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ON APPROXIMATION OF COMPACT MULTIVALUED MAPS IN TOPOLOGICAL VECTOR SPACES

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Abstract

In this paper we shall give a result on approximation of compact u.s.c. multivalued maps in topological vector spaces, using the theory of topological semifields and result of Kasahara [3].

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1. Introduction

First, we shall give some notations and definitions which will be used in the following text. Let X and Y be two Hausdorff topological spaces and 2^Y the system of nonempty subsets of Y. A multivalued (set-valued) mapping $T : X \to 2^Y$ is said to be upper semicontinuous (u.s.c) at a given point $x_0 \in X$ if and only if for each open subset W of Y such that $T(x_0) \subseteq W$, there exists a neighbourhood V of x_0 such that $T(V) \subseteq W$. A mapping $T : X \to 2^Y$ is u.s.c. on X if it is u.s.c. at each point of X. If $\overline{T(X)}$ is compact, T is said to be compact.

By R we shall denote the set of all real numbers. Further, let X be a vector space over R; R_Δ the set of all mappings from Δ into R with the Tychonoff product topology and the operations $+$ and scalar multiplication as usual. If $f, g \in R_\Delta$ we shall say that $f \leq g$ iff $f(t) \leq g(t)$, for every $t \in \Delta$ and $f \neq g$. By P_Δ we shall denote the cone of nonnegative elements in R_Δ.

Definition 1.1. *The triplet $(X, \|\cdot\|, \Phi)$ is a $\Phi-$paranormed space iff $\|\cdot\| : X \to P_\Delta$, Φ is a linear, continuous, positive mapping from R_Δ into R_Δ such that the following conditions are satisfied:*

1. $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\|\lambda x\| = |\lambda| \, \|x\|$, *for every $x \in X$ and every $\lambda \in R$;*
3. $\|x + y\| \leq \Phi(\|x\|) + \Phi(\|y\|)$, *for every $x, y \in X$.*

Let us denote by $\mathcal{U}$ the family of neighbourhoods of zero in R_Δ and for every $U \in \mathcal{U}$ we shall denote the set:

$$\{x : \; x \in X, \; \|x\| \in U\}$$

by V_U. Then X is a toplogical vector space in which $\{V_U\}_{U \in \mathcal{U}}$ is the family of neighbourhoods of zero in X.

In fact, then sets

$$U_{\mu,\varepsilon} = \{x \in X : \|x\|(t) < \varepsilon, \;\; \text{for every} \;\; t \in \mu\},$$

where μ is a finite subset of Δ and $\varepsilon > 0$, form a fundamental system of zero neighbourhoods in X.

In [3] it is proved that every Hausdorff topological vector space X is a paranormed space $(X, \|\cdot\|, \Phi)$ over a topological semifield R_Δ and we shall say that the triplet $(X, \|\cdot\|, \Phi)$ is *the associated paranormed space.*

Definition 1.2. [2] *Let X be a Hausdorff topological vector space and $(X, \|\cdot\|, \Phi)$ be the associated paranormed space. The set $K \subset X$ is of $\Phi-$type iff for every $n \in N$:*

$$\|\sum_{i=1}^{n} \lambda_i x_i\| \leq \sum_{i=1}^{n} \lambda_i \Phi(\|x_i\|), \;\; \textit{for every} \;\; x_i \in K - K (i = 1, 2, \ldots, n)$$

and every $\lambda_i \in [0,1]$, $\sum\limits_{i=1}^{n} \lambda_i = 1$.

In [2] O. Hadžić proved the following generalization of Lere-Schauder-Nagumo lemma for $\Phi-$type subsets.

Lemma 1.1. *Let C be a compact $\Phi-$type subset of X. Then for every $U \in \{U_{\mu,\varepsilon}\}_{\mu\subseteq\Delta,\varepsilon>0}$ there exists a continuous mapping p of C into X and a finite set $B \subset C$ such that*

$$p(C) \subset \overline{conv}\, B \qquad \text{and}$$

$$p(x) - x \in U \qquad \text{for every} \quad x \in C.$$

Remark. The map p in this lemma is in fact, a so called Schauder projection.

2. Results

First, we shall prove the following lemma.

Lemma 2.1. *Let $(Y, \|\,\|, \Phi)$ be a paranormed space, K convex $\Phi-$type subset of Y. Then for every compact convex subset C of K and every $U \in \{U_{\mu,\varepsilon}\}$ there exists $V \in \{U_{\mu,\varepsilon}\}$ such that*

$$conv\,((V + C) \cap K) \subset C + U$$

Proof. Let $U = U_{\mu,\varepsilon}$, $\mu \subset \Delta$ and $\varepsilon > 0$. Since the mapping Φ is linear and continuous it follows that $N_1 = (\Phi^2)^{-1}(U_{\mu,\varepsilon})$ is a neighbourhood of zero in $\mathbf{R}_\Delta$ and let $\mu' \subset \Delta$ and $\varepsilon' > 0$ be such that

$$U_{\mu',\varepsilon'} \subset \{x : \|x\| \in N_1\}.$$

Let $\mu'' \subset \Delta$ and $\varepsilon'' > 0$ be such that

$$U_{\mu'',\varepsilon''} + U_{\mu'',\varepsilon''} \subseteq U_{\mu',\varepsilon'}.$$

We shall prove that $conv((C + U_{\mu'',\varepsilon''}) \cap K) \subseteq U_{\mu,\varepsilon}$.
Since the set C is compact there exists a finit set $F = \{x_1, x_2, \ldots, x_n\} \subset C$ so that

$$C \subset \bigcup_{i=1}^{n}(x_i + U_{\mu'',\varepsilon''}).$$

Then $(C+U_{\mu'',\varepsilon''})\cap K\subset\bigcup\limits_{i=1}^{n}((x_i+U_{\mu',\varepsilon'})\cap K)$.
Let $\{\beta_k\}_{k=1}^{n}$ be a partition of the unity subordinated to the open covering $\{(x_i+U_{\mu',\varepsilon'})\}_{i=1}^{n}$.

Suppose now that $z\in conv((C+U_{\mu'',\varepsilon''})\cap K)$. Then there exist $\gamma_j\geq 0$, $j=1,2,\ldots,m$, $\sum\limits_{j=1}^{m}\gamma_j=1$, and $z_j\in((C+U_{\mu'',\varepsilon''})\cap K)$, $j=1,2,\ldots,m$, so that $z=\sum\limits_{j=1}^{m}\gamma_j z_j$

Further, for every $j\in\{1,2,\ldots,m\}$ let

$$c_j=\sum_{k=1}^{n}\beta_k(z_j)x_k.$$

Now, from the fact that C is convex it follows that $c_j\in C$, for every $j\in\{1,2,\ldots,m\}$, and so $c=\sum\limits_{j=1}^{m}\gamma_j c_j\in C$. Now, we shall prove that $z-c\in U_{\mu,\varepsilon}$, which implies that $z\in U_{\mu,\varepsilon}+C$. Indeed, since the set K is of $\Phi-$type it follows that for any $t\in\mu$:

$$\begin{aligned}
\|z-c\|(t) &= \|\sum_{j=1}^{m}\gamma_j z_j-\sum_{j=1}^{m}\gamma_j c_j\|(t)\leq\\
&\leq \sum_{j=1}^{m}\gamma_j\Phi(\|z_j-c_j\|)(t)=\\
&= \sum_{j=1}^{m}\gamma_j\Phi(\sum_{k=1}^{n}\beta_k(z_j)\Phi(\|z_j-x_k\|))(t)\\
&\leq \sum_{j=1}^{m}\gamma_j\ (\sum_{k=1}^{n}\beta_k(z_j)\Phi^2(\|z_j-x_k\|)(t))=\\
&= \sum_{j=1}^{m}\gamma_j\ (\sum_{\substack{k=1\\ \beta_k(z_j)\neq 0}}^{n}\beta_k(z_j)\Phi^2(\|z_j-x_k\|)(t)).
\end{aligned}$$

Since $\beta_k(z_j)\neq 0$ implies that $z_j-x_k\in U_{\mu',\varepsilon'}$ and so $\|z_j-x_k\|\in N_1$ we have that:

$$\|z-c\|(t)\leq\sum_{j=1}^{m}\gamma_j(\sum_{\beta_k(z_j)\neq 0}\beta_k(z_j)\varepsilon)=\varepsilon$$

and so $z - c \in U_{\mu,\varepsilon}$.

Now, we can prove our theorem.

Theorem 2.1. *Let X be a Hausdorff topological space, E a Hausdorff topological vector space and $(E, \|\cdot\|, \Phi)$ be the associated paranormed space, $T : X \to 2^E$ compact u.s.c. mapping with $T(x)$ a closed convex nonempty subset, $\overline{T(X)} \subset C_0$, C_0 convex $\Phi-$type subset. For $U \in \{U_{\mu,\varepsilon}\}$ let $p : \overline{T(X)} \to L$ be a Schauder projection into a finite-dimensional linear subspace L of E such that*

$$p(z) - z \in U', \qquad (U' \in \{U_{\mu,\varepsilon}\} \text{ and } \Phi^2(U') \subset U),$$

for every $z \in \overline{T(X)}$.

For each $x \in X$ let $PT(x) = \overline{conv}p(T(x))$. Then:

a) $\overline{conv}PT(X)$ and each $PT(x)$ is compact and convex subset of C_0;

b) $PT : X \to 2^E$ is u.s.c. and finite-dimensional;

c) $PT(x) \subset T(x) + U$ for each $x \in X$.

Proof. (a) As in Dugundji [1] because $p(T(x)) \subset L(\dim L < +\infty)$ is compact it's convex closure $PT(x)$ is a compact and convex subset of L and C_0. For the same reason, the compactness of $\overline{T(X)}$ implies that of $\overline{conv}PT(X)$.

(b) Clearly, only that PT is u.s.c. requires proof. Choose any $x \in X$ and let $W \subset X$ be open with $PT(x) \subset W$. Since $PT(x)$ is compact there is a $U_1 \in \{U_{\mu,\varepsilon}\}$ such that $PT(x) + U_1 \subset W$ and let $U \in \{U_{\mu,\varepsilon}\}$ be so that $U + U \subset U_1$. Since C_0 is of $\Phi-$type there is a zero neighbourhood $V \in \{U_{\mu,\varepsilon}\}$ so that

$$conv((PT(x) + V) \cap C_0) \subset PT(x) + U$$

Being the composition of two point-compact and u.s.c set functions, $x \mapsto pT(x)$ is also point-compact and u.s.c, so there is a neighbourhood $V(x)$ of x with $pT(y) \subset PT(x) + V$ for all $y \in V(x)$. According to the choose of V we find:

$$\begin{aligned} PT(y) &= \overline{conv}(pT(y) \cap C_0) \subset \overline{conv((PT(x) + V) \cap C_0)} \\ &\subset \overline{PT(x) + U} \subset PT(x) + U_1 \subset W \end{aligned}$$

for all $y \in V(x)$, so $PT(V(x)) \subset W$ and because x is arbitrary PT is u.s.c.

(c) Let $z \in PT(x)$. We have $z = \sum_{i=1}^{n} \lambda_i z_i$ for suitable $z_i \in pT(x)$ and real $0 \leq \lambda_i \leq 1$ with $\sum_{i=1}^{n} \lambda_i = 1$. For each i choose $y_i \in T(x)$ so that $p(y_i) = z_i$. Then

$$p(y_i) - y_i = \nu_i \in V \quad (i = 1, 2, \ldots, n)$$

so we have $y_i + \nu_i \in T(x) + V$ for each i and

$$\begin{aligned} z &= \sum_{i=1}^{n} \lambda_i p(y_i) = \sum_{i=1}^{n} \lambda_i (y_i + \nu_i) \\ &\in conv((T(x) + V) \cap C_0) \subset T(x) + U. \end{aligned}$$

As application of this result one can easy prove next fixed point theorem.

Theorem 2.2. *Let A be a nonvoid convex (not necesserly closed) subset of topological vector space X, let $T : A \to 2^A$ be u.s.c. compact mapping with $T(x)$ a closed convex subset for each $x \in A$ and C_0 $\Phi-$type convex subset such that $\overline{T(A)} \subseteq C_0 \subseteq A$. Then there exists $a \in A$ such that $a \in T(a)$.*

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