Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 2 (1995), 49-55

Review of Research
Faculty of Science
Mathematics Series

STRUCTURAL THEOREMS FOR S-BOUNDED DISTRIBUTIONS

Bogoljub Stanković
University of Novi Sad, Institute of Mathematics,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Two structural theorems for $S-$bounded distributions have been proved. The first one contains equivalent statements that a distribution is $S-$bounded and the second gives the subspaces of $\mathcal{D}'$ to which belongs a distribution being $S-$bounded relative to a special function c and a cone Γ.

AMS Mathematics Subject Classification 1991: 46F10.
Key words: Distribution, asymptotic behaviour, boundness

1. Introduction

Asymptotic behaviour of distributions at infinity has been introduced in various ways (see [2], [3], [4] and [8]). Such definitions give precise information on distributions, but restrict the class of distributions for which these definitions can be applied. This limitation appears particularly when we analyze the asymptotic behaviour at infinity of solutions of partial differential equations. That was the reason to introduce the notion "$S-$bounded distribution" [6], which generalized the well-defined asymptotic behaviour of numerical functions at infinity.

2. Notion and definitions

We shall denote by

$\mathcal{N}_0 = \mathcal{N} \cup \{0\}$,

Γ a convex, closed in $\mathcal{R}^m$ with the vertex at zero,

P the set of functions $c : \mathcal{R}^m \to \mathcal{R}_+$,

$\check{f}$ the function $\check{f}(x) = f(-x)$, $x \in \mathcal{R}^m$,

$\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{B}'$ the spaces of Schwartz's distributions, of tempered distributions and of bounded distributions, respectively.

$\mathcal{K}'_1$ the set of distributions of exponential growth introduced by M.Hasumi [1].

A sequence $\{\delta_n\}$ from $\mathcal{D}$, $supp\delta_n \subset \{x \in \mathcal{R}^m : \|x\| \le r_n\}$, $r_n \to 0$ when $n \to \infty$, called a $\delta-$ sequence if it converges to δ in $\mathcal{D}'$.

The sign $*$ denotes the convolution.

Definition 1. *$T \in \mathcal{D}'$ is said to be $S-$bounded in $\mathcal{D}'$ relative to $c \in P$ and to the cone Γ if the set $G_T = \{T(x+h)/c(h),\ h \in \Gamma\}$ is bounded in $\mathcal{D}'$.*

Since the strong and the weak boundness are rquivalent notions in $\mathcal{D}'$, G_T is bounded in $\mathcal{D}'$ if and only if for every $\varphi \in \mathcal{D}$ there exists $M_\varphi \in \mathcal{R}_+$ such that

$$|\langle T(x+h)/c(h), \varphi(x)\rangle| \le M_\varphi, \ \ h \in \Gamma.$$

Let us remember that for $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$

$$\langle T(x+h), \varphi(x)\rangle = (T * \check{\varphi})(h), \ \ h \in \mathcal{R}^m.$$

3. Structural theorems

Theorem 1. *Suppose that $T \in \mathcal{D}'$ and $c \in P$. The following statements are equivalent:*

a) T is $S-$bounded relative to c and to the cone Γ.

b) For every $\varphi \in \mathcal{D}$, $T * \varphi$ is $S-$bounded relative to c and to the cone Γ.

c) For a $\delta-$sequence $\{\delta_n\}$ and every $\varphi \in \mathcal{D}$ there exists $M_\varphi \in \mathcal{R}_+$ such that

$$|\langle (T * \delta_n)(x+h)/c(h),\ \varphi(x)\rangle| \leq M_\varphi,\ \ n \in \mathcal{N},\ \ h \in \Gamma.$$

Proof. Implication $a) \to b)$ follows from the relation

$$\langle (T * \varphi)(x+h)/c(h),\ \psi(x)\rangle = \langle T(x+h)/c(h), (\check{\varphi} * \psi)(x)\rangle,\ h \in \Gamma$$

and the property of $\mathcal{D} : \mathcal{D} * \mathcal{D} \subset \mathcal{D}$.

$b) \to a)$. It has been proved in [5] that any $\psi \in \mathcal{D}$ can be written in the form: $\psi = \psi_1 * \phi_1 + \ldots + \psi_k * \phi_k$, where ψ_i and ϕ_i, $i = 1, \ldots, k$, are from $\mathcal{D}$. For the proof we have only to use the following relation

$$\begin{aligned}\langle T(x+h)/c(h), \psi(x)\rangle = \ & \sum_{i=1}^{k} \langle T(x+h)/c(h),\ (\psi_i * \phi_i)(x)\rangle \\ & \sum_{i=1}^{k} \langle (T * \check{\psi}_i)(x+h)/c(h), \phi_i(x)\rangle,\ h \in \Gamma.\end{aligned}$$

$a) \to c)$. Let $\{\delta_n\}$ be a $\delta-$ sequence and $\varphi \in \mathcal{D}$. Then $(\delta_n * \varphi) \in \mathcal{D}$, $n \in \mathcal{N}$, as well and the sequence $\{\delta_n * \varphi\}$ converges in $\mathcal{D}$ to φ. Consequently, the set $\{\delta_n * \varphi,\ n \in \mathcal{N}\}$ is bounded in $\mathcal{D}$. For any $\varphi \in \mathcal{D}$ we have

$$\langle (T * \delta_n)(x+h)/c(h), \varphi(x)\rangle = \langle T(x+h)/c(h), (\delta_n * \check{\phi})\check{}(x)\rangle.$$

We have seen that the set $\{(\delta_n * \check{\varphi})\check{},\ n \in \mathcal{N}\}$ is bounded in $\mathcal{D}$ and that the set $\{T(x+h)/c(h),\ h \in \Gamma\}$ is bounded in $\mathcal{D}'$. By definition of a bounded set in $\mathcal{D}'$, it follows that there exists M_φ such that

$$|\langle T(x+h)/c(h),\ (\delta_n * \check{\varphi})\check{}(c)\rangle| \leq M_\varphi,\ h \in \Gamma.$$

$c) \to a)$. For a $\varphi \in \mathcal{D}$ we have

$$|\langle T(x+h)/c(h), \varphi(x)\rangle| = |\lim_{n\to\infty} \langle T(x+h)/c(h), (\delta_n * \check{\varphi})\check{}(x)\rangle|$$

$$= \lim_{n\to\infty} |\langle (T * \delta_n)(x+h)/c(h), \varphi(x)\rangle| \leq M_\varphi,\ h \in \Gamma.$$

We shall give, now, a sufficient condition for the $S-$boundness. Let e be a positive function defined on $\mathcal{R}^m$ and such that for every compact set $K \in \mathcal{R}^m$ there exists $C_K \geq 0$ such that $e(x+h)/c(h) \leq C_K$, $x \in K$ and $h \in \Gamma$.

Proposition 1. *A sufficient condition that the distribution T is $S-$bounded relative to e and to the cone Γ is that for a $\delta-$sequence $\{\delta_n\}$ there exists $M \geq 0$ such that*

$$|(T * \delta_n)(h)/e(h)| \leq M, \quad n \in \mathcal{N}, \ h \in K + \Gamma$$

for every compact set $K \subset \mathcal{R}^m$.

Proof. By Theorem 1 it is enough to prove that from the suppositions of Proposition 1 it follows the statement c) of Theorem 1. This is a consequence of the relation

$$|\langle \frac{(T*\delta_n)(x+h)}{e(h)}, \varphi(x)\rangle| = \quad |\int_{\mathcal{R}^m} \frac{(T*\delta_n)(x+h)}{e(x+h)} \frac{e(x+h)}{e(h)} \varphi(x) dx|$$

$$\leq MC_K \int_K |\varphi(x)| dx < \infty, \quad h \in \Gamma,$$

where supp$\varphi \subset K$.

Remark. The sufficient condition in Proposition 1 that T is $S-$ bounded is not, at the same time, a necessary condition. To show it, let H be an unbounded function which belongs to $\mathcal{C}^\infty(\mathcal{R}^m) \cap \mathcal{L}_1(\mathcal{R}^m)$. The regular distribution defined by H is $S-$ bounded relative to $e = 1$ because of

$$|\int_{\mathcal{R}^m} H(x+h)\varphi(x)dx| \leq \max_{t \in \mathcal{R}^m} |\varphi(t)| \int_{\mathcal{R}^m} |H(x)| dx < \infty,$$

where $\varphi \in \mathcal{D}$. Consequently, $1 + H$ is $S-$boubded, as well. But $(1 + H) * \delta_n$ does not satisfy condition of Proposition 1. Since $1 + H \in \mathcal{C}^\infty(\mathcal{R}^m)$ and δ_n defines distributions such that $supp\delta_n \subset B(0,1)$, $n \geq n_0$, by the property of convolution we have

$$\lim_{n\to\infty} ((1 + H) * \delta_n) = 1 + H \quad \text{in} \quad \mathcal{C}^\infty(\mathcal{R}^m).$$

Hence, for every compact set $K \subset \mathcal{R}^m$ and $\epsilon > 0$ we have

$$|((1 + H) * \delta_n)(h)| \geq |1 + H(h)| - \epsilon, \quad n \geq n_0(\epsilon) \ \ h \in K.$$

It follows that the sequence $\{(1 + H) * \delta_n\}$ can not satisfy conditions of Proposition 1 because we supposed that H was an unbounded function.

The second problem we shall discuss in this paper is to find the "natural" subspace of distributions to which belongs a distribution T being

$S-$bounded. Since the $S-$boundness is a local property (see [6]) we shall suppose that T has its support in the cone Γ.

In [6] we proved the following theorem which will be used in the proof of Theorem 2.

Theorem A. *$T \in \mathcal{D}'$ is $S-$bounded in $\mathcal{D}'$ relative to $c \in P$ and to the cone Γ if and only if for every ball $B(0,r) = \{x \in \mathcal{R}^m : \|x\| < r\}$, $r < \infty$,*

$$T = \sum_{|i| \leq k} D^i F_i \quad \textit{on} \quad B(0,r) + \Gamma,$$

where $i = (i_1, ..., i_m) \in \mathcal{N}_0^m$, $|i| = \sum_{k=1}^{m} i_k$ and $D^i = D^{i_1}...D^{i_m}$. The functions F_i are continuous on $B(0,r) + \Gamma$ and the sets $\{F_i(x+h)/c(h),\ x \in B(0,r),\ h \in \Gamma\}$ are bounded for every i, $|i| \leq k$.

Theorem 2. *Suppose that $T \in D'$, $suppT \subset \Gamma$ and that T is $S-$bounded relative to $c \in P$ and to the cone Γ. Let w be a positive and continuous function such that $0 < q \leq w(x) \leq Q < \infty$, $x \in \mathcal{R}^m$.*

a) If $c = w$, then $T \in \mathcal{B}'$.

b) If $c(x) = (1 + \|x\|^2)^{p/2} w(x)$, $p > 0$, $x \in \mathcal{R}^m$, then $T \in \mathcal{S}'$.

c) If $c(x) = w(x) \exp(k\|x\|)$, where $k \in \mathcal{R}_+$, $x \in \mathcal{R}^m$, then $T \in \mathcal{K}'_1$.

In all these three cases T is also $S-$bounded in the subspace of $\mathcal{D}'$ to which it belongs.

Proof. By Theorem A

$$T = \sum_{|i| \leq k} D^i F_i \quad \text{on} \quad B(0,r) + \Gamma.$$

Since $suppT \subset \Gamma$, then $suppF_i$, $|i| \leq k$, belongs to a neighbourhood Ω_ϵ, $\epsilon < r$, of Γ (Ω_ϵ is an open set such that $d(\Omega_\epsilon, \Gamma) < \epsilon$). Let f_i denote the function F_i/c, $|i| \leq k$. The functions f_i are continuous and bounded on R^m, and T can be written in the form

$$T = \sum_{|i| \leq k} D^i (c f_i) \quad \text{on} \quad \mathcal{R}^m.$$

Now a), b) and c) follow from the structural theorems for the spaces $\mathcal{B}', \mathcal{S}'$ and $\mathcal{K}'_1$ (see [1] and [7]).

To prove the last part of Theorem 1, we denote by $\mathcal{A}$ one of the spaces $\mathcal{D}_{L^1}$ (case a)), $\mathcal{S}$ (case b)) or $\mathcal{K}_1$ (case c)).

In all these three cases $(c(x+h)/c(h))\varphi^{(i)}(x) \in \mathcal{L}^1(\mathcal{R}^m)$, $i \in \mathcal{N}_0^m$, $h \in \Gamma$ and $\varphi \in \mathcal{A}$. In case a) this is obvious because $\varphi \in \mathcal{D}_{L^1}$ and $c = w$. In the case b), $\varphi \in \mathcal{S}$ and

$$\left|\frac{c(x+h)}{c(h)}\varphi^{(i)}(x)\right| \leq \frac{Q}{q}\Big(\frac{1+\|x+h\|^2}{1+\|h\|^2}\Big)^{p/2}\left|\varphi^{(i)}(x)\right|$$

$$\leq \frac{Q}{q}(\|x\|^2 + \|x\|)^{p/2}|\varphi^{(i)}(x)| \in \mathcal{L}^1(\mathcal{R}^m),$$

where $h \in \Gamma$ and $x \in \mathcal{R}^m$.

In the last case $\varphi \in \mathcal{K}_1$ and

$$\left|\frac{c(x+h)}{c(h)}\varphi^{(i)}(x)\right| \leq \frac{Q}{q}|\varphi^{(i)}(x)|\exp(k\|x\|) \in \mathcal{L}^1(\mathcal{R}^m),\ h \in \Gamma,\ x \in \mathcal{R}^m.$$

Suppose now that $\varphi \in \mathcal{A}$, then

$$|\langle T(x+h)/c(h), \varphi(x)\rangle| \leq \sum_{|i|\leq k}\int_{\mathcal{R}^m}|F_i(x+h)/c(h)||\varphi^{(i)}(x)|dx$$

$$\leq \sum_{|i|\leq k}\int_{\mathcal{R}^m}|f_i(x+h)|\frac{c(x+h)}{c(h)}|\varphi^{(i)}(x)|dx < \infty,\ h \in \Gamma.$$

This completes the proof of Theorem 2.

References

[1] Hasumi, M., Note on the n-dimensional tempered ultra-distributions, Tohoko, Math. J. (2), 13 (1961), 94-104.

[2] Lavoine, J. Misra, O.P., Théorèmes Abelians pour la transformation de Stieltjes des distributions, C. R. Acad. Sci. Paris 279 (1974), 99-102.

[3] Lighthill, M.J., Fourier analysis and generalized functions, Cambridge University Press, 1960.

[4] Pilipović, S., Stanković, B. Takači, A., Asymptotic behaviour and Stieltjes Transformation of Distributions, Teubner Verlagsgesellschaft, 1990.

[5] Rubel,L.A., Squires, W.A. Taylor, B.A., Irreducibility of certain entire functions with applications to harmonic analysis, Ann. Math., 108 (1978), 553-567.

[6] Stanković, B., S-bounded distributions, application to partial differential equations, Applicable Analysis , 48 (1993), 99-112.

[7] Schwartz, L., Théorie des distributions, Hermann, Paris, 1966.

[8] Vladimirov, V.S., Drozhinov, Yu. N. and Zav'yalov, B.I., Tauberian Theorems for Generalized Functions, Kluwer Academic Publishers, Maine, 1988.

Received by the editors March 5, 1992