Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 2 (1995), 131-140

Review of Research
Faculty of Science
Mathematics Series

SPECIAL ELEMENTS GENERALIZING MODULARITY IN A LATTICE

Svetozar Milić and Andreja Tepavčević[1]
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper results on modular lattices from [2] are generalized, using special elements of a lattice (modular, standard, co-standard and cancelable). Isomorphisms of some intervals which are generated by special elements are discussed, and a proposition analogue to the Zassenhaus lemma for arbitrary lattices is given. A number of applications in algebra are also given, in particular some corollaries in group theory.

AMS Mathematics Subject Classification (1991): 06C05, 08A30
Key words and phrases: modular lattices, special elements, weak congruences.

1. Preliminaries

1.1

Special elements of lattices have been considered by Ore, Birkhoff, Gratzer, Schmidt, and others [1].

[1] Work supported by Science Fund of Serbia.

An element a of a complete lattice $\mathcal{L} = (L, \wedge, \vee, \leq)$ is **distributive** iff $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$, for all $x, y \in L$.

A **co-distributive** element of $\mathcal{L}$ is defined dually.

An element a of $\mathcal{L}$ is **standard** iff $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$, for all $x, y \in L$.

A **co-standard** element of $\mathcal{L}$ is defined dually.

An element a of $\mathcal{L}$ is **modular** iff $a \leq y$ implies $a \vee (x \wedge y) = (a \vee x) \wedge y$, for all $x, y \in L$.

We said that an element a of a lattice $\mathcal{L}$ is **cancelable** iff $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$ imply $x = y$, for all $x, y \in L$.

It is well known [1] that an element is standard if and only if it is distributive and cancelable.

1.2

For elements $x, y \in L$, if $x \leq y$, then $[x, y]$ denotes an interval

$$\{z \mid x \leq z \leq y\} \text{ (as a sublattice).}$$

The following proposition is also well known.

Proposition 1. *If $\mathcal{L}$ is a modular lattice and $a, b \in L$ then*

$$[a, a \vee b] \cong [a \wedge b, b],$$

under the isomorphism $f : x \longrightarrow x \wedge b$. □

Isomorphisms of some other intervals in modular lattices are proved in [2]:

Proposition 2. *Let a, a', b, b' and d be elements of modular lattice $\mathcal{L}$, satisfying the following conditions:*

1. *$a' \leq a$ and $b' \leq b$;*
2. *$a \leq a' \vee d$ and $b \leq b' \vee d$;*
3. *$a \wedge d = b \wedge d$.*

Then, the interval $[a, a' \vee d]$ *is projectively isomorphic to the interval* $[b, b' \vee d]$. □

A corollary of this proposition, having an application in group theory, is the following (Zassenhaus lemma).

Proposition 3. ([2]) *If* a, a', b, b' *are elements of a modular lattice* $\mathcal{L}$ *such that* $a' \leq a$ *and* $b' \leq b$, *then*

$$[a' \vee (a \wedge b'), a' \vee (a \wedge b)] \cong [b' \vee (a' \wedge b), b' \vee (a \wedge b)]. \quad \square$$

1.3

A **weak congruence lattice** $Cw\mathcal{A}$, of an algebra $\mathcal{A} = (A, F)$ is a lattice of all weakly reflexive, symmetric, transitive and compatible relations on $\mathcal{A}$, under the set inclusion, i.e. the lattice of all congruences on subalgebras of $\mathcal{A}$. By $Con\mathcal{A}$ and $Sub\mathcal{A}$ we shall denote the lattices of congruences and subalgebras of an algebra $\mathcal{A}$, respectively. It is well known that for an algebra $\mathcal{A}$:

a) $Con\mathcal{A}$ is a sublattice of the lattice $Cw\mathcal{A}$;

b) $Sub\mathcal{A}$ is isomorphic with $\{(x, x) \mid x \in B, \text{ for } B \in Sub\mathcal{A}\}$, which is the sublattice of $Cw\mathcal{A}$;

c) Δ (the diagonal relation) is always a co-distributive element of $Cw\mathcal{A}$.

An algebra $\mathcal{A}$ has the **congruence extension property (CEP)** iff every congruence on a subalgebra of $\mathcal{A}$ is a restriction of a congruence on $\mathcal{A}$.

An algebra $\mathcal{A}$ has the **congruence intersection property (CIP)** iff

$$(\rho \cap \theta)_A = \rho_A \cap \theta_A, \text{ for all } \rho, \theta \in Cw\mathcal{A},$$

where ρ_A is the least congruence on $\mathcal{A}$ containing ρ.

An algebra $\mathcal{A}$ has the **weak congruence intersection property (w CIP)** iff

$$(\rho \cap \theta)_A = \rho_A \cap \theta \text{ for all } \rho \in Cw\mathcal{A} \text{ and } \theta \in Con\mathcal{A}.$$

In the lattice theoretic therms, using Δ as a special element, the former definitions are equivalent to the following properties of Δ [4,5].

An algebra $\mathcal{A}$ has the CEP iff Δ is a cancelable element of $Cw\mathcal{A}$ iff Δ is a co-modular element of $Cw\mathcal{A}$ iff Δ is a co-standard element of $Cw\mathcal{A}$.

An algebra $\mathcal{A}$ has the CIP iff Δ is a distributive element of $Cw\mathcal{A}$.

An algebra $\mathcal{A}$ has the wCIP iff Δ is a modular element of $Cw\mathcal{A}$.

2. Special elements generalizing modularity in a lattice

Modularity is not a necessary condition in propositions 1 and 2. Instead of modularity of $\mathcal{L}$ it is sufficient to assume that some of the elements from the proposition have special properties, as it is shown in the following example.

Example 1.

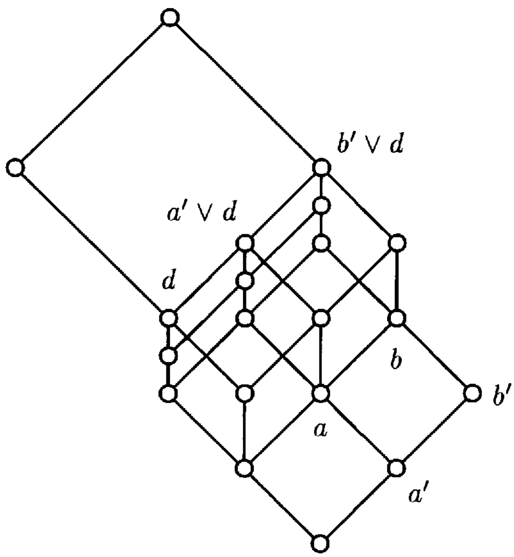

Figure 1.

Let $\mathcal{L}$ be a lattice in Fig.1. Obviously, $\mathcal{L}$ is not a modular lattice, but taking a, a', b, b' and d as in Fig.1, the conditions of Proposition 2 are satisfied, and the interval $[a, a' \vee d]$ is projectively isomorphic to the interval

$[b, b' \vee d]$. Note that d is a co-standard element of $\mathcal{L}$.

The following three lemmas generalize Proposition 1.

Lemma 1. *If a is a standard element of a lattice $\mathcal{L}$, and $b \in L$, then $[a \wedge b, b] \cong [a, a \vee b]$, under the isomorphism $f : x \longrightarrow x \vee a$.*

Proof. f is a mapping from the interval $[a \wedge b, b]$ into the interval $[a, a \vee b]$. As a is a standard element, it is a distributive and cancelable element as well. Since a is a distributive element, f is a homomorphism. If $f(x) = f(y)$ i.e. $x \vee a = y \vee a$, and $x, y \in [a \wedge b, b]$ then $x \wedge a = y \wedge a$ (since $x \leq b$, and $x \wedge a \leq b \wedge a$, and $b \wedge a \leq x$, $x \wedge a = b \wedge a$, for all $x \in [a \wedge b, b]$). Since a is a cancelable element, $x = y$, i.e. the mapping f in an injection. If $y \in [a, a \vee b]$, then $f(b \wedge y) = (b \wedge y) \vee a = (b \vee a) \wedge (y \vee a) = (b \vee a) \wedge y = y$. Hence, f is a surjection. □

Lemma 2. *If b is a co-standard element of a lattice $\mathcal{L}$ and $b \in L$ then $[a, a \vee b] \cong [a \wedge b, b]$, under the isomorphism $f : x \longrightarrow x \wedge b$.*

Proof. Similar to the one of Lemma 1. □

Lemma 3. *If a is a modular and cancelable element of a lattice $\mathcal{L}$, and $b \in L$, then $[a \wedge b, b] \cong [a, a \vee b]$, under the isomorphism $f : x \longrightarrow x \vee a$.*

Proof. Similarly as in Lemma 1, f is an injection and a surjection. If $x, y \in [a \wedge b, b]$ then

$$(x \vee a) \wedge (y \vee a) = a \vee ((x \vee a) \wedge y) = a \vee ((y \vee a) \wedge x).$$

Since $a \wedge x = a \wedge y$,

$$a \wedge ((x \vee a) \wedge y) = a \wedge ((y \vee a) \wedge x).$$

Hence, $(x \vee a) \wedge y = (y \vee a) \wedge x$ (a is a cancelable element).

Since $(x \vee a) \wedge y \wedge (y \vee a) \wedge x = x \wedge y$,

$$(x \vee a) \wedge y = (y \vee a) \wedge x = x \wedge y,$$

and,

$$(x \vee a) \wedge (y \vee a) = a \vee (x \wedge y).$$

Hence, f is an isomorphism. □

Each of the following four propositions generalizes Proposition 2 and its dual proposition.

Proposition 4. *If $a, a', b, b' \in L$ and d is a co-standard element of a lattice $\mathcal{L}$, such that,*

1. $a' \leq a$ *and* $b' \leq b$;
2. $a \leq a' \vee d$ *and* $b \leq b' \vee d$;
3. $a \wedge d = b \wedge d$,

then an interval $[a, a' \vee d]$ is projectively isomorphic to the interval $[b, b' \vee d]$.

Proof. By Lemma 2, $[a, a \vee d] \cong [a \wedge d, d]$, under the isomorphism $f : x \longrightarrow x \wedge d$, and $[b, b \vee d] \cong [b \wedge d, d]$, under the isomorphism $f : x \longrightarrow x \wedge d$. By condition 3, $[a \wedge d, d] \cong [b \wedge d, d]$, hence, $[a, a \vee d] \cong [b, b \vee d]$. By conditions 1 and 2, $a' \vee d = a \vee d$ and $b' \vee d = b \vee d$, hence, $[a, a' \vee d] \cong [b, b' \vee d]$. □

The following proposition is the dual of Proposition 3.

Proposition 5. *If $a, a', b, b' \in L$ and d is a standard element of a lattice $\mathcal{L}$, such that,*

1. $a \leq a'$ *and* $b \leq b'$;
2. $a \geq a' \wedge d$ *and* $b \geq b' \wedge d$;
3. $a \vee d = b \vee d$,

then an interval $[a' \wedge d, a]$ is projectively isomorphic to the interval $[b' \wedge d, b]$. □

Proposition 6 *If $a', b', d \in L$ and a and b are standard elements of a lattice $\mathcal{L}$, such that,*

1. $a' \leq a$ *and* $b' \leq b$;
2. $a \leq a' \vee d$ *and* $b \leq b' \vee d$;

3. $a \wedge d = b \wedge d$,

then an interval $[a, a' \vee d]$ *is projectively isomorphic to the interval* $[b, b' \vee d]$.

Proof. Since a and b are standard elements, $[a \wedge d, d] \cong [a, a \vee d]$ and $[b \wedge d, d] \cong [b, b \vee d]$. Since $a \vee d = a' \vee d$, and $b \vee d = b' \vee d$ the interval $[a, a' \vee d]$ is projectively isomorphic to the interval $[b, b' \vee d]$. □

The dual proposition is also satisfied.

Proposition 7. *If* $a, a', b, b' \in L$ *and* d *is a modular and cancelable element of a lattice* $\mathcal{L}$, *such that,*

1. $a \leq a'$ *and* $b \leq b'$;

2. $a \geq a' \wedge d$ *and* $b \geq b' \wedge d$;

3. $a \vee d = b \vee d$,

then an interval $[a' \wedge d, a]$ *is projectively isomorphic to the interval* $[b' \wedge d, b]$.

Proof. By Lemma 3 instead of Lemma 2, the proof is similar to the one of Proposition 4. □

The following proposition generalizes Proposition 3 and is analogous to the Zassenhaus lemma for arbitrary lattices.

Proposition 8. *If* c, c', f, f' *are elements of lattice* $\mathcal{L}$ *such that* $c \wedge f$ *is a co-standard element of* $\mathcal{L}$, *and* $c' \leq c$ *and* $f' \leq f$, *then*

$$[c' \vee (c \wedge f'), c' \vee (c \wedge f)] \cong [f' \vee (c' \wedge f), f' \vee (c \wedge f)].$$

Proof. By Proposition 4, taking $a = c' \vee (c \wedge f')$, $b = f' \vee (f \wedge c')$, $a' = c'$, $b' = f'$ and $d = c \wedge f$, since d is a co-standard element, and $a' \leq a$, $b' \leq b, a \leq a' \vee d$ and $b \leq b' \vee d$, the only thing left to prove is that $a \wedge d = b \wedge d$.

Since d is a co-standard, it is a co-distributive element, too, and, $a \wedge d = (c' \vee (c \wedge f')) \wedge (c \wedge f) = (c' \wedge c \wedge f) \vee (c \wedge f' \wedge c \wedge f) = (c' \wedge f) \vee (c \wedge f') = (f' \wedge c \wedge f) \vee (f \wedge c' \wedge c \wedge f) = (f' \vee (f \wedge c')) \wedge (c \wedge f) = b \wedge d$.

Hence, the interval $[a, a' \vee d]$ is projectively isomorphic to the interval $[b, b' \vee d]$, i.e.

$$[c' \vee (c \wedge f'), c' \vee (c \wedge f)] \cong [f' \vee (c' \wedge f), f' \vee (c \wedge f)]. \quad \square$$

3. Application to universal algebra

Proposition 9. *If an algebra $\mathcal{A} = (\mathcal{A}, \mathcal{F})$ has the CEP and $\rho, \theta \in Con\mathcal{B}$, and $\mathcal{B} \in Sub\mathcal{A}$, then*

$$[\rho, \rho \vee \Delta] \cong [\theta, \theta \vee \Delta],$$

and both of the intervals are isomorphic with the interval $[\mathcal{B}, \mathcal{A}]$ in the lattice $Sub\mathcal{A}$.

Proof. By the CEP, Δ is a co-standard element in the lattice $Cw\mathcal{A}$. Hence, the proof is a direct consequence of Proposition 4. □

Corollary 1. *If an algebra $\mathcal{A}$ has the CEP, then the interval $[B^2, B^2 \vee \Delta]$ is isomorphic with the interval $[\mathcal{B}, \mathcal{A}]$ in $Sub\mathcal{A}$.* □

As the CEP is hereditary on subalgebras, the Corollary 1 is valid on every subalgebra of $\mathcal{A}$, as well.

Proposition 10. *If the algebra $\mathcal{A}$ has the CEP and the wCIP, then for $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, for $\mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$, if $\rho \vee \Delta = \theta \vee \Delta$, then $[B, \rho] \cong [C, \theta]$ in $Cw\mathcal{A}$, and both of the intervals are isomorphic with the interval $[\Delta, \rho \vee \Delta]$ in $Con\mathcal{A}$.*

Proof. Straightforward, by Proposition 7, since the CEP implies that Δ is cancelable, and the wCIP that Δ is a modular element in the lattice $Cw\mathcal{A}$. □

Corollary 2. **[5]** *If an algebra $\mathcal{A}$ has the CEP and the wCIP and $B \in Sub\mathcal{A}$ then, $Con\mathcal{B} \cong [\Delta, B^2 \vee \Delta]$ in $Con\mathcal{A}$.* □

Corollary 3. *If an algebra $\mathcal{A}$ has the wCIP and the CEP, then an arbitrary lattice identity, satisfied on $Con\mathcal{A}$, is satisfied on $Con\mathcal{B}$ as well, for every $B \in Sub\mathcal{A}$.* □

4. Corollaries in group theory

In the following three corollaries, $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is a group. The fact that $\mathcal{H}$ is a subgroup of $\mathcal{G}$ is denoted by $\mathcal{H} < \mathcal{G}$. $\overline{\mathcal{H}}$ (or $\overline{\mathcal{H}}^G$) is a normal closure of a subgroup $\mathcal{H}$ in $\mathcal{G}$. $\mathcal{H} \lhd \mathcal{G}$ denotes that $\mathcal{H}$ is a normal subgroup of $\mathcal{G}$. Note that a part of the results is known, but not as the consequences of the previous considerations.

Corollary 4. *If a group $\mathcal{G}$ has the CEP and the wCIP, and $\mathcal{H}$ and $\mathcal{K}$ are subgroups of $\mathcal{G}$ such that $\overline{\mathcal{H}} = \overline{\mathcal{K}} = \mathcal{L}$, then $Con\mathcal{H} \cong Con\mathcal{K}$ and each of these congruence lattices is isomorphic with the lattice of all normal subgroup of $\mathcal{G}$ which are contained in $\mathcal{L}$.*

Proof. Straightforward, by Proposition 10, using the fact that $\mathcal{H}^2 \vee \Delta = \mathcal{K}^2 \vee \Delta$. □

Corollary 5. *If group $\mathcal{G}$ has the CEP, and $\mathcal{B} < \mathcal{C} < \mathcal{G}$, then there exists one and only group $\mathcal{X}$, such that $\mathcal{X} \lhd \mathcal{C}$, and $\mathcal{B} < \mathcal{X} < \overline{\mathcal{B}}$.*

Proof. By Corollary 1, $[B^2, B^2 \vee \Delta] \cong [\mathcal{B}, \mathcal{G}]$ in $Sub\mathcal{G}$. Therefore, for every $\mathcal{C} \in Sub\mathcal{G}$, if $\mathcal{B} < \mathcal{C}$, then $\mathcal{C} \vee \mathcal{B}^2$ is the only element from $Con\mathcal{C}$, such that $\mathcal{C} \vee \mathcal{B}^2 \vee \Delta = \mathcal{B}^2 \vee \Delta$. The element $\mathcal{C} \vee \mathcal{B}^2$ corresponds to a normal subgroup $(\mathcal{X})$ of a group $\mathcal{C}$. It is obvious that $\mathcal{B} < \mathcal{X}$, and since $\mathcal{C} \vee \mathcal{B}^2 \vee \Delta = \mathcal{B}^2 \vee \Delta$, $\mathcal{X} < \overline{\mathcal{B}}$ is satisfied. □

Corollary 6. *If group $\mathcal{G}$ has the CEP, the relation $\lhd$ ("to be normal subgroup of") is a transitive relation on $Sub\mathcal{G}$.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be subgroups of $\mathcal{G}$, such that $\mathcal{C} \lhd \mathcal{B}$ and $\mathcal{B} \lhd \mathcal{A}$. Since the CEP is hereditary on subalgebras, $\mathcal{A}$ has the CEP, as well, and Corollary 1 can be applied. By Corollary 5, since $\mathcal{C} < \mathcal{B}$, there is exactly one subgroup $\mathcal{X}$, such that $\mathcal{X} \lhd \mathcal{B}$ and $\mathcal{C} < \mathcal{X} < \overline{\mathcal{C}}^A$. Since $\mathcal{C} < \mathcal{B}$ and $\mathcal{B} \lhd \mathcal{A}$, $\overline{\mathcal{C}}^A < \mathcal{B}$ ensues, even, $\overline{\mathcal{C}}^A \lhd \mathcal{B}$. Hence, $\mathcal{C} = \overline{\mathcal{C}}^A$, i.e. $\mathcal{C} \lhd \mathcal{A}$. □

References

[1] Grätzer, G., General lattice theory, Academie-Verlag Berlin, 1978.

[2] Milić, S., On the isomorphism of modular lattices, Matematički vesnik 2(17) (1965), 153-155.

[3] Milić, S., Ob odnom dokazateljstve teoremi Šreiera v strukturah, Matematički vesnik 6(21) (1969), 161-162. (in Russian).

[4] Šešelja, B., Tepavčević, A., Infinitely distributive elements in the lattice of weak congruences, General Algebra 1988, R. Mlitz (Editor), Elsevier Publishers B. V. (North-Holland) 1990, 241-253.

[5] Šešelja, B., Tepavčević, A., On CEP and semimodularity in the lattice of weak congruence, Rev. of Res. Fac. of Sci. Univ. of Novi Sad, 22, 1 (1992), 95-106.

Received by the editors November 16, 1992.