F} : \mathcal{S} \times \mathcal{S} \to D_+$ being defined by $\mathcal{F}(A,B) = F_{AB} = \gamma_{A\Delta B}(\mathcal{S}, \mathcal{F}, T)$ is a θ-semi Menger space [10] and the θ-Menger semimetric on $\mathcal{S}$ is translation invariant.

(iii) the (ϵ,λ) metrizable uniformity on $\mathcal{S}$ induced by the θ - Menger semimetric $\mathcal{F}$ and the uniformity on $\mathcal{S}$ induced by the topology $\tau(\gamma)$ are identical.

(iv) if $\theta = \theta_\infty$ and $T = \min$ then $(\epsilon,\lambda)-$ uniformity is ultra-metrizable.

Proof. We show first that $\mathcal{F}$ is a θ-semi Menger space [10], on $\mathcal{S}$. Clearly, if $A = B$, then $F_{AB} = \gamma_{A\Delta B} = \gamma_\emptyset = H_0$, and $F_{AB} = F_{BA}$ for every $A, B \in \mathcal{S}$. Now, if $A, B, C \in \mathcal{S}$, since $A\Delta B = (A\Delta C)\Delta(C\Delta B) \subset (A\Delta C) \cup (C\Delta B)$, we have:

$$F_{AB}(\theta(x,y)) = \gamma_{A\Delta B}(\theta(x,y)) \geq \gamma_{(A\Delta C)\cup(C\Delta B)}(\theta(x,y)) \geq$$

$$\geq T(\gamma_{A\Delta B}(x), \gamma_{C\Delta B}(y)) = T(F_{AC}(x), F_{CB}(y))$$

and the triangle inequality is proved. Thus, $(\mathcal{S}, \mathcal{F}, T)$ is a θ-semi Menger space. Since T is continuous, it follows as in [13] that the family $\{\mathcal{U}(\epsilon,\lambda); \epsilon > 0, \lambda > 0\}$, where:

$\mathcal{U}(\epsilon,\lambda) = \{(A,B) \in \mathcal{S} \times \mathcal{S}; F_{AB}(\epsilon) > 1-\lambda\}$ is a base for the (ϵ,λ) - uniformity on $\mathcal{S}$ induced by $\mathcal{F}$; and the family:

$\{S_B(\epsilon,\lambda); \epsilon > 0, \lambda > 0\}$ where $S_B(\epsilon,\lambda) = \{A \in \mathcal{S}; F_{AB}(\epsilon) > 1-\lambda\}$ is a base of neighborhoods of the topology induced by (ϵ,λ) - uniformity generated by $\mathcal{F}$. Then, since $\mathcal{U}(\epsilon,\lambda) = S_\emptyset(\epsilon,\lambda)$, it follows that $\mathbf{U}$ is a base of neighborhoods of $\emptyset$ (for the (ϵ,λ) - topology induced by $\mathcal{F}$.) The fact that $\mathbf{U}$ is a normal base follows easily from the monotonicity of $\gamma, (m_3)$. Since $A\Delta B = (A\Delta C)\Delta(B\Delta C), \mathcal{F}$ is translation invariant, i.e. $\mathcal{F}(A\Delta C, B\Delta C) =$

$\mathcal{F}(A,B)$. Thus, to prove (i-iii) it suffices to show that the operation $\cap$ is continuous (relative) to the product topology on $\mathcal{S}\times\mathcal{S}$). For this, we remark that:

$$(A_1\cap B_1)\Delta(A_2\cap B_2)\subset(A_1\Delta A_2)\cup(B_1\Delta B_2)$$

Hence, from the definition of θ - Menger submeasure we have:

$$(1)\quad F_{A_1\cap B_1,A_2\cap B_2}(x)=\gamma_{(A_1\cap B_1)\Delta(A_2\cap B_2)}(x)\geq\gamma_{(A_1\Delta A_2)\cup(B_1\Delta B_2)}(\theta(x',x'))$$
$$\geq T(\gamma_{A_1\Delta A_2}(x'),\gamma_{B_1\Delta B_2}(x'))=T(F_{A_1A_2}(x'),F_{B_1B_2}(x'))$$

where $\theta(x',x')<x$.

Now if $(A_n,B_n)\to(A,B)$ in the $\tau(\gamma)$ - topology (or, equivalently in the (ϵ,λ) - topology induced by $\mathcal{F}$), then $A_n\to A$ and $B_n\to B$, hence $F_{A_nA}(x')\to 1$ and $F_{B_n}B(x')\to 1$. By (1) and the continuity of T at $x=1$, it follows that $F_{A_n\cap A,B_n\cap B}(x')\to 1$ for each $x>0$, and the continuity of $\cap$ is proved.

Remark 4.2. As we observed in Theorem 3.1., if $\sup_{x<1}T(x,x)=1$ then the family $\{\mathcal{U}(\epsilon,\lambda));\epsilon>0,\ \lambda\in(0,1)\}$, $\mathcal{U}(\epsilon,\lambda)=\{(A,B);\ \gamma_{A\Delta B}(\epsilon)>1-\lambda\}$ or $\{\mathcal{V}(\lambda);\lambda\in(0,1)\}$, $\mathcal{V}(\lambda)=\{(A,B);\ \gamma_{A\Delta B}(\lambda)>1-\lambda\}$ is a base for a metrizable uniformity on $\mathcal{S}$, which is called the $(\epsilon,\lambda)-$ uniformity or the $\mathcal{F}-$ uniformity.

If $(\mathcal{S},\gamma)$ is a ring with probabilistic $f-$ submeasure, then the family $\{V_f(\lambda);\lambda>0\},V_f(\lambda)=\{(A,B);f\circ\gamma_{A\Delta B}(\lambda)<\lambda\}$ generates a metrizable uniformity on S which coincides with the $\mathcal{F}$ - uniformity.

Theorem 4.3. *Let $(\mathcal{S},\gamma,T)$ be a θ-Menger submeasure such that $\theta\leq\theta_1$, and $T\geq T_f$ (where $T_f(a,b)=f^{(-1)}(f(a)+f(b))$. Then the mapping $\eta_f:\mathcal{S}\to D_+$ defined by:*

$$(2)\qquad \eta_f(A)=\sup\{t;f\circ\gamma_A(t)\geq t\}$$

is a numerical submeasure on $\mathcal{S}$ such that $\tau(\eta)=\tau(\gamma)$.

Proof. We will prove first that the two place function d_f defined on $\mathcal{S}\times\mathcal{S}$ by $d_f(A,B)=\sup\{t;f\circ\gamma_{A\Delta B}(t)\geq t\}$ is a semi-metric on $\mathcal{S}$ which semimetrices the $\mathcal{F}$ - uniformity. We will prove the triangle inequality for d_f. Suppose that $d_f(A,C)<x,d_f(C,B)<y$. Then, from (2) we have that $f\circ\gamma_{A\Delta C}(x)<x$ and $f\circ\gamma_{C\Delta B}(y)<y$. Since $T\geq T_f$, then from $(\theta m_3')$ it results that:

$$\gamma_{A\Delta B}(\theta(x,y))\geq\gamma_{(A\Delta C)\cup(C\Delta B)}(\theta(x,y))\geq T(\gamma_{A\Delta C})(x),\ \gamma_{C\Delta B}(y))\geq$$

$$\geq T_f(\gamma_{A\Delta C}(x), \gamma_{C\Delta B}(y)) = f^{(-1)}(f \circ \gamma_{A\Delta C}(x), f \circ \gamma_{C\Delta B}(y))$$

or equivalent $f \circ \gamma_{A\Delta B}(\theta(x,y)) \leq f \circ \gamma_{A\Delta C}(x) + f \circ \gamma_{C\Delta B}(y)$ and so $f \circ \gamma_{A\Delta B}(\theta(x,y)) < x+y$. Therefore, since $\theta(x,y) \leq x+y$, f is decreasing and $F_{A\Delta B}$ is non-decreasing, $f \circ \gamma_{A\Delta B}(x+y) < x+y$ which shows that $d_f(A,B) < x+y$, and the inequality $d_f(A,B) \leq d_f(A,C)+d_f(C,B)$ follows. The last part of the above assertion follows from the fact that:

$$(3) \qquad d_f(A,B) < t \text{ iff } f \circ \gamma_{A\Delta B}(t) < t$$

that is $V_f(t) = \{(A,B), d_f(A,B) < t\}$.

The semimetric d_f on $\mathcal{S}$ is translation invariant i.e.

$d_f(A,B) = d_f(A\Delta C, B\Delta C), A, B, C \in \mathcal{S}$ and $\eta_f(A) = d_f(A, \emptyset)$. It is obvious that:

(i) $\eta_f(\emptyset) = 0$;

(ii) $A \subset B \Rightarrow \eta_f(A) \leq \eta_f(B)$.

We will prove the inequality:

(iii) $\eta_f(A \cup B) \leq \eta_f(A) + \eta_f(B)$. For $A, B \in \mathcal{S}, A \cap B = \emptyset$, we have $A \cup B = A\Delta B$ and: $\eta_f(A \cup B) = d_f(A \cup B, \emptyset) \leq d_f(A \cup B, B) + d_f(B, \emptyset) = d_f((A \cup B)\Delta B, B\Delta B) + d_f(B, \emptyset) = d_f(A\Delta B\Delta B, B\Delta B) + d_f(B, \emptyset) = d_f(A, \emptyset) + d_f(B, \emptyset) = \eta_f(A) + \eta_f(B)$. If $A, B \in \mathcal{S}, A \cap B \neq \emptyset$ it results that: $\eta_f(A \cup B) = \eta_f(A \cup (B - A)) \leq \eta_f(A) + \eta_f(B - A) \leq \eta_f(A) + \eta_f(B)$.

5. The probabilistic metrizability of the topological set rings

Theorem 5.1. *Let $(\mathcal{S}, \tau)$ be a topological ring of subsets of a fixed set S such that there exists: $\eta : \mathcal{S} \to R_+, \eta(A) = 0$ iff $A = \emptyset$ and such that the collection $\{\mathcal{B}_\epsilon(A)\}_{\epsilon > 0}, \mathcal{B}_\epsilon(A) = \{E; \eta(E\Delta A) < \epsilon\}$ is a base of neighborhoods of $A \in \mathcal{S}$ for the τ topology.*

Then, there exists a mapping $\gamma : \mathcal{S} \to D_+$ with the properties:

(i) $\gamma_A = H_0$ iff $A = \emptyset$

(ii) $\gamma_{A\Delta B}(x+y) \geq T(\gamma_A(x), \gamma_B(y)), x, y > 0, A, B \in \mathcal{S}$ *where:*

$$T(a,b) = \begin{cases} a, & b = 1 \\ b, & a = 1 \\ 0, & a \neq 1,\ b \neq 1 \end{cases}$$

and such that the family $\{\mathcal{U}_A(\epsilon, \lambda)\}_{\epsilon>0,\lambda>0}$, $\mathcal{U}_A(\epsilon,\lambda) = \{E; \gamma_{E\Delta A}(\epsilon) > 1-\lambda\}$ *is a base of neighborhoods of* $A \in \mathcal{S}$ *for the* τ *topology.*

Proof. For each $A \in \mathcal{S}$, we define $\gamma : \mathcal{S} \to D_+$

$$\gamma_A(x) = \begin{cases} 1 - \dfrac{\eta(A)}{x + \eta(A)}, & x > 0 \\ \\ 0 & x \leq 0 \end{cases}$$

which satisfies (i) and (ii). The map $\mathcal{F} : \mathcal{S} \times \mathcal{S} \to D_+, \mathcal{F}(A,B) = \gamma_{A\Delta B}$ is a Menger metric which is translation invariant.

In addition $\mathcal{U}_A(\epsilon, \lambda) = \mathcal{B}_{\frac{\lambda\epsilon}{1-\lambda}}(A)$.

Theorem 5.2. *Let* $(\mathcal{S}, \tau)$ *be a topological ring such that there exists a map* $\gamma : \mathcal{S} \to D_+$ *with the properties:*

(i) $\gamma_A = H_0$ *iff* $A = \emptyset$

(ii) $\gamma_{A\Delta B}(x+y) \geq T(\gamma_{A\Delta C}(x), \gamma_{C\Delta B}(y)), x, y > 0, A, B \in \mathcal{S} (\sup_{x>0} T(x,x) = 1)$ *and such that the family* $\{\mathcal{U}_\emptyset(\epsilon, \lambda)\}_{\epsilon>0,\lambda>0}$ $\mathcal{U}_\emptyset(\epsilon,\lambda) = \{E; \gamma_E(\epsilon) > 1-\lambda\}$ *is a base of neighborhoods of* $\emptyset \in \mathcal{S}$ *for the* τ *- topology.*

Then there exists a mapping $\eta : \mathcal{S} \to R_+, \eta(A) = 0$ *iff* $A = \emptyset$ *such that the collection* $\{\mathcal{B}_\epsilon\}_{\epsilon>0} : \mathcal{B}_\epsilon = \{E; \eta(E) < \epsilon\}$ *is a base of neighborhoods of* $\emptyset$ *for the* τ *- topology.*

Proof. For each $A \in \mathcal{S}$, we define $\eta : \mathcal{S} \to R_+$ by

$$\eta(A) = \inf_{\epsilon>0}\{1 + \epsilon - \gamma_A(\epsilon)\}$$

Then $\eta(\emptyset) = 0$, and if $\eta(A) = 0$ then for each $\delta > 0$ there exists $\epsilon > 0$ such that $1 + \epsilon - \gamma_A(\epsilon) < \delta$. Thus $\epsilon < \delta$ and $A \in \mathcal{U}_\emptyset(\epsilon, \delta) \subset \mathcal{U}_\emptyset(\delta, \delta)$. By (i) we have $\bigcap_{\delta>0} \mathcal{U}_\emptyset(\delta,\delta) = \{\emptyset\}$ and therefore $A = \emptyset$. We have $\mathcal{B}_\delta(\emptyset) \supset \mathcal{U}_\emptyset(\frac{1}{2}\delta, \frac{1}{2}\delta)$

because $\gamma_A(\frac{1}{2}\delta) > 1-\frac{1}{2}\delta$ implies $1+\frac{1}{2}\delta-\gamma_A(\frac{1}{2}\delta) < \delta$. $\mathcal{B}_\epsilon(\emptyset) \subset \mathcal{U}_{\emptyset(\epsilon,\epsilon)}$, because $\eta(A) < \epsilon$ implies $1+\epsilon'-\gamma_A(\epsilon') < \epsilon$ for some ϵ' thus $\epsilon' < \epsilon$ and we have:

$$\gamma_A(\epsilon) \geq \gamma_A(\epsilon') > 1+\epsilon'-\epsilon > 1-\epsilon,$$

which implies $A \in \mathcal{U}_\emptyset(\epsilon,\epsilon)$. If $\{\mathcal{U}_\emptyset(\epsilon,\lambda)\}_{\epsilon>0,\lambda>0}$ is a neighborhood base at $\emptyset$ for a topology, then so is the collection $\{\mathcal{B}_\epsilon(\emptyset)\}_{\epsilon>0}$.

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