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ON THE COLLECTION OF LATTICES DETERMINED BY THE SAME POSET OF MEET-IRREDUCIBLES

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Abstract

The collection of all finite lattices with the same poset of meet-irreducible elements is a lattice, as proved in [5]. In the present paper, it is proved that every finite Boolean lattice is isomorphic to such collection-lattice of a particular poset. It is also proved that no chain could be represented by a collection-lattice, unless it has at most two elements. However, some direct products of a chain and a lattice (or lattices) can be represented by a collection-lattice. Such a representation is proved for the direct product of a three-element chain and a finite Boolean lattice.

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1. Preliminaries

We shall use the following notions and results, which can be found in [1] and [2].

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1. An element a of the lattice L distinct from the top ($\mathbf{1}$) is **meet-irreducible** if $b \wedge c = a$ implies $b = a$ or $c = a$.

The dual notion, considering an element different from $\mathbf{0}$, is a **join-irreducible** element.

If a is both, meet and join-irreducible, than it is said to be **irreducible.**

Finally, an element of a lattice L is **reducible** if it is neither meet, nor join irreducible.

2. Every element of a finite lattice can be represented by an infimum of meet-irreducible elements. If the lattice is distributive then the representation by meet-irreducible elements is unique if it is minimal (which means that no proper subset of the collection of meet irreducible elements has this element as the infimum).

3. Every element of the finite lattice is representable by the infimum of all meet-irreducible elements above it.

4. ([1]) Every finite distributive lattice is isomorphic with the lattice of all isotone functions from the collection of its meet irreducible elements into $\mathbf{2} = (\{\mathbf{0}, \mathbf{1}\}, \leq)$.

In the paper [4] a representation of all lattices with the same poset of meet-irreducible elements by isotone functions in terms of fuzzy sets was given.

Let $(X, \leq)$ be a finite partially ordered set, and $L_D(X)$ (or, briefly, L_D) the distributive lattice for which X is a poset of meet-irreducible elements. The poset $(X, \leq)$ generates the lattice L_D, in the sense of Birkhoff's theorem (4), and also some other, non-distributive lattices (by collections of isotone functions, or, equivalently, by the corresponding sets of ideals in X). In the paper [5], the collection of all such lattices was investigated, as follows.

Let $(X, \leq)$ and L_D be given as above, and let $\mathcal{L}(X)$ be the collection of all subsets of L_D which are lattices under the ordering from L_D, and which contain X:

$$\mathcal{L}(X) := \{L \subseteq L_D \mid X \subseteq L, \text{ and } L \text{ is a lattice under } \leq \text{ from } L_D\}.$$

In the cited paper, the following was proved:

I *$\mathcal{L}(X)$ is a lattice under the set inclusion, consisting of lattices determined by X as the poset of meet-irreducibles.*

Let

$$X' = X \cup \{z_x \mid x \in X\} \cup \{y \in L_D \mid y = \bigvee x_i, \text{ for some } x_i \in X\},$$

where z_x stands for the meet in L_D of all meet-irreducible elements above x (which is also meet-irreducible).

II *If $L_m = X' \cup \{\mathbf{0}, \mathbf{1}\}$, then L_m is a lattice contained in every lattice from $\mathcal{L}(X)$ (it is the bottom element of the collection-lattice $\mathcal{L}(X)$). In addition, every lattice from $\mathcal{L}(X)$ is closed under joins from L_D.*

If X is a poset of meet-irreducible elements in a finite distributive lattice L_D and $a \in L_D$, then, as it was defined in [5], a is said to be X**-independent** if it is not in L_m.

III *$\mathcal{L}(X)$ is a one-element lattice if and only if L_D has no X-independent elements.*

IV *$\mathcal{L}(X)$ is a Boolean lattice if and only if L_D contains no reducible elements which are X-independent.*

V *$\mathcal{L}(X)$ is a distributive lattice if and only if L_D does not contain an X-independent reducible element, which is a join of other X-independent elements.*

VI *If the lattice $\mathcal{L}(X)$ is modular, then it is also distributive.*

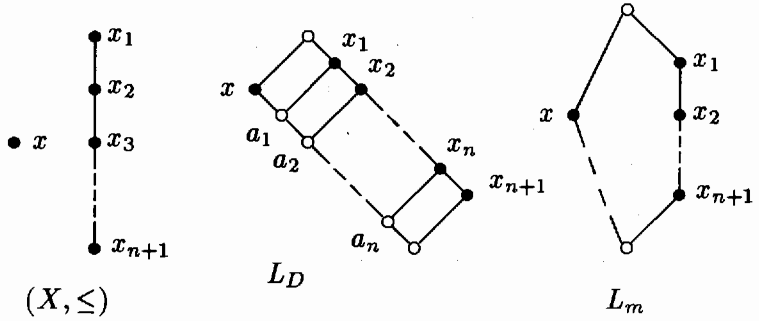
2. Results

Throughout this section, $(X, \leq)$ is a finite partially ordered set, and $L_D(X)$ (briefly L_D) is the distributive lattice in which $(X, \leq)$ is the poset of meet-irreducible elements (to be precise, elements of X are identified with meet-irreducibles from L_D). $\mathcal{L}(X)$ is a collection of all lattices - subsets of L_D, in which X is also the poset of meet-irreducible elements. By **I**, $\mathcal{L}(X)$ is a lattice under the set inclusion. If the minimum lattice in $\mathcal{L}(X)$ is denoted by L_m, then every lattice in this collection can be represented by $L_m \cup Y$, where Y is a suitable subset of $L_D \setminus L_m$ (for more details, see [5]). Hence, the lattice $\mathcal{L}(X)$ is isomorphic with the poset of these subsets, ordered by the set inclusion.

We shall say that a finite lattice L has a **representation** by $\mathcal{L}(X)$, if there is a finite poset $(X, \leq)$, such that L is isomorphic to the collection-lattice $\mathcal{L}(X)$.

Theorem 1. *Every finite Boolean lattice has a representation by $\mathcal{L}(X)$.*

Proof. Let $\mathcal{B}$ be a finite Boolean algebra isomorphic to the power set $\mathcal{P}(n)$, $n \in N$, and let $(X, \leq)$ be a disjoint union of two finite chains, with 1 and $n+1$ elements, respectively, (its Hasse-diagram is given in Figure 1 a).



a) b) c)

Fig. 1

Isotone functions from X to the two-element poset ordered componentwise, form the distributive lattice L_D (Fig. 1 b)), generated by X. The minimum lattice L_m, generated by the same poset, is obtained by adding the top and the bottom element to X (Fig. 1 c)). Hence, the set $Y = L_D \setminus L_m$ contains n elements: $Y = \{a_1, ..., a_n\}$. None of these elements is reducible, and by the proposition **IV** in Preliminaries, $\mathcal{L}(X)$ is isomorphic to the power set of Y, i.e., to the Boolean lattice $\mathcal{B}$. □

The collection-lattice $\mathcal{L}(X)$ is distributive under the conditions given by **V**. On the other hand, not every finite distributive lattice can be represented by a collection-lattice of some (finite) poset. In the following, we shall prove that such a representation exists for a finite chain if and only if this chain has at most two elements. In spite of that negative answer to the question of a representation of a chain, it is possible to represent a lattice which is a product of a chain and some other lattice(s). This shall also be proved in the sequel.

Let X, L_D and L_m be given as above, at the beginning of this section, and let $X' \subseteq L_m$ be the set defined in Preliminaries (see **II**). Now, if $a, b \in L_D$, then we shall say that a **depends** on b if these two elements do not belong to L_m and $b = x \vee a$, for some $x \in L_m$. Consequently, two elements a and b from L_D are said to be **independent** if a does not depend on b and b

does not depend on a. Again, a and b are not supposed to be in L_m (in other words, dependence and independence apply only to X-independent elements, as defined in Preliminaries).

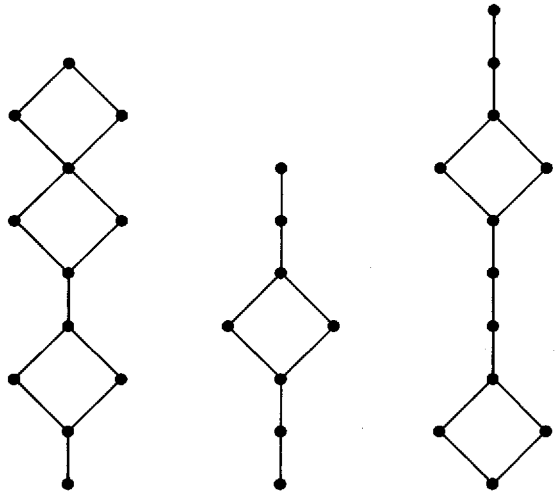
Lemma 1. *a) The relation of "being dependent" is transitive on L_D.*

b) An element a from $L_D \setminus L_m$ depends on b (also belonging to $L_D \setminus L_m$), if and only if b belongs to every lattice in $\mathcal{L}(X)$ in which is a.

Proof. a) Indeed, if a depends on b i.e., $a \vee x = b$, and b depends on c i.e., $b \vee y = c$, then obviously a depends on c, since $a \vee (x \vee y) = c$. (Since $x, y \in L_m, x \vee y \in L_m$.)

b) If a depends on b, and a belongs to the lattice L from $\mathcal{L}(X)$, then b also has to be in L, since by **II**, L is closed under joins from L_D.

On the other hand, if $a, b \in L_D \setminus L_m$ and a does not depend on b, then there is a lattice $L \in \mathcal{L}(X)$, such that $a \in L$ and $b \notin L$. Namely, $L = L_m \cup \langle a \rangle$, where $\langle a \rangle$ is a set of all elements from L_D which closes $\{a\}$ under "being dependent". By a), L is closed under joins from L_D. Further on, $b \notin L$, since otherwise, again by a), a would depend on b. □



Examples of singlets

Fig. 2

Recall that the **width** of a lattice is the cardinality of its maximal antichain. Now, consider distributive lattices of the width 2 and in which antichains consist of irreducible elements only. We shall call such lattices **singlets**. In other words, a singlet is a linear sum of finitely many finite chains and two-element antichains, including the top and the bottom element (Figure 2).

Lemma 2. *If L_D is a singlet, then the corresponding collection-lattice has exactly one element.*

Proof. Obvious, since all elements of a singlet are meet-irreducible, or joins of meet-irreducible elements. □

Lemma 3. *In a finite distributive lattice L_D which is not a singlet, there is at least one element which is independent of any other element in L_D.*

Proof. Any finite distributive lattice L_D can be represented as a union of an ideal $p\downarrow$ and a filter $p\uparrow$ ($p \in L_D$), so that $p\downarrow$ is a maximal such singlet, which can be a one-element lattice. Obviously, $p\downarrow$ is contained in L_m, and elements that we are looking for belong to $p\uparrow$. Hence, we consider this filter as a lattice with the bottom element p. Let a be an atom in $p\uparrow$, which is not meet-irreducible. By the above assumption, such an atom exists. By the same argument, it is not the only atom in this lattice, and hence it is not in L_m.

We claim that a is independent of any other element in $p\uparrow$, and thus in L_D. Suppose that $a \vee x = b$, for some $b \in L_D \setminus L_m, (b \neq a)$, and $x \in L_m$. Then also $a \wedge x = p$, since a covers p. Now, x could be: (a) meet-irreducible; (b) join of meet-irreducibles; (c) meet of all meet-irreducibles above some y which is meet-irreducible.

If (a) holds, then there is x', $x < x' \leq b$ and x' is the meet of all meet-irreducibles above x. Then, since b is not in L_m and is different from x', there is a pentagon sublattice in L_D ($\{p, a, b, x', x\}$), which is a contradiction.

(b) If y is any of meet-irreducibles, the join of which is x, a pentagon $\{p, a, b, y, x\}$ yields to a contradiction.

(c) In this case, x covers a meet-irreducible element y, and a pentagon arises again.

Thus, a does not depend on any other element in L_D. In addition, since a covers p, no element in L_D depends on a. □

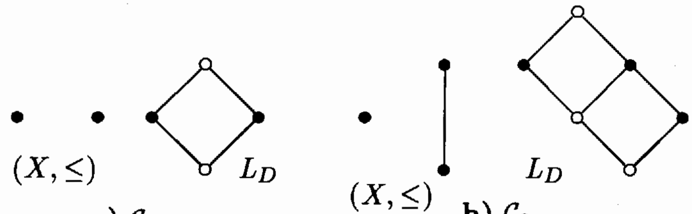
Corollary 1. *L_D is a singlet if and only if the corresponding collection-lattice has exactly one element.*

Proof. The "only if" part follows by Lemma 2. On the other hand, if L_D is not a singlet, then by Lemma 3 there is at least one X-independent element in L_D, and by **III** the corresponding collection-lattice has more than one element. □

Theorem 2. *A chain has a representation by $\mathcal{L}(X)$ if and only if it has at most two elements.*

Proof. By the previous lemma, every finite distributive lattice L_D which is not a singlet, has an element independent of any other element in $L_D \setminus L_m$. Hence, if a chain C has a representation by a collection-lattice, then this lattice is a singlet, in which case C has one element, or there is only one element a in $L_D \setminus L_m$ (the one whose existence is proved by the previous lemma). In this case, C has two elements, since $\mathcal{L}(X)$ contains two lattices: L_m and $L_m \cup \{a\}$.

If there are two or more elements in $L_D \setminus L_m$, where a is the one which is independent of e.g. b, by Lemma 1 b) there are two uncomparable lattices in $\mathcal{L}(X)$, each containing only one of these two elements. Hence, a chain with more than two elements could not be represented by $\mathcal{L}(X)$.



a) C_1 **b)** C_2

Fig. 3

On the other hand, if a chain has one or two elements, then obviously it has a representation by a collection-lattice: for a one-element-chain C_1, L_D can be any singlet, and if a chain C_2 has two elements, then L_D can be a lattice of divisors of 12 (these lattices and the corresponding posets of meet-irreducibles are represented in Figure 3 a) and b), respectively). □

As proven above, even a three-element chain can not be represented by a collection-lattice. However, it is possible to represent a direct product of that chain and any Boolean lattice, as it will be proven in the sequel.

Theorem 3. *If $\mathcal{B}$ is a finite Boolean lattice and $\mathcal{C}_3$ is a three-element chain, then the lattice $\mathcal{B} \times \mathcal{C}_3$ has a representation by $\mathcal{L}(X)$.*

Proof. Let $\mathcal{B}$ be a Boolean lattice with 2^n elements. Let also $(X, \leq)$ be a poset represented in Figure 4 a). The corresponding distributive lattice L_D is given in Figure 4 b). Now, the element b, such that $b = x_1 \wedge y_2$, depends on $a = x_1 \wedge y_1$, since for $x = x_2 \wedge y_1$, $a = x \vee b$. It is easy to check that all other elements in $L_D \setminus L_m$, namely $c_1, ..., c_n$ are independent of others. Hence, $\mathcal{L}(X)$ contains lattices of the form $L_m \cup Y$, where Y runs over a set isomorphic to $\mathcal{P}(\{c_1, ..., c_n\}) \times \{\emptyset, \{a\}, \{a, b\}\}$; precisely, every Y is a union of the coordinates in an ordered pair from that product. Indeed, Y can be a subset of $L_D \setminus L_m$, with the only restriction that b depends on a, and thus a has to be in Y, if b is.

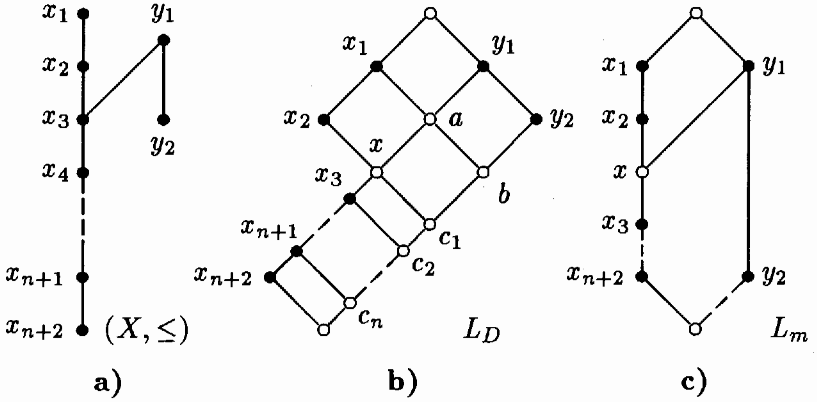


Fig. 4

Hence, $\mathcal{L}(X)$ is isomorphic to a direct product of the power set $\mathcal{P}(n)$ and a three-element chain. □

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