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A SPLINE DIFFERENCE SCHEME FOR BOUNDARY VALUE PROBLEM WITH SMALL PARAMETER

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Abstract

For the problem: $\varepsilon y" + p(x)y = f(x)$, $y(0) = \alpha_0$, $y(0) = \alpha_1$, the spline difference scheme having the second order of uniform convergence and fourth order of classical convergence is given. The scheme is a linear combination of two exponential spline difference schemes: the scheme from [2] and the one from the family given in [6]. Both of them have a second order of classical convergence. Numerical results are presented.

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1. Introduction

Let us consider the following singularly perturbed problem

$$(1)\qquad \begin{cases} Ly = -\varepsilon y'' + p(x)y = f(x), & x \in I = [0,1], \\ y(0) = \alpha_0, \quad y(1) = \alpha_1, \end{cases}$$

where $\varepsilon \in (0, \varepsilon_0), \varepsilon_0 \ll 1$, is a small perturbation parameter. The functions p and f are given and we assume

$$p, f \in C^4(I), \quad p(x) \geq \beta > 0, \quad x \in [0,1].$$

It is known that the problem (1) has a unique solution y, which in general displays boundary layers at x=0 and x=1. The difference scheme having the second order of the classical and second order of the uniform convergence is given in [2]. In [7] and [8] are presented the schemes having the convergence in respect to ε.

The family of difference schemes derived in [6] contains two schemes which have the second order of the classical and second order of the uniform convergence. One of them is the scheme from [2]. The analysis of the truncation errors schows that the constant near the leading member for the new scheme is four times smaller than the corresponding constant for the scheme in [2]. This fact suggested the linear combination which gives a better accuracy when the step of discretization is smaller than the perturbation parameter. The scheme preserves tridiagonal form.

2. Derivation of the scheme

Under the summoned assumptions the exact solution has the form ([9]):

$$y(x) = v(x) + w(x) + g(x), \tag{2}$$

where

$$v(x) = q_0 v_0(x), \quad w(x) = q_1 w_0(x),$$

$$v_0(x) = e^{-x\sqrt{\frac{p(0)}{\varepsilon}}} \quad w_0(x) = e^{(x-1)\sqrt{\frac{p(1)}{\varepsilon}}},$$

where q_0 and q_1 are bounded functions of ε independent of x and

$$\left|g^{(i)}(x)\right| \leq M\left(1 + \varepsilon^{\frac{1-i}{2}}\right), \quad i = 0, 1, 2, 3, 4.$$

In [6], a family of difference schemes via spline in tension on uniform mesh are derived.

One of them is the scheme from [2]:

$$(3) \qquad \begin{cases} r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = q^- f_{j-1} + q^c f_j + q^+ f_{j+1}, \\ u_0 = \alpha_0, \quad u_1 = \alpha_1, j = 1(1)n \end{cases}$$

where

$$r_j^- = -R(\rho_2^-), \quad r_j^+ = -R(\rho_2^+), \quad r_j^c = G(\rho_2^-) + G(\rho_2^+),$$

$$G(\rho) = \rho\coth(h\rho), \quad R(\rho) = \frac{\rho}{\sinh(h\rho)},$$

$$\rho_2^- = ((p_{j-1} + p_j)/(2\varepsilon))^{1/2} \quad \rho_2^+ = ((p_{j+1} + p_j)/(2\varepsilon))^{1/2},$$

$$q_j^+ = \frac{1}{2\varepsilon(\rho_2^+)^2}(G(\rho_2^+) - R(\rho_2^+)), \; q_j^- = \frac{1}{2\varepsilon(\rho_2^-)^2}(G(\rho_2^-) - R(\rho_2^-)), \; q_j^c = q_j^- + q_j^+.$$

The other scheme from [6] has the form:

$$(4) \qquad \begin{cases} r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = q^\mp f_{j-1/2} + q^\pm f_{j+1/2}, \\ u_0 = \alpha_0, \quad u_1 = \alpha_1, j = 1(1)n \end{cases}$$

where

$$r_j^- = -R(\rho_1^-), \quad r_j^+ = -R(\rho_1^+), \quad r_j^c = G(\rho_1^-) + G(\rho_1^+),$$

$$\rho_1^- = (p_{j-1/2}/\varepsilon)^{1/2}, \quad \rho_1^+ = (p_{j+1/2}/\varepsilon)^{1/2},$$

$$q_j^\mp = \frac{1}{\varepsilon(\rho_1^-)^2}(G(\rho_1^-) - R(\rho_1^-)), \; q_j^\pm = \frac{1}{\varepsilon(\rho_1^+)^2}(G(\rho_1^+) - R(\rho_1^+)).$$

The truncation error of the mentioned schemes $\tau_j(y)$,

$$\tau_j(y) = Ry_j - Q(Ly_j),$$

can be written in the form

$$\tau_j(y) = T_{j0}y_j + T_{j1}y_j' + T_{j2}y_j'' + T_{j3}y_j''' + R_{j4}(y),$$

where $T_{j0} = T_{j1} = 0$ for both schemes and $R_{j4}(y)$ contains the remainder terms. In the case $h^2 \leq \varepsilon$, after some Taylor's expansions, we obtain

$$(5) \qquad T_{j2} = \frac{-h^2}{6\varepsilon}(p'(\beta_1) + p'(\beta_2)) + O(\frac{h^3}{\varepsilon})$$

for the first scheme and

(6) $$T_{j2} = \frac{-h^2}{24\varepsilon}(p'(\beta_3) + p'(\beta_4)) + O(\frac{h^3}{\varepsilon})$$

for second scheme, where

$$x_{j-1} < \beta_1 < x_j < \beta_2 < x_{j+1},$$

$$x_{j-1/2} < \beta_3 < x_j < \beta_4 < x_{j+1/2}.$$

Further,

$$\tau_j(g) = -\frac{h^3}{6\varepsilon^{3/2}} + O(h^5/\varepsilon^{5/2})$$

for the first scheme and

$$\tau_j(g) = -\frac{h^3}{24\varepsilon^{3/2}} + O(h^5/\varepsilon^{5/2}).$$

for the second scheme. Since,

$$\tau_j(v) = O(h^5/\varepsilon^{5/2}), \quad \tau_j(w) = O(h^5/\varepsilon^{5/2})$$

for both schemes we obtain the new scheme using a suitable linear combination of the schemes in order to eliminate the members with $h^3/\varepsilon^{3/2}$. Namely, if we multiply the second scheme by -4 and add it to the first one we obtain the scheme:

$$r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = q^- f_{j-1} - 4q^{\mp} f_{j-1/2} + q^c f_j - 4q^{\pm} f_{j+1/2} + q^+ f_{j+1},$$
(7)

$u_0 = \alpha_0, \quad u_1 = \alpha_1, j = 1(1)n,$
where

$$r_j^- = -4r_1^- + r_2^-, \quad r_j^+ = -4r_1^+ + r_2^+,$$

$$r_j^c = -4r_1^c + r_2^c, \quad r_1^- = R(\rho_1^-), \quad r_2^- = R(\rho_2^-),$$

$$r_1^+ = R(\rho_1^+), \quad r_2^+ = R(\rho_2^+),$$

$$r_1^c = G(\rho_1^-) + G(\rho_1^+), \quad r_2^c = G(\rho_2^+) + G(\rho_2^+).$$

Throughout the paper M denotes any positive constant that may take different values in different formulas, but that are allways independent of ε and of discretization mesh.

3. Convergence of the scheme

Theorem 1. *Let $p, f \in C^4[I]$. Let u_j be the approximate value for $y(x_j)$ obtained using scheme (7). Then*

$$|y(x_i) - u_i| \leq M min(h^4/\varepsilon^{3/2}, h^2).$$

Proof. Let $h^2 \leq \varepsilon$. Let A be a matrix of the system (7). From the inequality

$$|y(x_j) - u_j| \leq ||A^{-1}|| \max_j |\tau_j(y)|$$

and estimates

$$||A^{-1}|| \leq M\varepsilon/h, \ |\tau_j(y)| \leq Mh^5/\varepsilon^{5/2}$$

we obtain the proof. When $\varepsilon < h^2$, the proof follows from the proofs of the uniform convergence for the first and second scheme [2] and [6].

Theorem 2. *Let the assumptions of the previous theorem are fulfilled. Let $p'(0) = p'(1) = 0$. Then*

$$|y(x_i) - u_i| \leq M min(h^4/\varepsilon, h^2).$$

Proof. Since ([1]) the estimate

$$\left|g^{(i)}(x)\right| \leq M\left(1 + \varepsilon^{\frac{2-i}{2}}\right), \quad i = 0, 1, 2, 3, 4.$$

is valid, we obtain the proof using the mentioned technique.

4. Numerical results

In this section we present results of some numerical experiments using the schemes described in the previous theorem. Our example is taken from [1],

$$-\varepsilon y'' + y + \cos^2 \pi x + 2\varepsilon\pi^2 \cos 2\pi x = 0, u(0) = 0, \quad u(1) = 0.$$

The exact solution has the form

$$y(x) = \frac{e^{\frac{-x}{\sqrt{\varepsilon}}} + e^{\frac{x-1}{\sqrt{\varepsilon}}}}{1 + e^{\frac{-1}{\sqrt{\varepsilon}}}} - \cos^2 \pi x.$$

We denote by E_n the maximum of $|y(x_j) - u_j|, j = 0(1)n+1$. Here $[u_0, u_1, \dots, u_{n+1}]^{\mathrm{T}}$ is the corresponding numerical solution obtained by using different schemes. Also, we define in the usual way the order of convergence Ord for two successive values of n with $Q_n = \max_j |u_j^n - u_j^{2n}|$ and Q_{2n}:

$$Ord = \frac{\log Q_n - \log Q_{n_2}}{\log n_2 - \log n},$$

where $n_2 = 2n$ and u_j^n is u_j calculated with $h = 1/n$. Different values of $\varepsilon = 2^k$ and n are considered.

k	n							
	16	32	64	128	256	512	1024	
3	1.86(−2)	4.68(−3)	1.17(−3)	2.92(−4)	7.32(−4)	1.83(−5)	4.56(−6)	E_n
			1.99	1.99	2.00	1.99	2.00	Ord
4	1.62(−2)	4.06(−3)	1.01(−3)	2.54(−4)	6.35(−5)	1.58(−5)	4.00(−6)	E_n
			1.99	1.99	2.00	1.99	1.34	Ord
5	1.43(−2)	3.59(−3)	8.97(−4)	2.24(−4)	5.61(−5)	1.40(−5)	3.50(−6)	E_n
			1.99	1.99	2.00	2.00	2.00	Ord
6	1.91(−2)	1.35(−3)	8.54(−4)	2.82(−4)	6.94(−5)	1.73(−5)	4.32(−6)	E_n
			2.00	1.99	2.00	2.00	1.00	Ord
7	1.30(−2)	2.24(−3)	8.09(−4)	2.02(−4)	5.05(−5)	1.26(−5)	3.15(−6)	E_n
			2.01	2.00	2.00	1.99	2.00	Ord
8	1.31(−2)	3.23(−3)	8.04(−4)	2.00(−4)	5.02(−5)	1.25(−5)	3.13(−6)	E_n
			2.02	2.00	2.00	1.99	1.99	Ord
9	1.33(−2)	3.24(−3)	8.05(−4)	2.00(−4)	5.02(−5)	1.25(−5)	3.13(−6)	E_n
			2.04	2.01	2.00	1.99	1.99	Ord
10	1.38(−2)	3.28(−3)	8.07(−4)	2.01(−4)	5.02(−5)	1.25(−5)	3.13(−6)	E_n
			2.08	2.02	2.01	1.99	1.99	Ord
11	1.46(−2)	3.35(−3)	8.12(−5)	2.01(−4)	5.02(−5)	1.25(−5)	3.13(−6)	E_n
			2.15	2.05	2.01	2.00	1.99	Ord
12	1.57(−2)	3.48(−3)	8.22(−4)	2.02(−4)	5.02(−5)	1.25(−5)	3.13(−6)	E_n
			2.20	2.10	2.03	2.00	2.00	Ord
13			2.21	2.16	2.06	2.01	2.00	Ord
14			2.16	2.21	2.10	2.03	2.00	Ord
15			2.07	2.23	2.16	2.05	2.01	Ord
16			2.01	2.17	2.22	2.10	2.02	Ord
17			1.98	2.09	2.23	2.16	2.05	Ord
18			1.98	2.02	2.18	2.22	2.10	Ord
19			1.98	2.00	2.10	2.22	2.16	Ord
20			1.98	1.99	2.03	2.18	2.22	Ord

Table 1. Scheme (3).

k	n							
	16	32	64	128	256	512	1024	
3	4.76(−3)	1.17(−3)	2.93(−4)	7.32(−5)	1.83(−5)	4.57(−6)	1.13(−6)	E_n
			2.01	2.00	2.00	2.00	1.99	Ord
4	4.14(−3)	1.02(−3)	2.54(−4)	6.35(−5)	1.58(−5)	3.96(−6)	1.02(−6)	E_n
			2.02	2.01	1.99	2.00	201	Ord
5	3.68(−3)	9.03(−4)	5.61(−5)	1.40(−5)	3.50(−6)	8.78(−7)	3.50(−6)	E_n
			2.02	2.01	1.99	2.00	2.00	Ord
6	3.45(−3)	8.40(−4)	2.08(−4)	5.20(−5)	1.30(−5)	3.25(−6)	8.09(−7)	E_n
			2.04	2.01	2.00	2.00	1.99	Ord
7	3.43(−3)	8.21(−4)	2.03(−4)	5.06(−5)	1.26(−5)	3.15(−6)	7.88(−7)	E_n
			2.07	2.02	2.00	2.00	2.00	Ord
8	3.56(−3)	8.26(−4)	2.02(−4)	5.03(−5)	1.25(−5)	3.13(−6)	7.84(−7)	E_n
			2.13	2.03	2.01	2.00	2.00	Ord
9	3.85(−3)	8.45(−4)	2.03(−4)	5.03(−5)	1.25(−5)	3.132(−6)	7.84(−7)	E_n
			2.22	2.06	2.01	2.00	2.00	Ord
10	4.36(−3)	8.83(−4)	2.05(−4)	5.05(−5)	1.25(−5)	3.13(−6)	7.84(−7)	E_n
			2.35	2.12	2.03	2.00	2.00	Ord
11	5.18(−3)	9.54(−4)	2.10(−4)	5.05(−5)	1.25(−5)	3.13(−6)	7.84(−7)	E_n
			2.50	2.22	2.05	2.01	2.00	Ord
12	6.30(−3)	1.08(−3)	2.20(−4)	5.14(−5)	1.26(−5)	3.14(−6)	7.84(−7)	E_n
			2.59	2.36	2.11	2.03	2.01	Ord
13			2.56	2.50	2.21	2.06	2.01	Ord
14			2.40	2.59	2.35	2.11	2.03	Ord
15			2.20	2.56	2.50	2.21	2.06	Ord
16			2.06	2.40	2.59	2.35	2.11	Ord
17			2.01	2.19	2.56	2.50	2.20	Ord
18			2.00	2.06	2.40	2.60	2.34	Ord
19			2.00	2.00	2.20	2.57	2.49	Ord
20			1.99	1.99	2.07	2.39	2.59	Ord

Table 2. Scheme (4)

k	n							
	16	32	64	128	256	512	1024	
3	1.23(−4)	7.70(−6)	4.81(−7)	3.05(−8)	4.76(−8)	1.56(−9)	2.58(−8)	E_n
			4.47	4.00	4.12	2.05	-1.95	Ord
4	1.20(−4)	7.47(−6)	4.67(−7)	2.90(−8)	4.39(−9)	5.98(−9)	3.23(−8)	E_n
			3.99	4.00	4.15	1.25	-1.84	Ord
5	1.28(−4)	8.00(−6)	5.00(−7)	3.14(−8)	1.99(−9)	2.45(−9)	2.08(−9)	E_n
			4.00	4.00	3.99	3.76	-1.03	Ord
6	1.60(−4)	1.00(−5)	6.26(−7)	3.92(−8)	2.31(−9)	4.40(−9)	1.28(−8)	E_n
			3.99	3.99	3.99	3.31	-2.20	Ord
7	2.34(−4)	1.47(−5)	9.23(−7)	5.77(−8)	3.72(−9)	1.32(−9)	8.42(−9)	E_n
			3.99	3.99	4.00	3.89	-0.96	Ord
8	3.85(−4)	2.45(−5)	1.54(−6)	9.66(−8)	6.07(−9)	1.15(−9)	8.71(−9)	E_n
			3.97	3.99	3.99	3.96	-0.38	Ord
9	6.72(−4)	4.40(−5)	2.78(−6)	1.74(−7)	1.09(−8)	1.51(−9)	3.95(−9)	E_n
			3.93	3.98	4.00	3.99	1.98	Ord
10	1.18(−3)	8.19(−5)	5.26(−6)	3.31(−7)	2.07(−8)	1.76(−9)	1.52(−9)	E_n
			3.84	3.95	3.99	3.99	3.94	Ord
11	2.01(−3)	1.53(−4)	1.01(−5)	6.42(−7)	4.03(−8)	2.52(−9)	1.25(−9)	E_n
			3.70	3.91	3.97	3.99	4.00	Ord
12	3.14(−3)	2.81(−4)	1.96(−5)	1.26(−6)	7.93(−8)	4.97(−9)	1.00(−9)	E_n
			3.45	3.83	3.95	3.98	4.00	Ord
13			3.09	3.68	3.91	3.98	3.99	Ord
14			2.68	3.44	3.83	3.95	3.98	Ord
15			2.32	3.10	3.67	3.91	3.97	Ord
16			2.11	2.68	3.44	3.82	3.95	Ord
17			2.02	2.32	3.08	3.68	3.91	Ord
18			2.01	2.09	2.68	3.43	3.83	Ord
19			2.01	2.01	2.32	3.09	3.68	Ord
20			2.01	2.01	2.10	2.68	3.44	Ord

Table 3. Scheme (7).

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