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REGULAR NULL-ADDITIVE MONOTONE SET FUNCTIONS

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Abstract

A singleton characterization of an atom of regular null-additive monotone set function is given. Autocontinuous from above monotone set function which is inner regular and exhaustive on the family of compact sets is always regular.

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1. Introduction

Fuzzy measures were introduced by Sugeno [4]. A fuzzy measure is a nonnegative extended real - valued set function μ defined on σ - algebra $\mathcal{F}$ and with the properties:

(FM_1) $\qquad \mu(\emptyset) = 0,$

(FM_2) $\qquad E \subset F \quad \Rightarrow \quad \mu(E) \leq \mu(F),$

(FM_3) $\qquad E_1 \subset E_2 \subset \dots \quad , E_n \in \mathcal{F} \quad \Rightarrow \quad \mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n),$

(FM_4) $E_1 \supset E_2 \supset ...$, $E_n \in \mathcal{F}$ and there exists n_0 such that $\mu(E_{n_0}) < \infty \Rightarrow \mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

Wang [7] and Suzuki [5],[6] have investigated fuzzy measures with special properties as null - additivity and autocontinuity. In this paper we shall investigate the regularity of set functions with properties (FM_1) and (FM_2) defined on σ - algebra $\mathcal{B}$ of Borel sets of a locally compact set. Using our previous methods from papers [1],[2] and [3] on $\perp$ - decomposable measures with respect to a t -conorm $\perp$ we shall prove a theorem on singleton characterization of an atom of regular null - additive set function and that inner regularity together with autocontinuity and exhaustivity on compact sets imply regularity of set function.

2. Null - additive set functions

Throughout this paper, let X be a locally compact Hausdorff topological space and let $\mathcal{K}$ be the lattice of all compact subsets of X. Borel σ - algebra $\mathcal{B}$ is the smallest σ - algebra containing $\mathcal{K}$. We shall denote by $\mathcal{O}$ the class of all open sets belonging to $\mathcal{B}$.
All the considered set functions in this paper are supposed to be with values in $[0, \infty]$, monotone and equal to zero on the empty set.
We have by Wang [7]

Definition 2.1.

A set function μ,

$$\mu : \mathcal{B} \to [0, \infty],$$

is called null - additive, if we have

$$\mu(E \cup F) = \mu(E)$$

whenever $E \in \mathcal{B}$, $F \in \mathcal{B}$, $E \cap F = \emptyset$ and $\mu(F) = 0$.

Example 1. *$\perp$ - decomposable measure $m : \mathcal{B} \to [0, 1]$ with respect to a t - conorm $\perp$ is always null - additive.*

Example 2. *Let $\mu(E) \neq 0$ whenever $E \in \mathcal{B}, E \neq \emptyset$. Then μ is null - additive.*

Example 3. *Let* $X = \{x, y\}$ *and define* μ *in the following way :* $\mu(X) = 1$ *and* $\mu(E) = 0$ *for* $E \neq X$. *Then* μ *is not null - additive.*

Definition 2.2. *A set function* μ *is called* σ *- finite if there exists a sequence* $\{X_n\}$ *such that*

$$X_1 \subset X_2 \subset ..., \qquad \cup_{n=1}^{\infty} X_n \in \mathcal{B} \qquad \text{and} \qquad \mu(X_n) < \infty \quad (n \in N).$$

Throughout this paper the set function μ always will be σ - finite.

Definition 2.3. *A set* $A \in \mathcal{B}$ *is an atom of* μ *iff* $\mu(A) > 0$ *and either* $\mu(B) = 0$ *or* $\mu(B) = \mu(A)$ *and* $\mu(A \setminus B) = 0$ *for* $B \subset A, B \in \mathcal{B}$.

Remark 1. *By* σ *- finiteness of* μ, *every atom has a finite measure.*

Definition 2.4. *A set function* μ *is regular if for each set* $A \in \mathcal{B}$ *and each* $\epsilon > 0$ *there exist* $K \in \mathcal{K}$ *and* $V \in \mathcal{O}$ *such that* $K \subset A \subset V$ *and*

$$\mu(V \setminus K) < \epsilon.$$

Remark 2. *A set function* μ *is regular iff*

$$inf\{\mu(V \setminus K) : K \subset A \subset V, K \in \mathcal{K}, V \in \mathcal{O}\} = 0$$

for each set $A \in \mathcal{B}$.

Proposition 1. *Regular monotone set function* μ *is* $\mathcal{O}$ *- exhaustive, i.e.,*

$$\lim_{n \to \infty} \mu(O_n) = 0$$

for each sequence $\{O_n\}$ *of open sets from* $\mathcal{O}$ *which are pairwise disjoint.*

Proof. or a sequence $\{O_n\}$ of open sets we have

$$\cup_{n=1}^{\infty} O_n \in \mathcal{O}.$$

Therefore by the regularity ofμ we have that for any $\epsilon > 0$ there exists a compact set K such that

$$K \subset \cup_{n=1}^{\infty} O_n \qquad \text{and} \qquad \mu(\cup_{n=1}^{\infty} O_n \setminus K) < \epsilon.$$

Since $\{O_n\}$ is an open cover of K and therefore exists $n_0 \in N$ such that $K \subset \cup_{n=1}^{n_0} O_n$. The monotonicity of μ implies

$$\mu(O_k) \leq \mu(O_k \cup (\cup_{n=1}^{n_0} O_n \setminus K)) \leq \mu(\cup_{n=1}^{\infty} O_n \setminus K)$$

for $k \geq n_0 + 1$.

We shall give a singleton characterization of atoms of regular null - additive set function.

Theorem 1. *Let μ be a regular null -additive set function. If $A \in \mathcal{B}$ is an atom of μ, then there exists a point $a \in A$ such that*

$$\mu(A) = \mu(\{a\}).$$

Proof. Let $A \in \mathcal{B}$ be an atom of μ. If we denote by $\mathcal{K}_1$ the family of all compact sets $K \subset A$ such that

$$\mu(A \setminus K) = 0,$$

then we shall prove that any K from $\mathcal{K}_1$ is an atom of μ. Since for any $B \subset K, B \in \mathcal{B}$ we have $K \setminus B \subset A \setminus B$, we obtain by the monotonicity of μ that either $\mu(B) = 0$ or $\mu(K \setminus B) = 0$ and so $\mu(K) = \mu((K \setminus B) \cup B) = \mu(B)$. For both cases the null - additivity of μ implies

$$\mu(K) = \mu((A \setminus K) \cup K) = \mu(A) > 0.$$

Let We have for$K_1, K_2 \in \mathcal{K}_1$ that

$$\mu((A \setminus K_1) \cup (A \setminus K_2)) = \mu(A \setminus (K_1 \cap K_2))$$

and $\mu(A \setminus K_2) = 0$. Hence by the null - additivity of μ

$$\mu(A \setminus (K_1 \cap K_2)) = \mu(A \setminus K_1) = 0,$$

i.e., $K_1 \cap K_2 \in \mathcal{K}_1$. We shall prove that

$$K_0 = \bigcap_{K \in \mathcal{K}_1} K$$

is a non - empty compact set . If we would suppose contrary, i.e., $K_0 = \emptyset$, then it would exist some finite subcollection of $\{K\}_{K \in \mathcal{K}_1}$ with the empty

intersection.This is impossible, since this finite subcollection would belong to $\mathcal{K}_1$, but it is an atom as an element of $\mathcal{K}_1$, which is non - empty.
We shall prove that $K_0 \in \mathcal{K}_1$. For $K \in \mathcal{K}_1$ it has to be $\mu(K \setminus K_0) = 0$. If we suppose that this is not true, then for $B \subset K \setminus K_0$ either $\mu(B) = 0$ or $\mu((K \setminus K_0) \setminus B) = 0$, which imply

$$\mu(K \setminus K_0) = \mu(((K \setminus K_0) \setminus B) \cup B) = \mu(B).$$

For both cases by the supposition $\mu(K \setminus K_0) > 0$ and so $K \setminus K_0$ would be an atom of μ.Since A and K are atoms of μ we have $\mu(A) > 0$ and $\mu(K_0) = 0$ (namely, $K_0 \subset K$ and $\mu(K \setminus K_0) > 0$). Null- additivity of μ implies

$$\mu(A) = \mu((A \setminus K) \cup (K \setminus K_0) \cup K_0) = \mu(K \setminus K_0).$$

By the supposition $\mu(K \setminus K_0) > 0$ and the preceding equality we obtain by the atomness of A

$$\mu(A \setminus (K \setminus K_0)) = 0.$$

Therefore $K \setminus K_0$ have to contain an element of $\mathcal{K}_1$, what is impossible, since K_0 is non - empty. Therefore $\mu(K \setminus K_0) = 0$. Hence by null - additivity of μ

$$\mu(A) = \mu((A \setminus K) \cup (K \setminus K_0) \cup K_0) = \mu(K_0),$$

i.e., $\mu(A \setminus K_0) = 0$ by the atomness of A. This implies $K_0 \in \mathcal{K}_1$.
Finally,we shall prove that K_0 is a singleton. If we suppose the contrary, i.e. that K_0 contains at least two distinct elements a_1 and a_2, then, since X is a locally compact Hausdorff topological space, there exists an open neighbourhood V of a_1 such that clV does not contain a_2. Hence

$$K_0 = (K_0 \setminus V) \cup (K_0 \cap clV).$$

Since one of the sets $K_0 \setminus V$ or $K_0 \cap clV$ have to belong to $\mathcal{K}_1$, but K_0 is the least element of $\mathcal{K}_1$, we obtain a contradiction. Hence there exists $a \in A$ such that

$$\mu(A) = \mu(K_0) = \mu(\{a\}).$$

In a special case we obtain the following result from [2]:

Corollary 2.1. *Let $m : \mathcal{B} \to [0,1]$ be a regular $\perp$ - decomposable measure with respect to an arbitrary but fixed t - conorm $\perp$. If $A \in \mathcal{B}$ is an atom of m, then there exist a point $a \in A$ such that*

$$m(A) = m(\{a\}).$$

Corollary 2.2. *Each continuous from above (i.e. with the property (FM_4) regular Borel null - additive set function μ , $\mu : \mathcal{B}(\mathcal{R}) \to [0, \infty]$, which has the property*

$$\mu((a,b)) = g(b-a)$$

on each finite interval (a,b) for some continuous at 0 function g with $g(0) = 0$, is non - atomic.

Proof. If we suppose that the theorem is not true, then there exists an atom $A \in \mathcal{B}(\mathcal{R})$ of μ. therefore by Theorem 1 there exists an element a from A such that $\mu(A) = \mu(\{a\})$. So we would have

$$\mu(\{a\}) = \mu(\cap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n})) = \lim_{n\to\infty} \mu((a - \frac{1}{n}, a + \frac{1}{n})) = \lim_{n\to\infty} g(\frac{2}{n}) = 0,$$

i.e., $\mu(A) = \mu(\{a\}) = 0$, what is a contradiction.

3. Autocontinuity from above

We have by Wang [7]

Definition 3.1. *A set function μ is called autocontinuous from above if we have*

$$\mu(A \cup B_n) \to \mu(A)$$

whenever $A \in \mathcal{B}$, $B_n \in \mathcal{B}$, $A \cap B_n = \emptyset$ $(n \in N)$, $\mu(B_n) \to 0$.

Proposition 2. *A monotone set function μ is autocontinuous from above iff for every $A \in \mathcal{B}$ and $\epsilon > 0$ there exists $\delta = \delta(\epsilon, A) > 0$ such that, whenever*

$$A \in \mathcal{B},\ B \in \mathcal{B},\ A \cap B = \emptyset,\ \mu(B) < \delta$$

implies

$$\mu(A) - \epsilon \le \mu(A \cup B) \le \mu(A) + \epsilon.$$

Remark 3. *Obviously, autocontinuity from above of μ implies null - additivity of μ.*

Proposition 3. *Let μ be an autocontinuous from above set function. If $\{E_n\}$ and $\{F_n\}$ are two decreasing sequences from $\mathcal{B}$ such that $E_1 \cap F_1 = \emptyset$ and*

$$\lim_{n\to\infty} \mu(E_n) = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \mu(F_n) = 0,$$

then

$$\lim_{n\to\infty} \mu(E_n \cup F_n) = 0.$$

We have that inner regularity and autocontinuity from above implies regularity:

Theorem 2. *Let μ be a set function which is autocontinuos from above, $\mathcal{K}$ - exhaustive and satisfies the equality*

$$(1) \qquad \mu(A) = sup\{\mu(K) : K \in \mathcal{K}, K \subset A\} \qquad (A \in \mathcal{B}).$$

Then μ is a regular Borel set function.

*Proof.*Suppose that the theorem is not true. Then there exist a set $A \in \mathcal{B}$ and a number $\epsilon > 0$ such that

$$(2) \qquad \mu(V \setminus K) > \epsilon$$

for each $K \in \mathcal{K}, V \in \mathcal{O}, K \subset A \subset V$. Let us fix such sets K_0 and V_0. The equality (1) implies that there exist

$$C_1, D_1 \in \mathcal{K},\ C_1 \subset V_0 \setminus A,\ D_1 \subset A \setminus K_0$$

such that

$$\mu(C_1) \geq \frac{1}{2}\mu(V_0 \setminus A) \qquad \textit{and} \qquad \mu(D_1) \geq \frac{1}{2}\mu(A \setminus K_0).$$

Let $V_1 = V_0 \setminus C_1$ and $K_1 = K_0 \cup D_1$.It is obvious that $V_1 \in \mathcal{O}$, $K_1 \in \mathcal{K}$ and $K_1 \subset A \subset V_1$.Then (2) implies

$$\mu(V_1 \setminus A_1) > \epsilon.$$

Repeating the preceding procedure after n - steps we obtain sets

$$C_n, D_n \in \mathcal{K},\ C_n \subset V_{n-1} \setminus A,\ D_n \subset A \setminus K_{n-1}$$

such that

$$(3) \qquad \mu(C_n) \geq \frac{1}{2}\mu(V_{n-1} \setminus A), \qquad \mu(D_n) \geq \frac{1}{2}\mu(A \setminus K_{n-1}).$$

We have obtained two sequences $\{C_n\}$ and $\{D_n\}$ of pairwise disjoint sets from $\mathcal{K}$. Now $\mathcal{K}$ - exhaustivity of μ implies

$$\lim_{n\to\infty} \mu(C_n) = 0 \qquad \lim_{n\to\infty} \mu(D_n) = 0.$$

Hence by (3)

$$(4) \qquad \lim_{n\to\infty} \mu(V_n \setminus A) = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \mu(A \setminus K_{n-1}) = 0.$$

In this way we have obtained decreasing sequences $\{V_n \setminus A\}$ and $\{A \setminus K_{n-1}\}$ from $\mathcal{B}$ (namely,$V_n \subset V_{n-1}$ and $K_n \supset K_{n-1}$) such that (4) holds and

$$(V_0 \setminus A) \cap (A \setminus K_0) = \emptyset.$$

Therefore by Proposition 3.

$$\lim_{n\to\infty} \mu(V_n \setminus K_n) = \lim_{n\to\infty} \mu((V_n \; A) \cup (A \setminus K_n)) = 0.$$

Contradiction with (2).

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