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# AN UNIVALENCE CRITERION AND THE SCHWARZIAN DERIVATIVE

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**Abstract**

In this note we obtain a univalence criterion for a class of functions defined by an integral operator and in a particular case we find the well-known condition for univalency established by Nehari [1].

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## 1. Introduction

We denote by $U_r$ the disk of $z$-plane, $U_r = \{z \in \mathbf{C} : |z| < r\}$ where $r \in (0, 1]$, $U_1 = U$, $U^* = U \setminus \{0\}$ and $I = [0, \infty)$.

Let $A$ be the class of functions $f$ which are analytic in $U$ with $f(0) = 0$ and $f'(0) = 1$.

**Theorem A.** *([1]). Let $f \in A$. If for all $z \in U$*

$$|\{f; z\}| \leq \frac{2}{(1 - |z|^2)^2}, \tag{1}$$

*where*

$$\{f; z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{2}$$

*then the function $f$ is univalent in $U$.*

## 2. Preliminaries

**Definition 1.** *A function $L : U \times I \to \mathbf{C}$ is called a Loewner chain if*

$$L(z,t) = e^t z + a_2(t)z^2 + \dots, \quad |z| < 1$$

*is analytic and univalent in $U$ for each $t \in I$ and if $L(z,s) \prec L(z,t)$, $0 \le s \le t < \infty$, where by $\prec$ we denote the relation of subordination.*

**Theorem B.** *([2]). Let $r$ be a real number, $r \in (0,1]$. Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots, a_1(t) \neq 0$, be analytic in $U_r$ for all $t \in I$, locally absolutely continuous in $I$ and locally uniform with respect to $U_r$.*

For almost all $t \in I$ suppose

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in $U$ and satisfies $Re\, p(z,t) > 0$, $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in $U_r$, then, for each $t \in I$, $L(z,t)$ has an analytic and univalent extension to the whole disk $U$.

## 3. Main results

**Theorem 1.** *Let $f \in A$ and let $\alpha$ be a complex number, $\mathrm{Re}\,\alpha > 0$. If*

$$\left|\frac{(1-|z|^{2\alpha})^2}{2\alpha^2|z|^{2\alpha}}(z^2\{f;z\} + (1-\alpha)\frac{zf''(z)}{f'(z)})\right| \le 1, \tag{3}$$

*for all $z \in U^*$, then the function $F_\alpha$,*

$$F_\alpha(z) = (\alpha\int_0^z u^{\alpha-1}f'(u)du)^{1/\alpha} \tag{4}$$

*is analytic and univalent in $U$.*

*Proof.* Let us prove that there exists a real number $r$, $r \in (0,1]$ such that the function $L : U_r \times I \to \mathbf{C}$, defined formally by

$$(5) \qquad L(z,t) = \left[\alpha \int_0^{e^{-t}z} u^{\alpha-1} f'(u)du + \frac{(e^{\alpha t} - e^{-\alpha t})z^\alpha f'(e^{-t}z)}{1 - \frac{e^{2\alpha t}-1}{2\alpha} \frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}}\right]^{1/\alpha}$$

is analytic in $U_r$ for all $t \in I$.

Because $f \in A$ we have

$$f(z) = z + a_2 z^2 + \ldots + a_n z^z + \ldots, \quad \forall z \in U.$$

Let us denote by

$$(6) \qquad g_1(z,t) = \alpha \int_0^{e^{-t}z} u^{\alpha-1} f'(u)du.$$

We obtain $g_1(z,t) = (e^{-t}z)^\alpha + \dfrac{2\alpha a_2}{\alpha+1}(e^{-t}z)^{\alpha+1} + \ldots$ and we observe that

$$(7) \qquad g_1(z,t) = z^\alpha g_2(z,t), \quad \text{where}$$

$$(8) \qquad g_2(z,t) = e^{-\alpha t} + \sum_{n=2}^{\infty} \frac{n\alpha}{n+\alpha-1} a_n e^{-(n+\alpha-1)t} z^{n-1}.$$

The function $g_2$ is analytic in $U$ for all $t \in I$, since

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|\frac{n\alpha}{n+\alpha-1} a_n e^{-(n+\alpha-1)t}\right|} = e^{-t} \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}.$$

It is clear that if $z \in U$, then $e^{-t}z \in U$ for all $t \in I$ and because $f'(0) = 1$, there is a disk $U_{r_1}$, $0 < r_1 \leq 1$ in which $f'(e^{-t}z) \neq 0$ for all $t \geq 0$. From the analyticity of $f$ it follows that the function $g_3$ is also analytic in $U_{r_1}$, where

$$(9) \qquad g_3(z,y) = 1 - \frac{e^{2\alpha t}-1}{2\alpha} \frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}.$$

We have $g_3(0,t) = 1$ and then there is a disk $U_{r_2}$, $0 < r_2 \leq r_1$ in which $g_3(z,t) \neq 0$ for all $t \geq 0$.

Then the function

$$g_4(z,t) = g_2(z,t) + (e^{\alpha t} - e^{-\alpha t}) \frac{f'(e^{-t}z)}{g_3(z,t)} \tag{10}$$

is also analytic in $U_{r_2}$ and $g_4(0,t) = e^{\alpha t}$.

Since $g_4(0,t) \neq 0$ for all $t \in I$, there is a disk $U_r$, $0 < r \leq r_2$ in which $g_4(z,t) \neq 0$ for all $t \in I$ and we can choose an analytic branch of $[g_4(z,t)]^{1/\alpha}$, denoted by $g(z,t)$. We choose the branch which is equal to $e^t$ at the origin.

From (5) - (10) it results that the relation (5) may be written as

$$L(z,t) = z \cdot g(z,t) \tag{11}$$

and hence we obtain that the function $L(z,t)$ is analytic in $U_r$,

$$L(z,t) = e^t z + a_2(t)z^2 + \dots, \quad \forall z \in U_r, \quad \forall t \in I.$$

$L(z,t)$ is an analytic function in $U_r$ for all $t \in I$ and then it follows that there is a number $r_3$, $0 < r_3 < r$ and a positive constant $K = K(r_3)$ such that

$$\left| L(z,t)/e^t \right| < K, \quad \forall z \in U_{r_3}, \ t \geq 0.$$

Then, by Montel's theorem, it results that $\{L(z,t)/e^t\}$ is a normal family in $U_{r_3}$.

From (11) we have

$$\frac{\partial L(z,t)}{\partial t} = z \cdot \frac{\partial g(z,t)}{\partial t}. \tag{12}$$

It is clear that $\dfrac{\partial g(z,t)}{\partial t}$ is an analytic function $U_{r_3}$ and then $\dfrac{\partial L(z,t)}{\partial t}$ is too. Then, for all fixed numbers $T > 0$ and $r_4$, $0 < r_4 < r_3$, there exists a constant $K_1 > 0$ (which depends on $T$ and $r_4$) such that

$$\left| \frac{\partial L(z,t)}{\partial t} \right| < K_1, \quad \forall z \in U_{r_4} \quad \text{and} \quad t \in [0,T].$$

Therefore, the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to $U_{r_4}$.

Since $\dfrac{\partial L(z,t)}{\partial t}$ is analytic in $U_{r_4}$, from (12) it results that there is a number $r_0$, $0 < r_0 < r_4$, such that $\dfrac{1}{z}\cdot\dfrac{\partial L(z,t)}{\partial t} \neq 0$, $\forall z \in U_{r_0}$, and then the function

$$p(z,t) = z\cdot\frac{\partial L(z,t)}{\partial z}\Big/\frac{\partial L(z,t)}{\partial t}$$

is analytic in $U_{r_0}$ for all $t \geq 0$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in $U$, to for all $t \geq 0$, it is sufficient to prove that the function $w(z,t)$ defined in $U_{r_0}$ by

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}$$

can be continued analytically in $U$, $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$.

After computation we obtain:

$$w(z,t) = \frac{(e^{\alpha t} - e^{-\alpha t})^2}{2\alpha^2}e^{-2t}z^2\{f; e^{-t}z\} + \tag{13}$$
$$+\frac{1-\alpha}{2\alpha^2}(e^{\alpha t} - e^{-\alpha t})^2\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}.$$

From (3) we deduce that $f'(z) \neq 0$ for all $z \in U$ and then the function $w(z,t)$ is analytic in the unit disk $U$.

We have

$$w(0,t) = 0 \quad \text{and} \quad w(z,0) = 0 \tag{14}$$

Let now a fixed number $t$, $t > 0$, $z \in U$, $z \neq 0$. In this case the function $w(z,t)$ is analytic in $\overline{U}$ because $|e^{-t}z| \leq e^{-t} < 1$, for all $z \in \bar{U}$. Using the maximum principle, for $z \in U$ and $t > 0$ we have

$$|w(z,t)| < \max_{|\xi|=1}|w(\xi,t)| = |w(e^{i\theta},t)|, \tag{15}$$

where $\theta = \theta(t)$ is a real number.

Let us denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t}$ and from (13) we obtain

$$|w(e^{i\theta}, t)| = \left| \frac{(1-|u|^{2\alpha})^2}{2\alpha^2 |u|^{2\alpha}} (u^2\{f; u\} + (1-\alpha)\frac{uf''(u)}{f'(u)}) \right|.$$

Because $u \in U$, the relation (3) implies $|w(e^{i\theta}, t)| \leq 1$ and from (14) and (15) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$.

From Theorem B it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disk $U$, for each $t \in I$. For $t = 0$ we conclude that the function

$$L(z,0) = \left( \alpha \int_0^z u^{\alpha-1} f'(u) du \right)^{1/\alpha} \equiv F_\alpha(z)$$

is analytic and univalent in $U$.

**Remark.** For $\alpha = 1$, the inequality (3) becomes (1), $F_1 = f$ and then Theorem 1 becomes Theorem A.

**Theorem 2.** *Let $f \in A$ and let $\alpha$ be a complex number,* $\operatorname{Re}\alpha > 0$. *If*

$$\frac{(1-|z|^{2\operatorname{Re}\alpha})^2}{2(\operatorname{Re}\alpha)^2} \left| z^2\{f; z\} + (1+\alpha)\frac{zf''(z)}{f'(z)} \right| \leq |z|^{2\operatorname{Re}\alpha} \tag{16}$$

*for all $z \in U$, then the function $F_\alpha$ defined by (4) is analytic and univalent in $U$.*

*Proof.* For all $z \in U$, $z \neq 0$ and $\operatorname{Re}\alpha > 0$ we have

$$\left| \frac{1-|z|^{2\alpha}}{\alpha} \right| \leq \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \tag{17}$$

Indeed,

$$\left| \frac{1-|z|^{2\alpha}}{\alpha} \right| = \left| \frac{1-e^{2\alpha \ln|z|}}{\alpha} \right| = \left| 2\ln|z| \cdot \int_0^1 e^{2\alpha t \ln|z|} dt \right| \leq$$

$$\leq -2\ln|z| \int_0^1 |e^{2\alpha t \ln|z|}| dt = -2\ln|z| \int_0^1 e^{2\operatorname{Re}\alpha t \ln|z|} dt = \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}$$

In view of (17), the hypothesis of Theorem 1 are satisfied and hence the function $F_\alpha$ defined by (4) is analytic and univalent in $U$.

For $\alpha = 1 + ib$, by Theorem 2 we obtain the following.

**Corollary 1.** *Let $f \in A$ and let $b$ by any real numbers. If*

$$\frac{(1-|z|^2)^2}{2}|z^2\{f;z\} - ib\frac{zf''(z)}{f'(z)}| \le |z|^2,$$

*for all $z \in U$, the function $F_{1+ib}$,*

$$F_{1+ib}(z) = \left[(1+ib)\int_0^z u^{ib} f'(u)du\right]^{1/(1+ib)}$$

*is analytic and univalent in $U$.*

**Example.** Let $\alpha$ be a natural number. The function

$$F(z) = \frac{z}{(1-z^\alpha)^{1/\alpha}} \tag{18}$$

is analytic and univalent in $U$.

*Proof.* Let us consider the function $f \in A$,

$$f'(z) = \frac{1}{(1-z^\alpha)^2}, \quad \forall z \in U. \tag{19}$$

We obtain

$$\{f;z\} = \frac{2\alpha(\alpha-1)z^{\alpha-2}}{1-z^\alpha} \quad \text{and} \quad z^2\{f;z\} + (1-\alpha)\frac{zf''(z)}{f'(z)} = 0.$$

Then, from Theorem 1 it results that the function

$$(\alpha\int_0^z u^{\alpha-1}f'(u)du)^{1/\alpha} = (\alpha\int_0^z \frac{u^{\alpha-1}}{(1-u^\alpha)^2}du)^{1/\alpha} = \frac{z}{(1-z^\alpha)^{1/\alpha}}$$

is analytic and univalent in $U$.

# References

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