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A MULTIVALUED FIXED POINT THEOREMS IN NON-ARCHIMEDEAN VECTOR SPACES

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Abstract

We prove that every multivalued contractive mapping on a spherically complete non-Archimedean normed space has a fixed point. Also for nonexpansive multivalued mapping, some fixed point theorem are proved.

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1. Introduction

Let $\langle X, \| \ \| \rangle$ be a non-Archimedean normed space (for the definition see [4]). We say that $\langle X, \| \ \| \rangle$ is spherically complete if every shrinking collection of balls in X has a nonempty intersection. If $T: X \to \text{comp}(X)$ (the space of all compact subsets of X with the Hausdorff distance H), then T is said to be a multivalued contractive (nonexpansive) mapping if

$$H(Tx, Ty) < \|x - y\| \text{ for any distinct points in } X$$

$$(H(Tx, Ty) \leq \|x - y\| \text{ for any } x, y \in X).$$

It is known that a contractive mapping in a complete metric space need not to have a fixed point (see for example ([1], [2], [3])).

In [2], the authors proved that in a non-Archimedean spherically complete normed space $\langle X, \| \ \| \rangle$ every contractive mapping has a unique fixed point. We extend this result for a multivalued mapping.

2. Main results

Theorem 2.1. *Let X be a non-Archimedean spherically complete normed space. If $T\colon X \to \operatorname{comp}(X)$ is a mapping such that*

$$H(Tx, Ty) < \|x - y\| \quad \text{for any distinct points } x \text{ and } y \text{ in } X.$$

Then T has a fixed point.

Proof. Let $B_a = B[a, d(a, Ta)]$ denote the closed spheres centered at a with the radius $d(a, Ta)$, and let $\mathcal{A}$ be the collection of these spheres for all $a \in X$. The relation

$$B_a \leq B_b \ \text{ iff } \ B_b \subseteq B_a$$

is a partial order. Consider a totally ordered subfamily $\mathcal{A}_1$ of $\mathcal{A}$. From the spherical completeness of X, we have

$$\bigcap_{B_a \in \mathcal{A}_1} B_a = B \neq \emptyset.$$

Let $b \in B$ and $B_a \in \mathcal{A}_1$. Using

$$\|x - a\| \leq \max\{\|a - b\|, \|b - x\|\}$$

we shall prove that $B_b \subseteq B_a$, for every $B_a \in \mathcal{A}_1$. Since $\|a - b\| \leq d(a, Ta)$, because $b \in B_a$ and

$$\begin{aligned} \|x - b\| &\leq d(b, Tb) \leq \inf_{c \in Tb} \|b - c\| \leq \\ &\leq \max\{\|b - a\|, \|a - d\|, \inf_{c \in Tb} \|d - c\|\}, \end{aligned}$$

where $d \in Ta$ be such that $\|a - d\| = d(a, Ta)$ (such point exists as Ta is compact).
Therefore

$$\begin{aligned} \|x - b\| &\leq \max\{d(a, Ta), d(d, Tb)\} \leq \\ &\leq \max\{d(a, Ta), H(Ta, Tb)\} \leq d(a, Ta). \end{aligned}$$

Hence

$$\|x-a\| \leq d(a,Ta).$$

So $x \in B_a$ and $B_b \subseteq B_a$ for every $B_a \in \mathcal{A}_1$. Thus B_b is the upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. By Zorn's lemma $\mathcal{A}$ has a maximal element, say B_z, for some $z \in B$. We claim that $z \in Tz$.
Let $z \notin Tz, \overline{z} \in Tz$ $(z \neq \overline{z})$ be such that $\|z-\overline{z}\| = d(z,Tz)$ (such point exists as Tz is compact).

Now we will show that $B_{\overline{z}} \subseteq B_z$.

If $y \in B_{\overline{z}}$ then $\|\overline{z}-y\| \leq d(\overline{z},T\overline{z})$.
Since $\overline{z} \in Tz$, we have

$$\|\overline{z}-y\| \leq d(\overline{z},T\overline{z}) \leq H(Tz,T\overline{z}) < \|z-\overline{z}\| = d(z,Tz).$$

Also

$$\|y-z\| \leq \max\{\|y-\overline{z}\|, \|\overline{z}-z\|\} \leq d(z,Tz).$$

This means that $y \in B_z$.
So $B_{\overline{z}} \subseteq B_z$. But as

$$\|z-\overline{z}\| > H(Tz,T\overline{z}) \geq d(\overline{z},T\overline{z}).$$

Hence $z \notin B_{\overline{z}}$ but $z \in B_z$. Therefore $B_{\overline{z}} \subset B_z$, and this contradicts the maximality of B_z. Thus, T has a fixed point.

Theorem 2.2. *Suppose X is a spherically complete non-Archimedean normed space and $T\colon X \to \mathrm{comp}(X)$ is a mapping such that*

$$H(Tx,Ty) \leq \|x-y\|. \tag{1}$$

Then, either T has at least one fixed point, or there exists a sphere B of a radius $r > 0$ such that $d(b,Tb) = r$ for each $b \in B$.

Proof. Defining B_a and $\mathcal{A}$ as in the proof of Theorem 2.1, we find similarly a maximal element B_z of $\mathcal{A}$.
For any $b \in B_z$, we have

$$\begin{aligned} d(b,Tb) &= \inf_{\overline{b}\in Tb} \|b-\overline{b}\| \leq \\ &\leq \max\{\|b-z\|, \|z-\overline{z}\|, \inf_{\overline{b}\in Tb} \|\overline{z}-\overline{b}\|\} \end{aligned}$$

As $b \in B_z$ then

$$\|b - z\| \leq d(z, Tz) \tag{2}$$

and for $\overline{z} \in Tz$ such that $\|z - \overline{z}\| = d(z, Tz)$ (such point exists as Tz is compact), we have:

$$\inf_{\overline{b} \in Tb} \|\overline{z} - \overline{b}\| = d(\overline{z}, Tb) \leq H(Tz, Tb) \leq \|z - b\| \leq d(z, Tz).$$

Therefore

$$d(b, Tb) \leq d(z, Tz). \tag{3}$$

Now we will show that $B_b \subseteq B_z$.
Firstly, if $y \in B_b$, then

$$\|y - b\| \leq d(b, Tb) \leq d(z, Tz). \tag{4}$$

Also

$$\|y - z\| \leq \max\{\|y - b\|, \|b - b_z\|, \|b_z - z\|\}$$

where $b_z \in Tb$ is such that $\|b - b_z\| = d(b, Tb)$.
Therefore

$$\begin{aligned} \|y - z\| &\leq \max\{d(b, Tb), d(b, Tb), \|b_z - z\|\} \leq \\ &\leq \max\{d(b, Tb), \|b_z - z_z\|, \|z_z - z\|\} \end{aligned}$$

where $z_z \in Tz$ be such that $\|z_z - z\| = d(z, Tz)$.
Hence

$$\begin{aligned} \|y - z\| &\leq \max\{d(z, Tz), \|b_z - b\|, \|b - z_z\|\} \leq \\ &\leq \max\{d(z, Tz), \|b - z_z\|\} \leq \\ &\leq \max\{d(z, Tz), \|b - z\|, \|z - z_z\|\} \leq d(z, Tz). \end{aligned}$$

Hence $y \in B_z$, and therefore $B_b \subseteq B_z$.
From (3) $d(b, Tb) \leq d(z, Tz)$ for any $b \in B_z$. If $z \in Tz$ then z is a fixed point of T.
Now, if $z \notin Tz$, then $d(b, Tb) = d(z, Tz)$ for every $b \in B_z$, because if for some $b \in B_z$, we have

$$d(b, Tb) < d(z, Tz), \tag{5}$$

then as $\|b - z\| \leq d(z, Tz)$, we have that

$$\begin{aligned} d(z,Tz) &= \inf_{c \in Tz} \|z - c\| \leq \max\{\|z - b\|, \inf_{c \in Tz} \|b - c\|\} = \\ &= \max\{\|z - b\|, d(b, Tz)\} \leq \\ &\leq \max\{\|z - b\|, d(b, Tb), H(Tb, Tz)\} \leq \\ &\leq \max\{\|z - b\|, d(b, Tb), \|b - z\|\} \leq \|b - z\|. \text{by } (3) \end{aligned}$$

Hence we obtain $d(z, Tz) = \|b - z\|$. As $\|b - z\| = d(z, Tz) > d(b, Tb)$, this implies that $z \notin B_b$. Since $z \in B_z$ we have that $B_b \subset B_z$, which is impossible from the maximality of B_z. Thus

$$d(b, Tb) = d(z, Tz) = r \quad \text{for any} \quad b \in B_z.$$

So, the proof is complete.

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