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A NOTE ON BABYLONIAN SQUARE-ROOT ALGORITHM AND RELATED VARIANTS

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Abstract

We connect the square-root algorithm developed by Old Babylonians before 4000 years and some iterative procedures as Newton and Weierstrass's methods. Special attention is devoted to the global convergence of these methods for the quadratic polynomial. We also give an accelerated method of the fourth order which is also connected with Babylonian's formula.

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1. Babylonian formula

An Old Babylonian tablet from the Yale Collection (No. 7289), old almost 4000 years, shows that Babylonian mathematicians possessed the skill in developing algorithmic procedure. One of their most known procedures was a square-root process described by C.B. Boyer in his *A History of Mathematics* [2]:

"Let $x = \sqrt{c}$ be the root desired and let a_1 be a first approximation to this root; let a second approximation h_1 be found from the equation $h_1 = c/a_1$. If a_1 is too small, then h_1 is too large, and vice versa. Hence, the arithmetic mean $a_2 = \frac{1}{2}(a_1 + h_1)$ is a plausible next approximation. Then one calculates $h_2 = c/a_2$ and one takes the arithmetic mean $a_3 = \frac{1}{2}(a_2 + h_2)$ to obtain a still better result; the procedure can be continued indefinitely".

Using a modern notation, the described iterative algorithm can be expressed by the sequence of arithmetic means $\{a_n\}$ and the sequence of harmonic means $\{h_n\}$ given by

(1) $$a_{n+1} = \frac{a_n^2 + c}{2a_n} \quad (n = 2, 3, \ldots)$$

and

(2) $$h_{n+1} = \frac{c}{a_{n+1}} = \frac{2a_n c}{a_n^2 + c} \quad (n = 2, 3, \ldots).$$

2. Cayley's problem

Now we recall the classical Newton method, one of the most prominent numerical methods for finding solutions of nonlinear equations. Let $p(x)$ be a function with continuous derivative and let $N(x) = x - p(x)/p'(x)$ be the Newton operator. As usual, we write N_k for the k-fold composition $N \circ \cdots \circ N$ of the function N so that $N_k(x)$ is the kth iterate $N(N(\cdots(N(x))))$ od x. If $N(x^*) = x^*$ we call x^* a *fixed point* of N. Let us note that the fixed points of N are the roots of p. Newton's iterative method reads

(3) $$x_{k+1} = N(x_k) = x_k - \frac{p(x_k)}{p'(x_k)} \quad (k = 0, 1, \ldots).$$

Remark. It is well known that the described Babylonian's square-root algorithm is actually Newton's method (3) applied to the function $p(x) = x^2 - c$. The iterative sequence $\{x_k\}$ obtained by (3) coincides with the sequence of arithmetic means $\{a_k\}$ given by (1). Furthermore, it is easy to see that the sequence of harmonic means $\{h_k\}$ given by (2) is obtained applying Newton method (3) to the function $f(x) = (x^2 - c)/x$.

In the sequel our attention will be paid to Cayley's problem from 1879. In his one-page paper [3] Cayley suggested the extension of the Newton method to complex polynomial p of the complex argument z. One assumes that the Newton method (3) is always a local method, that is, the initial value z_0 should be chosen sufficiently close to a zero z^* of the polynomial p to provide the convergence of the sequence $\{z_k\}$ to a zero z^* of p. But Cayley suggested to study the problem globally, which assumes the global set of initial points which converge to z^* under the Newton iteration. In terms of the fractal theory (see [6, Ch. 14], [9, Ch. 6]) it is necessary to find the global *basin of attraction* for a zero z^*

$$A(z^*) := \{z \in C : N_k(z) \to z^* \text{ as } k \to \infty\}.$$

As mentioned in [6] and [9], the solution of Cayley's problem is easy in the case of the quadratic equation $z^2 + \alpha z + \beta = 0$. As this equation can be transform to $z^2 - c = 0$, it suffices to consider the last equation. For the quadratic polynomial $p(z) = z^2 - c$ with the zeros $\pm\sqrt{c}$ Newton's method (3) gives Babylonian's formula

$$B(z) = \frac{z^2 + c}{2z}. \tag{4}$$

Since $B(z) \pm \sqrt{c} = (z \pm \sqrt{c})^2/2z$, we find

$$\frac{B(z) + \sqrt{c}}{B(z) - \sqrt{c}} = \left(\frac{z + \sqrt{c}}{z - \sqrt{c}}\right)^2.$$

From the last expression we state the following assertions:

Theorem 1. *Let $k \to \infty$, for $c \in C$ we have*

(i) *if* $\dfrac{|z + \sqrt{c}|}{|z - \sqrt{c}|} < 1$, *then* $\dfrac{|B_k(z) + \sqrt{c}|}{|B_k(z) - \sqrt{c}|} \to 0$ *and* $B_k(z) \to -\sqrt{c}$;

(ii) *if* $\dfrac{|z + \sqrt{c}|}{|z - \sqrt{c}|} > 1$, *then* $\dfrac{|B_k(z) - \sqrt{c}|}{|B_k(z) + \sqrt{c}|} \to 0$ *and* $B_k(z) \to \sqrt{c}$:

Let us note that the line $|z+\sqrt{c}| = |z-\sqrt{c}|$ (the limit case in Theorem 1) is the perpendicular bisector of $-\sqrt{c}$ and $\sqrt{c}$. Let ∂A denotes the boundary of the set A. One of basic notions in the fractal theory connected to iterative processes and convergence of an iterative function f is *Julia set* denoted by $J(f)$. We use the following assertion concerning the Julia set for the Newton operator N proved in [6, Lemma 14.11]:

Lemma 1. *Let z^* be an attractive fixed point of N. Then $J(N) = \partial A(z^*)$, where $A(z^*)$ is the basin of attraction of z^*.*

From Theorem 1 we conclude that the basins of attraction $A(-\sqrt{c})$ and $A(\sqrt{c})$ in the case of Babylonian's operator B are the half-planes on either side in relation to the line $|z+\sqrt{c}| = |z-\sqrt{c}|$, the perpendicular bisector of $-\sqrt{c}$ and $\sqrt{c}$ (see Fig. 1). Since $\pm\sqrt{c}$ are attractive fixed points of B, according to Lemma A the Julia set $J(B)$ is the boundary of the basins of attraction $A(-\sqrt{c})$ and $A(\sqrt{c})$, that is,

$$J(B) = \partial A(-\sqrt{c}) = \partial A(\sqrt{c}) = \{\gamma i\sqrt{c} : \gamma \in R\}.$$

Actually, the Julia set $J(B)$ is just the line $|z+\sqrt{c}| = |z-\sqrt{c}|$, as shown in Fig. 1.

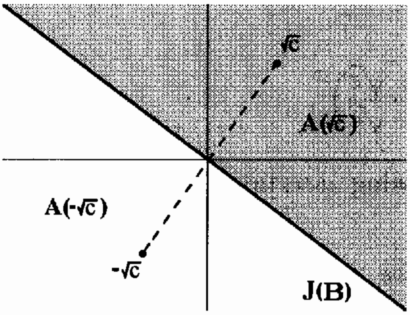

Fig. 1

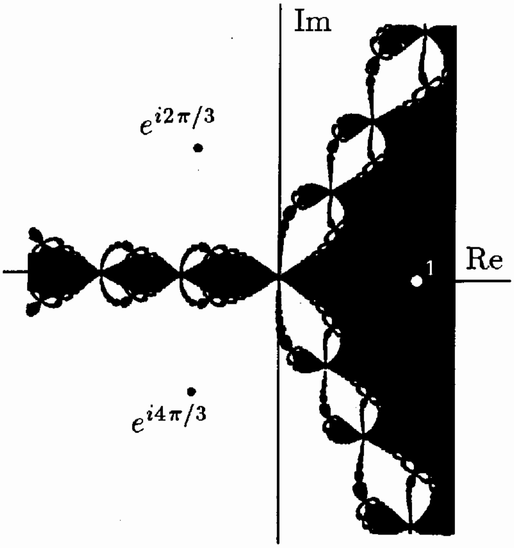

Fig. 2

According to the above consideration we conclude that Newton's method applied in the complex plane to the quadratic polynomial $p(z) = z^2 - c$ is globally convergent. But in the case of other polynomials of degree 3 or more, the situation is fundamentally different and more difficult. Domains of attraction and the Julia sets are very complicated with a great number of multiple points; moreover, the Julia sets very often are uncountable. Their highly intricate form makes that they are still far from being understood. More details about this problem can be found in [6] and [9], where the particular case $p(z) = z^3 - 1$ was analysed. For this reason we will not consider polynomials of higher degree. Instead of that, in order to visualize complexity of domains of attraction we displayed in Fig. 2 the domain of attraction of the zero $z = 1$ for Newton's method applied to $p(z) = z^3 - 1$.

In Fig. 3 we illustrate graphically the convergence behaviour of Newton's method (3) applied to the quadratic polynomial $p(z) = z^2 - c$. We have taken a specific value $c = 0.5 + 0.7i$ and considered the square $[-10, 10] \times [-10, 10]$. Darker colour corresponds to faster convergence (smaller number of iterative steps), the white line presents the set of divergent points, which is actually the Julia set $J(B)$ given in Fig. 1 for the same value of $c = 0.5 + 0.7i$. This is an interesting illustration of a theoretical result given above.

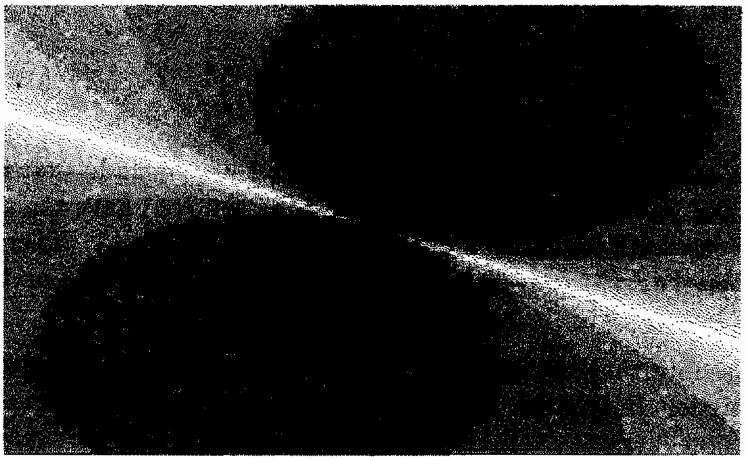

Fig. 3 Areas of convergence speed; the white line represents the Julia set

3. Accelerated method

According to Remark 1 it follows that both sequences (1) and (2), appearing in Babylonian's procedure, converge quadratically. Now we are interested in the acceleration of iterative process for finding the square root $\sqrt{c}$. Let us define a new approximation to the root $\sqrt{c}$ as the arithmetic mean of the terms given by (2) and (3), namely

$$(5)\quad z_{k+1} = \frac{1}{2}(a_{k+1} + h_{k+1}) = \phi(z_k) = \frac{z_k^4 + 6z_k^2c + c^2}{4z_k(z_k^2 + c)} \quad (k = 0, 1, \ldots).$$

Theorem 2. *The iterative sequence $\{z_k\}$ given by (5) converges globally to $\sqrt{c}$ or $-\sqrt{c}$ with the fourth order of convergence.*

Proof. First we find

$$(6)\qquad z_{k+1} \pm \sqrt{c} = \phi(z_k) \pm \sqrt{c} = \frac{1}{4z_k(z_k^2 + c)}(z_k \pm \sqrt{c})^4,$$

wherefrom we conclude that the order of convergence of the accelerated method (5) is four.

From (6) there follows

$$\frac{\phi(z_k) - \sqrt{c}}{\phi(z_k) + \sqrt{c}} = \left(\frac{z_k - \sqrt{c}}{z_k + \sqrt{c}}\right)^4.$$

Using the same argumentation as in Theorem 1 we conclude that the basins of attraction $A(-\sqrt{c})$ and $A(\sqrt{c})$ and the Julia set $J(\phi)$ have the same form as in the case of Babylonian's operator B shown in Fig. 1. Therefore, the iterative method (5) converge globally to $\sqrt{c}$ if $z_0 \in A(\sqrt{c})$ or to $-\sqrt{c}$ if $z_0 \in A(-\sqrt{c})$. □

Let us note that the iterative function $\phi(z)$ is connected with Babylonian's function $B(z)$ given by (4). Indeed, it is easy to see that

$$\phi(z) = B_2(z) = B \circ B = \frac{\left(\frac{z^2 + c}{2z}\right)^2 + c}{2\left(\frac{z^2 + c}{2z}\right)} = \frac{z^4 + 6z^2c + c^2}{4z(z^2 + c)}.$$

4. Convergence of Weierstrass' method

Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a complex polynomial of degree n with simple zeros. The Weierstrass method (also known in the literature as the Durand-Dočev-Kerner method), given by the iterative formula

$$(7) \qquad z_i^{(k+1)} = z_i^{(k)} - \frac{p(z_i^{(k)})}{\prod\limits_{j\neq i}(z_i^{(k)} - z_j^{(k)})} \quad (i = 1, \ldots, n;\ k = 0, 1, \ldots),$$

is one of the most efficient methods for the simultaneous approximation of all zeros of a polynomial p (see, e.g. [5], [4], [8]). According to a number experiments many authors have observed that the iterative method (7) is globally convergent outside of a set of measure zero for any starting disjoint approximations $z_1^{(0)}, \ldots, z_n^{(0)}$. This is proved by Small [7] for the case $n = 2$ and by Yamagishi [10] for the cubic polynomial $p(z) = z^3$.

We are going to present a simple proof of the global convergence of Weierstrass' method for a specific case $n = 2$. This proof is similar to that presented in [7]. Let us apply Weierstrass' method (7) to the quadratic polynomial $p(z) = z^2 - c$ assuming that the initial points $z_1^{(0)}$ and $z_2^{(0)}$ are disjoint. The first two iterations give

$$z_1^{(1)} = z_1^{(0)} - \frac{z_1^{(0)2} - c}{z_1^{(0)} - z_2^{(0)}} = \frac{c - z_1^{(0)} z_2^{(0)}}{z_1^{(0)} - z_2^{(0)}} = w^{(0)},$$

$$z_2^{(1)} = z_2^{(0)} - \frac{z_2^{(0)2} - c}{z_2^{(0)} - z_1^{(0)}} = -\frac{c - z_2^{(0)} z_1^{(0)}}{z_2^{(0)} - z_1^{(0)}} = -w^{(0)}$$

and

$$z_1^{(2)} = z_1^{(1)} - \frac{z_1^{(1)2} - c}{z_1^{(1)} - z_2^{(1)}} = \frac{c - z_1^{(1)} z_2^{(1)}}{z_1^{(1)} - z_2^{(1)}} = \frac{w^{(0)2} + c}{2w^{(0)}},$$

$$z_2^{(2)} = z_2^{(1)} - \frac{z_2^{(1)2} - c}{z_2^{(1)} - z_1^{(1)}} = -\frac{c - z_2^{(1)} z_1^{(1)}}{z_2^{(1)} - z_1^{(1)}} = -\frac{w^{(0)2} + c}{2w^{(0)}} = -z_1^{(2)}.$$

Thus, we again meet Babylonian's formula (4). The sequences $\{z_1^{(k)}\}$ and $\{z_2^{(k)}\}$ behave as Newton's iteration (4) providing, in addition, $z_1^{(k)} = -z_2^{(k)}$

for all $k \geq 2$. Since Newton's method is globally convergent for $n = 2$ we conclude that Weierstrass' method (7) is also globally convergent in a particular case $n = 2$. In practice, only a few iterations is sufficient to produce approximations to $-\sqrt{c}$ and $\sqrt{c}$ with very high accuracy, independently on starting points.

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