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PROPERTIES OF TOPOLOGICAL n-PARTITIONS

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Abstract

A topological n–partition of a Hausdorff space $(X, \mathcal{O})$ is an n–partition $\Pi \subset P(X)$ (in the sense of Hartmanis) which satisfies an additional topological condition. A subbase for the topology $\mathcal{O}_\Pi$ of Π consists of all sets $O^* = \{p \in \Pi \mid p \cap O \neq \emptyset\}$ where $O \in \mathcal{O}$. The central question in the paper is: if $\mathcal{P}$ is a topological property and X has $\mathcal{P}$, does Π have $\mathcal{P}$? Preserving of topological properties is investigated for an arbitrary X and in the special situations when X is a locally compact or a compact space, or $\Pi = [X]^n$.

If X is compact or $\Pi = [X]^n$, then Π is a subspace of 2^X with the Vietoris topology. Topological projective and Euclidean planes are the special topological 2–partitions.

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1. Introduction

According to Hartmanis (see [3]), an n–partition on a set X (where $|X| \geq n$) is a family of its subsets Π, such that each subset contains at least n elements of X and any n distinct elements $x^1, ..., x^n \in X$ are contained in exactly one subset, denoted by $p(x^1, ..., x^n)$.

In [5] are considered special $n-$partitions of Hausdorff spaces - topological $n-$partitions. Such partitions are equipped with topology and investigated as topological spaces.

This paper deals with the question: which topological properties do transfer from X to Π. Also, it will be proved that for some special cases the partition Π is a subspace of 2^X with the Vietoris topology as defined in [6].

The topological projective (and Euclidean) planes studied in [7] are special cases of the partitions mentioned above (for $n = 2$) and all of the results are valid in these spaces.

A few words about the notation. If X is a set, $\mid X \mid$ will denote its cardinality, $P(X)$ the power set of X and $[X]^n$ the family of subsets of X with exactly n elements. If $(X, \mathcal{O})$ is a topological space, the letters $\mathcal{P}$ and $\mathcal{B}$ will be reserved for a subbase and a base for the topology $\mathcal{O}$. Finally, the fact that the net $< x_\sigma \mid \sigma \in \Sigma >$ converges to x we will denote by $< x_\sigma > \rightarrow x$. All the facts from general topology used here can be found in [2].

The main idea of [5] and this paper arised in a discussion with professor J.Ušan. I am grateful to him for many valuable suggestions.

2. Preliminaries

Firstly, because of completness, we list some definitions and facts proved in [5], which are used in this paper.

To simplify the following definition, if $x^1, ..., x^n$ are not distinct, we define $p(x^1, ..., x^n) = X$.

Definition 2.1. *Let X be a Hausdorff space. An $n-$partition Π on X is topological iff it holds: if $x^1, ..., x^n, z \in X$ are distinct points and if $< x^i_\sigma \mid \sigma \in \Sigma >$ are nets, such that $< x^i_\sigma > \rightarrow x^i$ for $i = 1, ..., n$; where $p(x^1_\sigma, ..., x^n_\sigma) = p_\sigma$ and $p(x^1, ..., x^n) = p$, then*

(a) if $< z_\sigma > \rightarrow z$ and $z_\sigma \in p_\sigma$ for all $\sigma \in \Sigma$, then $z \in p$

(b) if $z \in p$, then there is a net $< z_\sigma \mid \sigma \in \Sigma >$ such that $< z_\sigma > \rightarrow z$ and for all $\sigma \in \Sigma$, $z_\sigma \in p_\sigma$.

It is easy to show that the elements of Π are closed subsets of X.

If for $O \in \mathcal{O}$ we define $O^* = \{p \in \Pi \mid p \bigcap O \neq \emptyset\}$, then $\mathcal{P}_\Pi = \{O^* \mid O \in \mathcal{O}\}$ is a subbase for some topology on Π, denoted by $\mathcal{O}_\Pi$. From now on we will consider the space $(\Pi, \mathcal{O}_\Pi)$. One can show that Π is a Hausdorff space (since X is). Also, it holds

Theorem 2.1. *Let $p \in \Pi$ and let $x^1, ..., x^n \in p$ be distinct points. If $\mathcal{B}(x^i)$ is a neighbourhood base at the point x^i for $i = 1, ..., n$; then*

(i) if $k \in N$ and $O_j \in \mathcal{O}$ for $j = 1, ..., k$ and $p \in \bigcap_{j=1}^k O_j^*$, then there is $(B_1, ..., B_n) \in \prod_{i=1}^n \mathcal{B}(x^i)$ such that $p \in \bigcap_{i=1}^n B_i^* \subset \bigcap_{j=1}^k O_j^*$.

(ii) $\mathcal{B}(p) = \{\bigcap_{i=1}^n B_i^* \mid (B_1, ..., B_n) \in \prod_{i=1}^n \mathcal{B}(x^i)\}$ is a local base at the point $p \in \Pi$.

(iii) if $\mathcal{B}$ is a topology base for X, then $\mathcal{B}_\Pi = \{\bigcap_{i=1}^n B_i^* \mid (B_1, ..., B_n) \in \mathcal{B}^n, B_i \bigcap B_j = \emptyset$ for $i \neq j\}$ is a topology base for Π. □

We will say that a topological property $\mathcal{P}$ is a Π-invariant iff for each Hausdorff space X and for each topological $n-$partition Π on X it holds: if X has $\mathcal{P}$, then Π has $\mathcal{P}$.

According to [5], the following properties are Π-invariants: $w \leq \alpha, \chi \leq \alpha, d \leq \alpha$ (where α is a cardinal, $\alpha \geq \aleph_0$), T_2, discretness ... . In this paper we investigate some more properties.

At last, something about the examples. In [5] it is proved that the space of lines of a topological projective (or Euclidean) plane is a $2-$partition of the space of points. For each Hausdorff space X, $[X]^n$ is a topological $n-$partition on X. Also, the set of all 1-dimensional manifolds in a normed vector space is a topological $n-$partition. Finally, the set of all circles and lines in R^2 is a topological 3-partition of R^2.

3. What is not preserved

Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be Hausdorff spaces, where $X_1 \bigcap X_2 = \emptyset$ and $| X_1 |, | X_2 |> 1$. For the space $X = X_1 \bigoplus X_2$ we define:

$$\Pi_{X_1 \bigoplus X_2} = \{\{x_1, x_2\} \mid x_1 \in X,\ \ x_2 \in X_2\} \bigcup \{X_1, X_2\}.$$

Theorem 3.1. *$\Pi_{X_1 \bigoplus X_2}$ is a topological 2– partition on X. The space $\Pi_{X_1 \bigoplus X_2}$ is homeomorphic to $(X_1 \times X_2) \bigoplus D_2$, where D_2 is the two-element discrete space.*

Proof. (i) Obviously, each element of $\Pi_{X_1 \bigoplus X_2}$ contains at least two points and every two points are contained in the unique element of $\Pi_{X_1 \bigoplus X_2}$. Thus, $\Pi_{X_1 \bigoplus X_2}$ is a 2–partition of X.

Let $x, y, z \in X$ be distinct points and $< x_\sigma > \rightarrow x, \quad < y_\sigma > \rightarrow y$.

(a) Let $< z_\sigma > \rightarrow z$ and $z_\sigma \in p(x_\sigma, y_\sigma), \quad \sigma \in \Sigma$. If $x, y \in X_1$, then $p(x, y) = X_1$, and since X_1 is open there is $\sigma_0 \in \Sigma$ such that for $\sigma \geq \sigma_0, \quad x_\sigma, y_\sigma \in X_1$, that is $p(x_\sigma, y_\sigma) = X_1$. Now, $z_\sigma \in X_1$ for $\sigma \geq \sigma_0$, and since X_1 is closed too, and $< z_\sigma > \rightarrow z$, we have $z \in X_1 = p(x, y)$. Similarly if $x, y \in X_2$.

If $x \in X_1, \quad y \in X_2$, then $p(x, y) = \{x, y\}$. The openness of X_1 and X_2 in X and convergence of $< x_\sigma >$ and $< y_\sigma >$ gives $\sigma_0 \in \Sigma$ such that for $\sigma \geq \sigma_0, \quad x_\sigma \in X_1$ and $y_\sigma \in X_2$. Thus, for $\sigma \geq \sigma_0, \quad z_\sigma \in \{x_\sigma, y_\sigma\}$. Since X is a Hausdorff space, we have $< z_\sigma > \rightarrow x$ or $< z_\sigma > \rightarrow y$. But the points x, y and z are distinct. A contradiction! Similarly, if $x \in X_2$ and $y \in X_1$.

(b) Let $z \in p(x, y)$. Since x, y and z are distinct, we have $p(x, y) = X_1$ or $p(x, y) = X_2$. If $p(x, y) = X_1$, then since X_1 is open, there is $\sigma_0 \in \Sigma$ such that for $\sigma \geq \sigma_0, \quad x_\sigma, y_\sigma \in X_1$, that is $p(x_\sigma, y_\sigma) = X_1$. Let us define $z_\sigma = z$ for $\sigma \geq \sigma_0$ and $z_\sigma = x_\sigma$ otherwise. Now, for all $\sigma \in \Sigma, \quad z_\sigma \in p(x_\sigma, y_\sigma)$ and $< z_\sigma > \rightarrow z$. Similarly, if $p(x, y) = X_2$. Thus, $\Pi_{X_1 \bigoplus X_2}$ is a topological 2–partition.

(ii) Since $| X_1 | \geq 2$, there are $a, b \in X_1, \quad a \neq b$. X_1 is a Hausdorff space, so there are U and V open in X_1 and disjoint, such that $a \in U, \quad b \in V$. It is easy to prove that $U^* \bigcap V^* = \{X_1\}$, thus $\{X_1\}$ is clopen in $\Pi_{X_1 \bigoplus X_2}$. Similarly, $\{X_2\}$ is clopen too. Now, we have $\Pi_{X_1 \bigoplus X_2} = \Pi' \bigoplus \{X_1\} \bigoplus \{X_2\}$ where $\Pi' = \{\{x_1, x_2\} \mid x_1 \in X_1, \quad x_2 \in X_2\}$.

(iii) Let us prove that Π' is homeomorphic to $X_1 \times X_2$. If $\mathcal{B}_i$ is a topology base for $X_i, \quad i = 1, 2$, then $\mathcal{B}_X = \mathcal{B}_1 \cup \mathcal{B}_2$. According to Theorem 2.1 $\mathcal{B}_\Pi = \{B_1^* \bigcap B_2^* \mid B_1, B_2 \in \mathcal{B}_X, \quad B_1 \bigcap B_2 = \emptyset\}$ is a topology base for $\Pi_{X_1 \bigoplus X_2}$. If $B_1, B_2 \in \mathcal{B}_1$, according to (ii), it holds $B_1^* \bigcap B_2^* = \{X_1\}$. Similarly , if $B_1, B_2 \in \mathcal{B}_2$, then $B_1^* \bigcap B_2^* = \{X_2\}$. Thus $\mathcal{B}_\Pi = \{B_1^* \bigcap B_2^* \mid B_1 \in \mathcal{B}_1, \quad B_2 \in \mathcal{B}_2\} \cup \{\{X_1\}, \{X_2\}\}$. Since for $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ it holds $B_1^* \bigcap B_2^* \subset \Pi'$, we have that

$$\mathcal{B}_{\Pi'} = \{B_1^* \bigcap B_2^* \mid B_1 \in \mathcal{B}_1, \quad B_2 \in \mathcal{B}_2\}$$

is a topology base for the subspace Π' of $\Pi_{X_1 \bigoplus X_2}$.

Let us define $\varphi : X_1 \times X_2 \to \Pi'$ with $\varphi(x_1, x_2) = \{x_1, x_2\}$. Let (x_1, x_2), $(y_1, y_2) \in X_1 \times X_2$ be distinct. For $x_1 \neq y_1$, since $x_1 \neq y_2$ (because $X_1 \bigcap X_2 = \emptyset$), we have $x_1 \notin \{y_1, y_2\}$, that is $\{x_1, x_2\} \neq \{y_1, y_2\}$. Similarly for $x_2 \neq y_2$. Thus φ is "$1 - 1$". If $\{x_1, x_2\} \in \Pi'$, where $x_1 \in X_1, \ \ x_2 \in X_2$, then $\{x_1, x_2\} = \varphi(x_1, x_2)$. Therefore φ is onto.

Let $B_1 \in \mathcal{B}_1, \ \ B_2 \in \mathcal{B}_2$. Now, $(a, b) \in \varphi^{-1}(B_1^* \bigcap B_2^*)$ iff $a \in X_1, \ \ b \in X_2$ and $\{a, b\} \in B_1^* \bigcap B_2^*$, iff $(a, b) \in B_1 \times B_2$. So, it holds $\varphi^{-1}(B_1^* \bigcap B_2^*) = B_1 \times B_2$. φ is continuous. Since φ is a bijection we have $\varphi(B_1 \times B_2) = B_1^* \bigcap B_2^*$, so φ is open . □

A topological property $\mathcal{P}$ is finitely additive iff for each finite collection $X_1, ..., X_k$ of spaces it holds: if $X_1, ..., X_k$ have $\mathcal{P}$, then $\bigoplus_{i=1}^{k} X_i$ has $\mathcal{P}$.

$\mathcal{P}$ is finitely inversely additive iff the opposite implication holds.

Theorem 3.2. *Let $\mathcal{P}$ be a topological property which is finitely additive, finitely inversely additive and which is not finitely multiplicative. Then $\mathcal{P}$ is not Π-invariant.*

Proof. Since $\mathcal{P}$ is not finitely multiplicative, there are two spaces X_1 and X_2 such that X_1, X_2 have $\mathcal{P}$, but $X_1 \times X_2$ has not. $\mathcal{P}$ is finitely additive, so $X_1 \bigoplus X_2$ has $\mathcal{P}$. Suppose that $\mathcal{P}$ is $\Pi-$ invariant. Then $\Pi_{X_1 \bigoplus X_2}$ has $\mathcal{P}$. But $\Pi_{X_1 \bigoplus X_2} \cong (X_1 \times X_2) \bigoplus D_2$. Since $\mathcal{P}$ is inversely additive, $X_1 \times X_2$ has $\mathcal{P}$. A contradiction! Thus, $\mathcal{P}$ is not $\Pi-$ invariant . □

Let us notice that if $\mathcal{P}$ is hereditary to open (closed) sets, then $\mathcal{P}$ is finitely inversely additive.

Theorem 3.3. *The following properties are not Π-invariant: T_4, T_5, T_6, Frechét-Urison space, sequential space, $k-$ space, countable compactness, pseudocompactness, Lindelofness, strong zero-dimensionality, extremal disconnectedness, paracompactness, collectionwise-normality, countable paracompactness, weak paracompactness and strong paracompactness.*

Proof. All of these properties are finitely-additive and are not finitely multiplicative. Except of pseudocompactness, strong zero-dimensionality and extremal disconnectedness, all of the properties are hereditary to closed subspaces, and these properties are finitely inversely additive ([2]). □

The compactness and sequential compactness are not Π-invariants as it is shown by the following example.

Example 1. Let Π be the set of all lines in R^2 and $X = [0,1]^2$. It is easy to show that $\Pi_X = \{p \bigcap X \mid p \in \Pi, \mid p \bigcap X \mid \geq 2\}$ is a topological 2−partition on X (see [5]). Obviously, X is a compact, sequentially compact space. The sets $O_n = X \bigcap \{(x,y) \in R^2 \mid y > \frac{1}{n} - x\}$, $n \in N$ are open in X, thus O_n^*, $n \in N$ are open in Π_X. Since $X = \bigcup_{n \in N} O_n \cup \{(0,0)\}$ and for each $p \in \Pi_X$, $\mid p \mid \geq 2$, we have $\Pi_X = \bigcup_{n \in N} O_n^*$. But this open cover does not contain a finite subcover. (Suppose that $O_{n_1}^*, ..., O_{n_k}^*$ is a finite cover and $m = \max\{n_i \mid i = 1, ..., k\}$. Then, the line $y = \frac{1}{m} - x$ is not covered. A contradiction!). Thus Π_X is not compact. On the other hand, the sequence of lines $p_n \bigcap X$, where p_n is given by $y = \frac{1}{n} - x$, does not contain a convergent subsequence. Π_X is not sequentially compact.

4. Local properties

For the proof of the preserving of some local properties we need the following lemma.

Lemma 4.1. *Let $(X, \mathcal{O})$ be a Hausdorff space and let Π be a topological n−partition of X. If $A_i \subset X$, $i = 1, ..., n$, where $A_i \bigcap A_j = \emptyset$ for $i \neq j$, then $\bigcap_{i=1}^n A_i^*$ is a continuous image of the space $\prod_{i=1}^n A_i$.*

Proof. We define $\varphi : \prod_{i=1}^n A_i \to \bigcap_{i=1}^n A_i^*$ by $\varphi(a_1, ..., a_n) = p(a_1, ..., a_n)$. Since the sets A_i are disjoint, $a_1, ..., a_n$ are distinct elements of X, so φ is well-defined.

If $p \in \bigcap_{i=1}^n A_i^*$, then there are $a_i \in p \bigcap A_i$, $i = 1, ..., n$. Now, $p = p(a_1, ..., a_n) = \varphi(a_1, ..., a_n)$, hence φ is onto.

Let $O = O_1 \bigcap (\bigcap_{i=1}^n A_i^*)$ be open in $\bigcap_{i=1}^n A_i^*$, where $O_1 \in \mathcal{O}_\Pi$, and $(a_1, ..., a_n) \in \varphi^{-1}(O)$. Then $p(a_1, ..., a_n) \in O_1$ and, according to Theorem 2.1, there are neighbourhoods $U_i \in \mathcal{B}(a_i)$, $i = 1, ..., n$, such that $p(a_1, ..., a_n) \in \bigcap_{i=1}^n U_i^* \subset O_1$. Now, for $i = 1, ..., n$, the set $V_i = U_i \bigcap A_i$ is open in A_i and, since $V_i \subset U_i$, we have $\bigcap_{i=1}^n V_i^* \subset \bigcap_{i=1}^n U_i^* \subset O_1$. Also, since $V_i \subset A_i$, it holds $\bigcap_{i=1}^n V_i^* \subset \bigcap_{i=1}^n A_i^*$, which gives $\bigcap_{i=1}^n V_i^* \subset O$. On the other hand, if $(x_1,...,x_n) \in \prod_{i=1}^n V_i$, then $p(x_1,...,x_n) = \varphi(x_1,...,x_n) \in \bigcap_{i=1}^n V_i^* \subset O$, so $(x_1,...,x_n) \in \varphi^{-1}(O)$. Thus $\prod_{i=1}^n V_i \subset \varphi^{-1}(O)$, and clearly $(a_1,...,a_n) \in \prod_{i=1}^n V_i$. Since $\prod_{i=1}^n V_i$ is open in $\prod_{i=1}^n A_i$, $\varphi^{-1}(O)$ is open. φ is continuous . □

Theorem 4.1. *If $(X,\mathcal{O})$ is a Hausdorff space and Π is a topological $n-$partition of X, then:*

(i) if X is locally compact, then Π is locally compact;

(ii) if X is locally sequentially compact, then Π is locally sequentially compact;

(iii) if X is locally connected, then Π is locally connected.

Proof. (i) Let $p \in \Pi$ and $x_1,...,x_n \in p$. Let U_i, $i = 1,...,n$ be disjoint neighbourhoods at the points $x_1,...,x_n$. Since X is locally compact, for each $i \in \{1,...,n\}$ there is a neighbourhood V_i of x_i such that $\overline{V}_i \subset U_i$ and $\overline{V}_i$ is compact. According to Lemma 4.1, $\bigcap_{i=1}^n \overline{V_i^*}$ is a continuous image of the compact space $\prod_{i=1}^n \overline{V}_i$, so it is a compact neighbourhood of p.

(ii) Similarly to (i).

(iii) Let $p \in \Pi$ and let U be an arbitrary neighbourhood of p. For $x_1,...,x_n \in p$, according to Theorem 2.1, there are disjoint neighbourhoods U_i, $i = 1,...,n$, such that $\bigcap_{i=1}^n U_i^* \subset U$. Since X is locally connected, for each $i \in \{1,...,n\}$ there is a connected neighbourhood V_i of x_i, contained in U_i. By the last lemma , $\bigcap_{i=1}^n V_i^*$ is a connected neighbourhood of p contained in U. □

5. Connectedness

Let n be a positive integer and let X be a nonempty set such that $\mid X \mid\geq n$. A mapping $\Gamma : [X]^n \to \{U,V\}$ is a colouring of $[X]^n$ into two colours (U and V). A colouring is monochromatic if Γ is a constant function. Let us define

$$\Gamma^c(\alpha) = \begin{cases} U & \text{if } \Gamma(\alpha) = V \\ V & \text{if } \Gamma(\alpha) = U \end{cases} .$$

for all $\alpha \in [X]^n$.

The set $\beta \in [X]^{n-1}$ is separating in X iff there are $a, b \in X - \beta$ such that $\Gamma(\beta \cup \{a\}) \neq \Gamma(\beta \cup \{b\})$. We define $s(X) = \{\beta \in [X]^{n-1} \mid \beta \text{ is separating}\}$ and for $\beta \in s(X)$ $G_U^\beta = \{x \in X - \beta \mid \Gamma(\beta \cup \{x\}) = U\}$, $G_V^\beta = \{x \in X - \beta \mid \Gamma(\beta \cup \{x\}) = V\}$.

The smallest family $\mathcal{O} \subset P(X)$ satisfying: (i) $G_U^\beta, G_V^\beta \in \mathcal{O}$, for each $\beta \in s(X)$; (ii) if $A, B \in \mathcal{O}$, then $A \bigcap B \in \mathcal{O}$; (iii) if $A, B \in \mathcal{O}$, then $A \cup B \in \mathcal{O}$; is the family of open sets (generated by the colouring Γ). The use of the same word in topology would not make a confusion.

Theorem 5.1. *If the colouring Γ of $[X]^n$ is not monochromatic, then X is a finite union of disjoint open sets.*

Proof. Let Γ be nonmonochromatic. We divide the proof into several lemmas.

Lemma 5.1. *There is $A \subset X$ where $| A |= n+1$ such that $\Gamma \mid [A]^n$ is not monochromatic.*

Proof. Suppose that for each $\{x_1, ..., x_n\} \in [X]^n$ coloured in U for every $t \notin \{x_1, ..., x_n\}$, $\Gamma \mid [\{x_1, ..., x_n, t\}]^n$ is monochromatic (coloured in U). Let $\Gamma(\{x_1, ..., x_n\}) = U$ and $\{y_1, ..., y_n\} \in [X]^n$. Then, by assumption, $[\{x_1, ..., x_n, y_1\}]^n$ is coloured in U, so $\Gamma(\{x_1, ..., x_{n-1}, y_1\}) = U$. Now, $[\{x_1, ..., x_{n-1}, y_1, y_2\}]^n$ is coloured in U and we have $\Gamma(\{x_1, ..., x_{n-2}, y_1, y_2\}) = U$, etc. At last, we have $\Gamma(\{y_1, ..., y_n\}) = U$. If $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$ have common elements, the proof is similar. Now, Γ is monochromatic. A contradiction!

Thus, there is $\{x_1, ..., x_n\} \in [X]^n$ and $t \notin \{x_1, ..., x_n\}$ such that $\Gamma \mid [\{x_1, ..., x_n, t\}]^n$ is not monochromatic. □

For the set A from the previous lemma we define

$$s(A) = \{\beta \in [A]^{n-1} \mid \beta \quad \text{is a separating set in } A\}$$

$$s(x) = \{\beta \in s(A) \mid x \notin \beta\}, \quad x \in A.$$

Lemma 5.2. *(i) $s(A) \neq \emptyset$; (ii) for each $x \in A$, $s(x) \neq \emptyset$.*

Proof. (i) Two elements of $[A]^n$ of different colour have $n-1$ common points.

(ii) Let $A = \{a_1, ..., a_{n+1}\}$. Suppose $s(a_1) = \emptyset$, that is for all $\beta \in s(A)$, $a_1 \in \beta$. Let $\Gamma(\{a_1, ..., a_n\}) = U$. Then $\Gamma(\{a_2, ..., a_{n+1}\}) = U$ (otherwise $\{a_2, ..., a_n\} \in s(a_1)$). Now, $\{a_1, a_{k_1}, ..., a_{k_{n-1}}\}$ and $\{a_2, ..., a_{n+1}\}$ have $n-1$ common elements different from a_1, hence $\Gamma(\{a_1, a_{k_1}, ..., a_{k_{n-1}}\}) = U$. Thus $\Gamma \mid A$ is monochromatic. A contradiction! Similarly if $\Gamma(\{a_1, ..., a_n\}) = V$. Finally we have $s(a_1) \neq \emptyset$. For a_i, $i = 2, ..., n+1$ the proof is analogous. □

Lemma 5.3. *For all* $x \in A$, $x \in \bigcap_{\beta \in s(x)} G^{\beta}_{\Gamma^c(A-\{x\})}$.

Proof. Let $x \in A$ and $\beta \in s(x)$, where $A = \beta \bigcup \{x, y\}$. Since β is a separating set in A, we have $\Gamma(\beta \cup \{x\}) = \Gamma^c(\beta \cup \{y\}) = \Gamma^c(A - \{x\})$. Now, $x \in G^{\beta}_{\Gamma(\beta \cup \{x\})} = G^{\beta}_{\Gamma^c(A-\{x\})}$. □

We define $A_U = \{x \in A \mid \Gamma^c(A - \{x\}) = U\}$, $A_V = \{x \in A \mid \Gamma^c(A - \{x\}) = V\}$ and

$$O_U = \bigcup_{x \in A_U} \bigcap_{\beta \in s(x)} G^{\beta}_{\Gamma^c(A-\{x\})} \text{ and } O_V = \bigcup_{x \in A_V} \bigcap_{\beta \in s(x)} G^{\beta}_{\Gamma^c(A-\{x\})}.$$

The last lemma gives $A_U \subset O_U$, $A_V \subset O_V$, thus we have $A = A_U \cup A_V \subset O_U \cup O_V = O_A$, where $O_A = \bigcup_{x \in A} \bigcap_{\beta \in s(x)} G^{\beta}_{\Gamma^c(A-\{x\})}$.

Lemma 5.4. *O_U and O_V are disjoint, open sets.*

Proof. Suppose that $t \in O_U \bigcap O_V$. Then, there are $x \in A_U$ and $y \in A_V$ such that $t \in (\bigcap_{\beta \in s(x)} G^{\beta}_{\Gamma^c(A-\{x\})}) \bigcap (\bigcap_{\beta \in s(y)} G^{\beta}_{\Gamma^c(A-\{y\})})$. Since $\Gamma^c(A - \{x\}) = U$ and $\Gamma^c(A - \{y\}) = V$, we have $\beta_0 = A - \{x, y\} \in s(x) \cup s(y)$. Now, $\bigcap_{\beta \in s(x)} G^{\beta}_{\Gamma^c(A-\{x\})} \subset G^{\beta_0}_{\Gamma^c(A-\{x\})} = G^{\beta_0}_U$, and $\bigcap_{\beta \in s(y)} G^{\beta}_{\Gamma^c(A-\{y\})} \subset G^{\beta_0}_{\Gamma^c(A-\{y\})} = G^{\beta_0}_V$, so we have $t \in G^{\beta_0}_U \bigcap G^{\beta_0}_V = \emptyset$. A contradiction! □

Lemma 5.5. $X - O_A = \bigcap_{x \in A} \bigcup_{\beta \in s(x)} G^{\beta}_{\Gamma(A-\{x\})}$, *that is* $X - O_A$ *is open.*

Proof. Since $A \subset O_A$, we have

$$O_A^c = O_A^c - A = [\bigcap_{x \in A} \bigcup_{\beta \in s(x)} (G^{\beta}_{\Gamma(A-\{x\})} \cup \beta)] - A$$

$$= \bigcap_{x \in A} \bigcup_{\beta \in s(x)} [(G^{\beta}_{\Gamma(A-\{x\})} - A) \cup (\beta - A)] = \bigcap_{x \in A} \bigcup_{\beta \in s(x)} (G^{\beta}_{\Gamma(A-\{x\})} - A)$$

$$\subset \bigcap_{x \in A} \bigcup_{\beta \in s(x)} G^{\beta}_{\Gamma(A-\{x\})} \subset O^c_A,$$

which gives the equality. □

Now, $X = O_A \cup O^c_A = O_U \cup O_V \cup O^c_A$, that is X is the union of disjoint, open sets . □

Let $(X, \mathcal{O})$ be a Hausdorff space, and let Π be a topological n-partition of X. Then it holds

Theorem 5.2. *If X is connected, then Π is connected.*

Proof. Suppose that Π is disconnected, that is $\Pi = U \cup V$, where U and V are nonempty, disjoint, open sets. Let us define $\Gamma : [X]^n \to \{U, V\}$ by

$$\Gamma(\{x_1, ..., x_n\}) = \begin{cases} U & \text{if} \quad p(x_1, ..., x_n) \in U \\ V & \text{if} \quad p(x_1, ..., x_n) \in V. \end{cases}$$

Obviously, Γ is a well-defined colouring which is not monochromatic. By Lemma 5.1, there exists a separating set β in $[X]^{n-1}$, that is the set $s(X) = \{\beta \in [X]^{n-1} \mid \exists a, b \in X - \beta \quad \Gamma(\beta \bigcup \{a\}) \neq \Gamma(\beta \cup \{b\})\}$ is nonempty.

Let $\beta = \{a_1, ..., a_{n-1}\} \in s(X)$. We will prove that the sets G^{β}_U and G^{β}_V defined in the first part of this paragraph are open in X in topological sense.

Let $x \in G^{\beta}_U$. Then $p(a_1, ..., a_{n-1}, x) \in U$. According to Theorem 2.1, there is $(B_1, ..., B_n) \in \prod_{i=1}^{n-1} \mathcal{B}(a_i) \times \mathcal{B}(x)$ such that $p(a_1, ..., a_{n-1}, x) \in \bigcap_{i=1}^{n} B^*_i \subset U$. For $y \in B_n$ it holds $p(a_1, ..., a_{n-1}, y) \in \bigcap_{i=1}^{n} B^*_i \subset U$, thus $y \in G^{\beta}_U$, which proves $B_n \subset G^{\beta}_U$. Now, G^{β}_U is a neighbourhood of x, so G^{β}_U is open. Similarly for G^{β}_V.

According to Theorem 5.1, X is the union of disjoint, open (in topological sense) sets, so X is disconnected . □

Π can be connected although X is not connected, as will be shown by the following example.

Example 1. Let Π be the set of all lines in R^2, and let $X \subset R^2$ be the union of two disjoint, open discs, D_1 and D_2. Then (see [5]) $\Pi_X =$

$\{p \bigcap X \mid p \in \Pi, | p \bigcap X | \geq 2\}$ is a topological 2 -partition of X. Let us prove that the Π_X is connected. Suppose that $\Pi_X = U \cup V$, where U and V are open, disjoint and nonempty. Then (as in the previous theorem) there is a separating point x, such that $X = \{x\} \cup G_U^x \cup G_V^x$. Suppose that $x \in D_1$. Now, $D_1 - \{x\} \subset G_U^x \cup G_V^x$, and since $D_1 - \{x\}$ is connected, we have $D_1 \cup G_U^x = \emptyset$ or $D_1 \bigcap G_V^x = \emptyset$. The same conclusion holds for D_2. Let $D_2 \subset G_V^x$. Then $D_1 \bigcap G_V^x = \emptyset$ (otherwise $G_U^x = \emptyset$). Finally, we have $D_1 = \{x\} \cup G_U^x$, $D_2 = G_V^x$. Let $z \in D_2$. The line $p(x,z)$ have infinitely many common points with D_1. Choose $y \in p(x,z) \bigcap (D_1 - \{x\})$. Now, $p(x,y) = p(x,z) \in U \bigcap V = \emptyset$. A contradiction!

Example 2. Let X be as in the preceding example and $\Pi_X = [X]^2$. Define $U_1 = \{p \in \Pi_X \mid p \subset D_1\}$, $U_2 = \{p \in \Pi_X \mid p \subset D_2\}$ and $U_{12} = \{p \in \Pi_X \mid p \bigcap D_1 \neq \emptyset \wedge p \bigcap D_2 \neq \emptyset\}$. Then we have $\Pi_X = U_1 \cup U_2 \cup U_{12}$, where U_1, U_2 and U_{12} are disjoint open sets. Π_X is disconnected.

6. n-partition of a compact

Let 2^X denote the set of all nonempty, closed subsets of a space $(X, \mathcal{O})$. If $U_1, ..., U_k$ are open in X, we define:

$$\mathcal{V}(U_1, ..., U_k) = \{F \in 2^X \mid F \subset \bigcup_{i=1}^{k} U_i \wedge \forall i \in \{1, ..., k\} F \bigcap U_i \neq \emptyset\}.$$

The family $\mathcal{B}_{2^X} = \{\mathcal{V}(U_1, ..., U_k) \mid U_i \in \mathcal{O}, \ i = 1, ..., k; \ k \in N\}$ is the base for the Vietoris topology on 2^X, denoted by $2^{\mathcal{O}}$, which has been investigated by Michael in [6]. It is easy to verify that the family $\mathcal{P}_{2^X}$ which consists of all sets $U^*_{2^X} = \{F \in 2^X \mid F \bigcap U \neq \emptyset\}$ and $U^0_{2^X} = \{F \in 2^X \mid F \subset U\}$, where $U \in \mathcal{O}$, is a subbase for $2^{\mathcal{O}}$.

If Π is a topological $n-$partition of X, then $\Pi \subset 2^X$. Now we consider the induced topology $2^{\mathcal{O}}_{\Pi} = \{O \bigcap \Pi \mid O \in 2^{\mathcal{O}}\}$.

Theorem 6.1. *Let $(X, \mathcal{O})$ be a Hausdorff, compact space. Then Π is a subspace of 2^X with the Vietoris topology, that the is $\mathcal{O}_{\Pi} = 2^{\mathcal{O}}_{\Pi}$.*

Proof. ($\subset$) Obviously, for $U \in \mathcal{O}$ we have $U^* = U^*_{2^X} \bigcap \Pi \in 2^{\mathcal{O}}_{\Pi}$. Thus $\mathcal{P}_{\Pi} = \{U^* \mid U \in \mathcal{O}\} \subset 2^{\mathcal{O}}_{\Pi}$ and it holds $\mathcal{O}_{\Pi} \subset 2^{\mathcal{O}}_{\Pi}$.

($\supset$) Let us prove that $(\mathcal{P}_{2^X})_\Pi \subset \mathcal{O}_\Pi$. Since $U^*_{2^X} \bigcap \Pi = U^*$ it remains to prove that $U^0_{2^X} \bigcap \Pi \in \mathcal{O}_\Pi$, for each $U \in \mathcal{O}$. Let $p \in U^0_{2^X} \bigcap \Pi$, that is $p \subset U$, and let $x^1, ..., x^n \in p$ be distinct points. Since X is Hausdorff, we may assume that the elements of the local bases $\mathcal{B}(x^i)$, $i = 1, ..., n$, are disjoint (for different points) and contained in U. According to Theorem 2.1,

$$\mathcal{B}_{(p)} = \{\bigcap_{i=1}^{n} B_i \mid (B_1, ..., B_n) \in \prod_{i=1}^{n} \mathcal{B}(x^i)\}$$

is a local base at the point $p \in \Pi$.

Suppose $\forall (B_1, ..., B_n) \in \prod_{i=1}^n \mathcal{B}(x^i)$, $\exists p_{(B_1,...,B_n)} \in (\bigcap_{i=1}^n B_i^*) \bigcap (X - U)^*$. Let $x^i_{(B_1,...,B_n)} \in p_{(B_1,...,B_n)} \bigcap B_i$, for all $(B_1, ..., B_n)$ and all i. Then for all $i \in \{1, ..., n\}$ we have $< x^i_{(B_1,...,B_n)} > \to x^i$. Choose $y_{(B_1,...,B_n)} \in p_{(B_1,...,B_n)} \bigcap (X - U)$. Since the set $X - U$ is compact, the net $< y_{(B_1,...,B_n)} \mid (B_1, ..., B_n) \in \prod_{i=1}^n \mathcal{B}(x^i) >$ must have an accumulation point $y \in X - U$. Let $< y_\sigma \mid \sigma \in \Sigma >$ be a subnet of the net $< y_{(B_1,...,B_n)} >$ converging to y. Then for the corresponding subnets $< x^i_\sigma \mid \sigma \in \Sigma >$ we have $< x^i_\sigma > \to x^i$, $i = 1, ..., n$, while $y_\sigma \in p(x^1_\sigma, ..., x^n_\sigma)$, $\sigma \in \Sigma$. Since Π is a topological n-partition, we have $y \in p(x^1, ..., x^n) = p \subset U$. A contradiction!

Thus there is $(B_1, ..., B_n) \in \prod_{i=1}^n \mathcal{B}(x^i)$, such that $\bigcap_{i=1}^n B_i^* \subset U^0_{2^X} \bigcap \Pi$. Therefore $U^0_{2^X} \bigcap \Pi \in \mathcal{O}_\Pi$ and we have $(\mathcal{P}_{2^X})_\Pi \subset \mathcal{O}_\Pi$, which gives $2^{\mathcal{O}}_\Pi \subset \mathcal{O}_\Pi$. □

If X is a compact, then $2^X = C(X)$, where $C(X)$ is the set of all nonempty, compact subsets of X. By the previous theorem, Π is a subspace of $C(X)$ with the Vietoris topology.

Before the following theorem we mention that with

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \quad \sup_{b \in B} d(b, A)\}$$

is defined the Hausdorff metric for the space of all nonempty, closed and bounded subsets of a metric space (X, d).

Theorem 6.2. *Let $(X, \mathcal{O})$ be a compact Hausdorff space and let Π be a topological n-partition of X. Then*

(i) if X is metrizable, then Π is metrizable by d_H;

(ii) if X is totally disconnected, Π is too ;

(iii) if X is zero-dimensional, Π is too .

Proof. (i) By [6], if X is compact, then $(2^X, 2^{\mathcal{O}})$ is metrizable with d_H.

(ii) By [6], total disconnectedness of X imply total disconnectedness of $C(X)$. Since this property is hereditary, the proof is over.

(iii) Similar to (ii) . □

By [6], the compactness of X implies compactness of 2^X. Thus, Π must not be closed in 2^X, although X is compact (see Example 3.1). Π can be closed in 2^X as it will be shown by the following example.

Example 1. If $I = [0,1]$, then $X = I \bigoplus I$ is compact. According to Theorem 3.1, $\Pi_{I \bigoplus I} \cong I^2 \bigoplus D_2$ is also compact, so $\Pi_{I \bigoplus I}$ is closed in 2^X.

Example 2. In Example 3.1 X is a compact, metric space, but Π is not countably compact.

Since Π is metrizable, it is T_4. The pseudocompactness of Π with T_4, would imply countable compactness (see [2]). So, Π is not pseudocompact.

7. The partition $[X]^n$

If X is a Hausdorff space, $\Pi = [X]^n$ is a topological $n-$ partition of X (see [5]). Clearly, it holds that $\Pi \subset C(X) \subset 2^X$.

Theorem 7.1. *If* $\Pi = [X]^n$, *then* $\mathcal{O}_\Pi = 2^{\mathcal{O}}_\Pi$.

Proof. ($\subset$) This inclusion always holds (see paragraph 6).

($\supset$) According to Theorem 6.1, it remains to prove that $U^0_{2^X} \bigcap \Pi \in \mathcal{O}_\Pi$, for all $U \in \mathcal{O}$. Let $U \in \mathcal{O}$ and $p = \{x^1, ..., x^n\} \in U^0_{2^X} \bigcap \Pi$, that is $x^1, ..., x^n \in U$. Since X is Hausdorff, there are disjoint neighbourhoods $B_i \in \mathcal{B}(x^i)$, $i = 1, ..., n$; such that $B_i \subset U$. Then, $p \in \bigcap_{i=1}^n B_i^* \subset U^0_{2^X} \bigcap \Pi$, thus $U^0_{2^X} \bigcap \Pi$ is open. □

Theorem 7.2. *Let $\mathcal{P}$ be a topological property such that (i) if the space X has $\mathcal{P}$, then $C(X)$ (with the Vietoris topology) has $\mathcal{P}$; (ii) $\mathcal{P}$ is hereditary. Then, if X has $\mathcal{P}$, then $[X]^n$ is $\mathcal{P}$.*

Proof. Since $[X]^n$ is a subspace of $C(X)$, the assertion follows . □

Theorem 7.3. *Let X be a Hausdorff space. Then, if X is T_3($T_{3\frac{1}{2}}$, metrizable, totally disconnected, zero-dimensional, without isolated points), then $[X]^n$ has the same property.*

Proof. If X has any of these properties, then $C(X)$ has it too (see [6]). All of these properties are hereditary, and the previous theorem holds. □

For the proof of a stronger version of Theorem 7.2 we need:

Lemma 7.1. *(i) $[X]^{\leq n}$ is a closed subset of $C(X)$. (ii) $[X]^n$ is an open subset of $[X]^{\leq n}$.*

Proof. (i) Let $A \in 2^X - [X]^{\leq n}$. Then $\mid A \mid > n$, so we have $x^1, ..., x^{n+1} \in A$. If $U_i \in \mathcal{B}(x^i)$, $i = 1, ..., n+1$ are disjoint neighbourhoods of these points, we have $[\bigcap_{i=1}^{n+1} (U_i)^*_{2X}] \bigcap [X]^{\leq n} = \emptyset$. Now, $[X]^{\leq n}$ is closed in 2^X and, since $[X]^{\leq n} \subset C(X), [X]^{\leq n}$ is closed in $C(X)$.

(ii) Let $A = \{x^1, ..., x^n\} \in [X]^n$. If $U_i \in \mathcal{B}(x^i)$, $i = 1, ..., n$; are disjoint, then $\bigcap_{i=1}^{n} (U_i)^*_{2X} \bigcap [X]^{\leq n}$ is open in $[X]^{\leq n}$, it contains A and it is contained in $[X]^n$. □

Theorem 7.4. *Let $\mathcal{P}$ be a topological property such that: (i) if X has $\mathcal{P}$, then $C(X)$ has $\mathcal{P}$; (ii) $\mathcal{P}$ is hereditary to closed sets; (iii) $\mathcal{P}$ is hereditary to open sets. Then, if X has $\mathcal{P}$, then $[X]^n$ has $\mathcal{P}$.*

Proof. If X has $\mathcal{P}$ then, by (i),. $C(X)$ has $\mathcal{P}$. According to Lemma 7.1, then $[X]^n$ has $\mathcal{P}$. □

Theorem 7.5. *Let X be a Hausdorff space. Then, if X is Čech-complete, then $[X]^n$ is Čech-complete.*

Proof. If X is Čech-complete, so is $C(X)$ (see [1]). Čech-completness is hereditary to open and closed sets (see [2]) and we can apply the last theorem. □

Theorem 7.6. *Let Π be a topological $n-$partition of a Hausdorff space X. Then the mapping $\varphi : [X]^n \to \Pi$ given by $\varphi(\{x^1, ..., x^n\}) = p(x^1, ..., x^n)$ is an open surjection.*

Proof. Obviously, φ is a well-defined surjection. To avoid the confusion we introduce the following notation: for $O \in \mathcal{O}$, $O^* = \{p \in \Pi \mid p \bigcap O \neq \emptyset\}$ and $O^+ = \{\tau \in [X]^n \mid \tau \cup O \neq \emptyset\}$.

Let $B_1, ..., B_n \in \mathcal{O}$ be disjoint. Let us prove $\varphi(\bigcap_{i=1}^n B_i^+) = \bigcap_{i=1}^n B_i^*$.

$(\subset)$ If $\{x^1, ..., x^n\} \in \bigcap_{i=1}^n B_i^+$, since $\{x^1, ..., x^n\} \subset p(x^1, ..., x^n) = \varphi(\{x_1, ..., x_n\})$, we have $\varphi(\{x^1, ..., x^n\}) \in \bigcap_{i=1}^n B_i^*$.

$(\supset)$ If $p \in \bigcap_{i=1}^n B_i^*$, then there are distinct points $x^i \in p \bigcap B_i, i = 1, ..., n$; so we have $\{x^1, ..., x^n\} \in \bigcap_{i=1}^n B_i^+$, that is $p = \varphi(\{x^1, ..., x^n\}) \in \varphi(\bigcap_{i=1}^n B_i^+)$.

Suppose $O \in \mathcal{O}$ and $\{x^1, ..., x^n\} \in \varphi^{-1}(O^*)$. Then $p(x^1, ..., x^n) \in O^*$ and by Theorem 2.1 there are disjoint neighbourhoods $B_i \in \mathcal{B}(x^i), i = 1, ..., n$: such that $p(x^1, ..., x^n) \in \bigcap_{i=1}^n B_i^* \subset O^*$. Now, $\{x^1, ..., x^n\} \in \bigcap_{i=1}^n B_i^+ \subset \varphi^{-1}(\varphi(\bigcap_{i=1}^n B_i^+)) \subset \varphi^{-1}(\bigcap_{i=1}^n B_i^*) \subset \varphi^{-1}(O^*)$. Thus $\varphi^{-1}(O^*)$ is open, and φ is continuous.

The openness of φ follows directly from the proved equality. □

Theorem 7.7. *Let $\mathcal{P}$ be a topological property such that: (i) if X has $\mathcal{P}$, then $[X]^n$ has $\mathcal{P}$; (ii) $\mathcal{P}$ is preserved by open mappings. Then $\mathcal{P}$ is a $\Pi-$ invariant.*

Proof. Let X has $\mathcal{P}$ and let Π be an arbitrary topological $n-$partition of X. By (i), $[X]^n$ has $\mathcal{P}$ and by (ii) and Theorem 7.6, Π also has $\mathcal{P}$. □

Theorem 7.8. *The following properties do not transfer from X to $[X]^n$: (i) to be a Frechét-Urison space, (ii) sequential space, (iii) $k-$ space , (iv) compactness, (v) countable compactness, (vi) pseudocompactness , (vii) sequential compactness, (viii) Lindelofness, (ix) extremal disconnectedness.*

Proof. All of these properties are not Π invariants, but they are invariants of open mappings. □

Example 1. ($[X]^2$ is not T_4, although X is T_6). The Sorgenfrey line, denoted by X, is a T_6 (perfectly normal) space (see [2]). If $Y = \{(x, y) \in X^2 \mid x < y\}$, then the mapping $\varphi : Y \to [X]^2$ is obviously a bijection. One base for the topology on Y consists of the half-open squares of shape $[a, b) \times [c, d)$, where $a < b \leq c < d$. It is easy to show that $\varphi([a, b) \times [c, d)) = [a, b)^* \bigcap [c, d)^*$, so φ is a homeomorphism, that is $[X]^2 \cong Y$. But Y is not

T_4 (the closed sets $A = \{(-q,q) \mid q \in Q \bigcap (0,\infty)\}$ and $B = \{(-i,i) \mid i \in (R-Q)\bigcap(0,\infty)\}$ can not be separated by disjoint open sets). Therefore $[X]^2$ is not T_4.

8. Partition of a locally compact space

Let X be a locally compact Hausdorff space and let Π be a topological $n-$ partition of X. According to Theorem 4.1 , Π is locally compact, so it is $T_{3\frac{1}{2}}$. But if X is T_4 (or T_5), Π need not be a T_4 space, as will be shown by the following example.

Example 1. According to [2], the ordinal spaces $W_0 = \{\mu \mid \mu < \omega_1\}$ and $W = W_0 \cup \{\omega_1\}$ with the order topology are T_5 spaces. Moreover, W is compact, while W_0 is locally compact. Thus $X = W_0 \bigoplus W$ is a locally compact, T_5 space. Also (by [4]) $W_0 \times W$ is not T_4. By Theorem 1.1, $\Pi_{W_0 \bigoplus W}$ is a topological $n-$ partition of X, and it is homeomorphic to $(W_0 \times W) \bigoplus D_2$. Hence, $\Pi_{W_0 \bigoplus W}$ is not T_4, because its closed subspace $W_0 \times W$ is not T_4.

Some forms of disconnectedness are Π invariants if X is a Hausdorff locally compact space. But firstly, let us remind that the following fact holds (see [2]).

Let X be a Hausdorff, locally compact space. Then the following conditions are equivalent: (i) X is zero-dimensional (ii) X is totally disconnected and (iii) X is hereditarily disconnected. □

Now, we can prove:

Theorem 8.1. *Let X be a Hausdorff, locally compact space. If X is zero-dimensional (totally disconnected, hereditarily disconnected), then* Π *is zero-dimensional.*

Proof. Under the given conditions, for $p \in \Pi$ and distinct points $x^1, ..., x^n \in p$, there exist disjoint compact neighbourhoods $\overline{U}_i$, $i = 1, ..., n$. Let $\mathcal{B}$ be a base for the topology on X, which consists of clopen sets. Then $\mathcal{B}'(x^i) = \{B \in \mathcal{B} \mid x^i \in B \subset U_i\}$ is a local base at the point x^i, where $i = 1, ..., n$. The elements of $\mathcal{B}'(x^i)$ are closed in $\overline{U}_i$, so they are compact. Now, according to

Theorem 2.1,

$$\mathcal{B}'(p) = \{\bigcap_{i=1}^{n} B_i \mid (B_1, ..., B_n) \in \prod_{i=1}^{n} \mathcal{B}'(x^i)\}$$

is a local base at p in Π. The elements of $\mathcal{B}'(p)$ are open in Π, but, by Lemma 4.1, they are compact and closed (since Π is Hausdorff). Now $\mathcal{B}_\Pi = \bigcup_{p\in\Pi} \mathcal{B}'(p)$ is a base for the topology on Π which consists of clopen sets. □

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