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ON THE DOUBLE g-INTEGRAL[1]

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Abstract

In the framework of the Pap g-calculus, the notion of double g-integral is introduced. As the main results there are presented the Fubini theorem, the integral transformation formula, and the Green theorem.

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1. Introduction

The g-calculus of E. Pap ([3], see also [4]), is based on a function $g : [a,b] \to [0,\infty]$ which is strictly monotone, bijective and such that either $g(a) = 0$, or $g(b) = 0$. The g-integral of a measurable function $f : [\alpha,\beta] \to [a,b]$ is defined by the formula

$$\int\limits_{[\alpha,\beta]}^{\oplus} f(x)dx = g^{-1}\Big(\int_{\alpha}^{\beta} g(f(x))dx\Big).$$

In this paper we define analogously

$$\iint\limits_{D}^{\oplus} f(x,y)dxdy = g^{-1}\Big(\iint\limits_{D} g(f(x,y))dxdy\Big).$$

Of course, since we want to formulate and prove the Green formula, we must extend our means. Namely, the curvilinear integral

$$\int\limits_{\partial D} g(f(x,y))dx$$

need not be non-negative, although g is non-negative. Therefore, we shall work with the interval $[-\infty,\infty]$ instead of the interval $[0,\infty]$. Of course, if we have a bijection $g : [a,b] \to [-\infty,\infty]$, then there exists $c \in (a,b)$ such that $g(c) = 0$. If we restrict our considerations to the interval $[a,c]$, if g is decreasing, or to the interval $[c,b]$, if g is increasing, then we obtain the case of the Pap theory.

2. Preliminaries

There is given a generator $g : [a,b] \to [-\infty,\infty]$, which is a bijective mapping, decreasing or increasing. We define on the interval $[a,b]$ three binary operations: the pseudo-addition by the formula

$$u \oplus v = g^{-1}(g(u) + g(v)),$$

the pseudo-multiplication by the formula

$$u \otimes v = g^{-1}(g(u) \cdot g(v))$$

and the pseudo-difference by the formula

$$u \ominus v = g^{-1}(g(u) - g(v)).$$

While the first two operations could be introduced also in the case of a generator $g : [a, b] \to [0, \infty]$, the third would have sense only in the case $v \leq u$.

If $D \subset R^2$ is a measurable set and $f : D \to [a, b]$ a measurable function, then we define the double g-integral

$$\iint_D^{\oplus} f(x,y)dxdy = g^{-1}\Big(\iint_D g(f(x,y))dxdy\Big).$$

We shall say that f is g-integrable (on D), if $g \circ f$ is integrable.

3. Properties of the integral

Proposition 1. *Let D be a measurable set, $f_1, f_2 : D \to [a, b]$ be g-integrable functions on D, $\alpha \in [a, b]$, D_1, D_2 be measurable, non-overlaping sets with $D_1 \cup D_2 = D$. Then*

$$\iint_D^{\oplus} f_1(x,y)dxdy \;\oplus \iint_D^{\oplus} f_2(x,y)dxdy =$$
$$= \iint_D^{\oplus}((f_1(x,y)) \oplus (f(x,y)))dxdy,$$

$$\alpha \otimes \iint_D^{\oplus} f_1(x,y)dxdy = \iint_D^{\oplus}(\alpha \otimes f_1(x,y))dxdy,$$

$$\iint_D^{\oplus} f_1(x,y)dxdy = \Big(\iint_{D_1}^{\oplus} f_1(x,y)dxdy\Big) \oplus \Big(\iint_{D_2}^{\oplus} f_1(x,y)dxdy\Big).$$

Proof. By the definition

$$
\begin{aligned}
&\iint\limits_D^{\oplus} f_1(x,y)dxdy \oplus \iint\limits_D^{\oplus} f_2(x,y)dxdy = \\
= \;& [g^{-1}(\iint\limits_D g(f_1(x,y))dxdy)] \oplus [g^{-1}(\iint\limits_D g(f_2(x,y))dxdy)] = \\
= \;& g^{-1}\Big(g[g^{-1}(\iint\limits_D g(f_1(x,y))dxdy)] + g[g^{-1}(\iint\limits_D g(f_2(x,y))dxdy)]\Big) = \\
= \;& g^{-1}(\iint\limits_D g(f_1(x,y))dxdy + \iint\limits_D g(f_2(x,y))dxdy) = \\
= \;& g^{-1}\Big(\iint\limits_D [g^{-1}(g(f_1(x,y)) + g(f_2(x,y)))]dxdy\Big) = \\
= \;& g^{-1}(\iint\limits_D g[f_1(x,y) \oplus f_2(x,y)]dxdy) = \\
= \;& \iint\limits_D^{\oplus} f_1(x,y) \oplus f_2(x,y)dxdy\,.
\end{aligned}
$$

Further

$$
\begin{aligned}
&\alpha \otimes \iint\limits_D^{\oplus} f_1(x,y)dxdy = g^{-1}(g(\alpha) \cdot g(\iint\limits_D^{\oplus} f_1(x,y)dxdy)) = \\
= \;& g^{-1}\Big(g(\alpha) \cdot g(g^{-1}(\iint\limits_D g(f_1(x,y))dxdy])\Big) = \\
= \;& g^{-1}(g(\alpha) \cdot \iint\limits_D g(f_1(x,y))dxdy) = \\
= \;& g^{-1}(\iint\limits_D g[g^{-1}(g(\alpha)) \cdot g(f_1(x,y))]dxdy) = \\
= \;& g^{-1}(\iint\limits_D g(\alpha \otimes f_1(x,y))dxdy) = \iint\limits_D^{\oplus} \alpha \otimes f_1(x,y)dxdy\,.
\end{aligned}
$$

Finally

$$\overset{\oplus}{\iint\limits_D} f_1(x,y)dxdy = g^{-1}\Big(\iint\limits_D g(f_1(x,y))dxdy\Big) =$$

$$= g^{-1}\Big(\iint\limits_{D_1} g(f_1(x,y))dxdy + \iint\limits_{D_2} g(f_2(x,y))dxdy\Big) =$$

$$= g^{-1}\Big(g\big[g^{-1}\big(\iint\limits_{D_1} g(f_1(x,y))dxdy\big)\big] + g\big[g^{-1}\big(\iint\limits_{D_2} g(f_1(x,y)dxdy)\big)\big]\Big) =$$

$$= g^{-1}\Big(g\big[\overset{\oplus}{\iint\limits_{D_1}} f_1(x,y)dxdy\big] + g\big[\overset{\oplus}{\iint\limits_{D_2}} f_1(x,y)dxdy\big]\Big) =$$

$$= \big[\overset{\oplus}{\iint\limits_{D_1}} f_1(x,y)dxdy\big] \oplus \big[\overset{\oplus}{\iint\limits_{D_2}} f_1(x,y)dxdy\big] . \square$$

Theorem 1. *Let D be an elementary region defined by the equality $D = \{(x,y) \in \mathbf{R}^2;\ c \le x \le d,\ \varphi(x) \le y \le \psi(x)\}$, where φ, ψ are continuous on $[c,d]$ and $\varphi(x) \le \psi(x)$ for all $x \in [c,d]$. Let f be continuous on D. Then f is g-integrable on D and*

$$\overset{\oplus}{\iint\limits_D} f(x,y)dxdy = \overset{\oplus}{\int\limits_{[c,d]}} \Big[\overset{\oplus}{\int\limits_{[\varphi(x),\psi(x)]}} f(x,y)dy\Big]dx .$$

Proof. Since g is monotone and bijective, it is continuous. Since f is continuous on a compact set, it is finite. Therefore, the composite mapping $g \circ f$ is continuous, hence integrable on D and therefore f is g-integrable. In a similar way we can prove the integrability of the function $y \mapsto f(x,y)$ on $[\varphi(x),\psi(x)]$ for every $x \in [c,d]$ and the integrability of the function $x \mapsto \overset{\oplus}{\int\limits_{[\varphi(x),\psi(x)]}} f(x,y)dy$. Now, by the classical Fubini theorem we obtain

$$\overset{\oplus}{\iint\limits_D} f(x,y)dxdy = g^{-1}\Big(\iint\limits_D g(f(x,y))dxdy\Big) =$$

$$= g^{-1}\Big(\int_c^d \Big[\int_{\varphi(x)}^{\psi(x)} g(f(x,y))dy\Big]dx\Big) =$$

$$= g^{-1}\left(\int_c^d g\Big[g^{-1}\Big(\int_{\varphi(x)}^{\psi(x)} g(f(x,y))dy\Big)\Big]\,dx\right)$$

$$= g^{-1}\Big(\int_c^d g\Big[\overset{\oplus}{\int\limits_{[\varphi(x),\psi(x)]}} f(x,y)dy\Big]dx\Big) =$$

$$= \overset{\oplus}{\int\limits_{[c,d]}}\Big[\overset{\oplus}{\int\limits_{[\varphi(x),\psi(x)]}} f(x,y)dy\Big]dx\,.\square$$

Theorem 2. *Let $U \subset \mathbf{R}^2$ be a region and the mapping $\Phi : U \to \mathbf{R}^2$ determined by equations $x = \varphi(u,v)$, $y = \psi(u,v)$ is injective and regular on U. Let $D \subset U$ be a compact region, f be continuous on $\Phi(D)$. Then*

$$\overset{\oplus}{\iint\limits_{\Phi(D)}} f(x,y)dxdy = \overset{\oplus}{\iint\limits_{D}} f(\varphi(u,v),\psi(u,v)) \otimes g^{-1}(|J(u,v)|)dudv\,,$$

where $J(u,v)$ is the corresponding Jacobi determinant.

Proof. By the definition of g-integrals and the integral transformation formula we obtain

$$\overset{\oplus}{\iint\limits_{\Phi(D)}} f(x,y)dxdy = g^{-1}\Big(\iint\limits_{\Phi(D)} g(f(x,y))dxdy\Big) =$$

$$= g^{-1}\Big(\iint\limits_{D} g(f(\varphi(u,v)\,,\,\psi(u,v)))\big|J(u,v)\big|dudv\Big) =$$

$$= g^{-1}\left(\iint\limits_{D} g\big(g^{-1}[g(f(\varphi(u,v),\,\psi(u,v)))\cdot g(g^{-1}(\big|J(u,v)\big|))]\big)dudv\right) =$$

$$= g^{-1}\Big(\iint\limits_{D} g(f(\varphi(u,v)\,,\,\psi(u,v))\otimes g^{-1}(\big|J(u,v)\big|))dudv\Big) =$$

$$= \overset{\oplus}{\iint\limits_{D}} f(\varphi(u,v)\,,\,\psi(u,v))\otimes g^{-1}(\big|J(u,v,)\big|)dudv\,.\square$$

Remark. In a similar way the transformation formula for 1-dimensional case can be proved:

$$\int\limits_{[c,d]}^{\oplus} f(x)dx = \int\limits_{[\alpha,\beta]}^{\oplus} f(\varphi(u)) \otimes g^{-1}(\varphi'(u))du\,.$$

Of course, here $\varphi : [\alpha, \beta] \to [c, d]$ is a differentiable function, $\varphi(\alpha) = c$, $\varphi(\beta) = d$. This formula is a special case of another one presented for curvilinear integrals in Proposition 3.

4. Green theorem

Let $K \subset \mathbf{R}^2$ be a regular oriented curve, and P, Q be real functions such that their domains contain K, $P, Q : K \to [a, b]$. Then, we define

$$\int\limits_{K}^{\oplus} P(x,y)dx = g^{-1}\Big(\int\limits_{K} g(P(x,y))dx\Big)dy$$

$$\int\limits_{K}^{\oplus} Q(x,y)dy = g^{-1}\Big(\int\limits_{K} g(Q(x,y))dy\Big)\,.$$

Proposition 2. *If $g(P(x,y))\vec{i} + g(Q(x,y))\vec{j}$ is integrable on a regular oriented curve K, then*

$$\Big(\int\limits_{K}^{\oplus} P(x,y)dx\Big) \oplus \Big(\int\limits_{K}^{\oplus} Q(c,y)dy\Big) = g^{-1}\Big(\int\limits_{K} g(P(x,y))dx + g(Q(x,y))dy\Big)\,.$$

Proof. We have

$$\Big(\int\limits_{K}^{\oplus} P(x,y)dx\Big) \oplus \Big(\int\limits_{K}^{\oplus} Q(x,y)dy\Big) =$$

$$= \; g^{-1}\Big(g\Big(\int\limits_{K}^{\oplus} P(x,y)dx\Big) + g\Big(\int\limits_{K}^{\oplus} Q(x,y)dy\Big)\Big) =$$

$$= g^{-1}\Big(g\big(g^{-1}(\int_K g(P(x,y))dx)\big) + g\big(g^{-1}(\int_K g(Q(x,y))dy)\big)\Big) =$$

$$= g^{-1}(\int_K g(P(x,y))dx + g(Q(x,y))dy)\ .\square$$

Proposition 3. *Let K be a regular oriented curve, P be a real function continuous on K, $P : K \to [a,b]$, $x = \varphi(t), y = \psi(t)$, $t \in [\alpha,\beta]$ be a parametrization of K. Then*

$$\int_K^{\oplus} P(x,y)dx = \int_{[\alpha,\beta]} P(\varphi(t),\, \psi(t)) \otimes g^{-1}(\varphi'(t))dt\,.$$

Proof. Proof is straighforward:

$$\int_K^{\oplus} P(x,y)dx = g^{-1}(\int_K g(P(x,y))dx) =$$

$$= g^{-1}(\int_\alpha^\beta g(P(\varphi(t),\, \psi(t)))\varphi'(t)dt) =$$

$$= g^{-1}\Big(\int_\alpha^\beta g\big(g^{-1}[g(P(\varphi(t),\, \psi(t)))\cdot g(g^{-1}(\varphi'(t)))]\big)dt\Big) =$$

$$= g^{-1}(\int_\alpha^\beta (P(\varphi(t),\, \psi(t)) \otimes g^{-1}(\varphi'(t)))dt) =$$

$$= \int_{[\alpha,\beta]}^{\oplus} P(\varphi(t),\, \varphi(t)) \otimes g^{-1}(\varphi'(t))dt\ .\square$$

Theorem 3. *Let D be a closed, convex region whose boundary is a simple positively oriented, closed curve ∂D. Let g be differentiable on (a,b), and let P, Q have continuous partial derivatives. Then*

$$(\int_{\partial D}^{\oplus} P(x,y)dx) \oplus (\int_{\partial D} Q(x,y)dy) =$$

$$= \iint_D^{\oplus} (\frac{\partial^{\oplus} Q(x,y)}{\partial x} \ominus \frac{\partial^{\oplus} P(x,y)}{\partial y})dxdy.$$

Proof. By the definition

$$\frac{\partial^{\oplus} Q(x,y)}{\partial x} = g^{-1}\Big(\frac{\partial g(Q(x,y))}{\partial x}\Big),$$
$$\frac{\partial^{\oplus} P(x,y)}{\partial y} = g^{-1}\Big(\frac{\partial g(P(x,y))}{\partial y}\Big),$$

hence

$$\frac{\partial^{\oplus} Q(x,y)}{\partial x} \ominus \frac{\partial^{\oplus} P(x,y)}{\partial y} = g^{-1}\Big(g\Big(\frac{\partial^{\oplus} Q(x,y)}{\partial x}\Big) - g\Big(\frac{\partial^{\oplus} P(x,y)}{P\partial y}\Big)\Big) =$$
$$= g^{-1}\Big(g\Big(g^{-1}\Big(\frac{\partial g(Q(x,y))}{\partial x}\Big)\Big) - g\Big(g^{-1}\Big(\frac{\partial g(P(x,y))}{\partial y}\Big)\Big)\Big) =$$
$$= g^{-1}\Big(\frac{\partial g(Q(x,y))}{\partial x} - \frac{\partial g(P(x,y))}{\partial y}\Big).$$

Therefore,

(1)
$$\iint_D^{\oplus} \Big(\frac{\partial^{\oplus} Q(x,y)}{\partial x} \ominus \frac{\partial^{\oplus} P(x,y)}{\partial y}\Big) dx dy =$$
$$= g^{-1}\Big(\iint_D g\Big[g^{-1}\Big(\frac{\partial g(Q(x,y))}{\partial x} - \frac{\partial g(P(x,y))}{\partial y}\Big)\Big] dx dy\Big)$$
$$= g^{-1}\Big(\iint_D \Big(\frac{\partial g(Q(x,y))}{\partial x} - \frac{\partial g(P(x,y))}{\partial y}\Big) dx dy\Big).$$

On the other hand, by Proposition 2 and the Green formula

$$\Big(\int_{\partial D}^{\oplus} P(x,y)dx\Big) \oplus \Big(\int_{\partial D}^{\oplus} Q(x,y)dy\Big) =$$
$$= g^{-1}\Big(\int_{\partial D} g(P(x,y))dx + \int_{\partial D} g(Q(x,y))dy\Big)$$
$$= g^{-1}\Big(\iint_D \Big(\frac{\partial g(Q(x,y))}{\partial x} - \frac{\partial g(P(x,y))}{\partial y}\Big) dx dy\Big).$$

By the equality and (2) the stated formula follows. □

5. Generalized double g-integral

The Pap g-integral $\int^{\oplus}$ and its corresponding g-derivative $d^{\oplus}$,

$$d^{\oplus} f(x) = g^{-1}\Big(\frac{dg(f(x))}{dx}\Big),$$

cannot be derived from the common integral (derivative) via replacing the common arithmetical operations in the definition by the corresponding pseudo-arithmetical operations.

Based on the latter approach Marková [1] has introduced another type of g-integral,

$$\int_{[a,b]}^{g} f dx = \Big(\int_a^b g \circ f(x) \cdot g'(x) dx\Big),$$

where $g'(x)$ is an arbitrary non-negative real function coinciding with the first derivative of g on its domain. The corresponding g-derivative is given by

$$d^g f = g^{-1}\left(\frac{\frac{d(g\circ f)}{dx}}{g'}\right),$$

(and it is defined whenever the right side is defined).

Finaly, Mesiar [2] has introduced a generalized g-integral,

$$\int_{[a,b]}^{g,h} f dx = g^{-1}\Big(\int_a^b g \circ f(x) \cdot h(x) dx\Big),$$

where h is non-negative integrable real function. The generalized g-integral includes g-integrals as special cases (with $h = -1$ and $h = g'$, respectively), and it has all pseudo-linearity propeties of Pap's integral. The corresponding g-h-derivative is given by

$$d^{g,h} f = g^{-1}\left(\frac{\frac{d(g\circ f)}{dx}}{h}\right).$$

Consequently, a generalized double g-integral can be introduced and above results can be checked. Note that using Marková's approach

$$\iint_D^{g} f d\sigma = g^{-1}\Big(\iint_D g \circ f \cdot g'(x) \cdot g'(y) d\sigma\Big),$$

the resulting integral is the limit of the corresponding integral pseudo-sums. Similar result is true for the curve integrals

$$\int_K^g P dx = g^{-1}\Big(\int_K g \circ P \cdot g'(x) dx\Big),$$

$$\int_K^g Q dy = g^{-1}\Big(\int_K g \circ Q \cdot g'(y) dy\Big).$$

Further Green formula

$$\int_{\partial D}^g P dx \oplus Q dy = -\iint_D^g \Big(\frac{\delta^g Q}{\delta x} \ominus \frac{\delta^g P}{\delta y}\Big) dx dy$$

holds true, too.

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