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LATTICE POWERS OF UNARS

Branimir Šešelja
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Lattice power of an algebra is a kind of its extension by a suitable complemented lattice. If the lattice is Boolean, this extension coincides with the Boolean power.

In this paper, lattice powers of unary algebras are investigated. It turns out that any variety of unary algebras is preserved under the construction of lattice powers. It is proved that the lattice power of an algebra in such variety is isomorphic with the union of particular Boolean powers of the same algebra.

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1. Introduction

A lattice power of algebras has been introduced in [1] (under the name of a "fuzzy power"), as a generalization of a Boolean power.

Not every lattice is suitable for the construction of lattice powers. A detailed description of such lattices "allowing the power" was given in [1]. Some general properties of fuzzy powers were also formulated in [1], in particular the identities preserved under that construction.

In the present paper some general properties of lattices allowing the power are proved. It is also shown that lattice powers are closely related to some collections of Boolean powers. Finally, lattice powers of unars are investigated. It is shown that varieties of unary algebras are closed under the construction of lattice power. It is proved that lattice powers of unary algebras are particular unions of Boolean powers. Moreover, lattice powers of finite algebras in varieties of unars are, up to the isomorphism, subdirect products of these algebras.

2. Lattices allowing the power

As mentioned above, not every lattice is suitable for the power extension of algebras. To avoid too long definitions, we use a special class of lattices, but the main results about the powers are valid for all lattices described in [1].

Let L be a complete lattice in which 0 is the bottom, and 1 the top element. Recall that an equivalence relation on L is said to be a **0,1-equivalence** if the classes containing 0 and 1 are one-element sets. We assume that there is a complete 0,1-congruence relation θ on L, such that $(L/\theta, \wedge, \vee)$ is a complete Boolean algebra (a congruence on a complete lattice L is **complete** if it admits arbitrary meets and joins). The lattice L, satisfying the above conditions, is said to **allow the lattice power** of algebras (or, simply, to allow the power). We recall that the characterization of all lattices suitable for the construction of powers was given in [1].

In the sequel we prove some properties of lattices allowing the power, which have not (or not explicitely) been mentioned in [1].

Lemma 2.1. *If the lattice L allowing the power has no infinite antichains, then the Boolean algebra L/θ is finite.*

Proof. L/θ has no infinite antichains, since by assumption L has this property as well. Thus, L/θ is obviously a finite Boolean algebra. □

Lemma 2.2. *If the lattice L allows the power, then it is complemented.*

Proof. Let x be an element from L, and $[y]_\theta$ a complement of $[x]_\theta$ in the Boolean algebra L/θ, $y \in L$. Then, obviously $[x]_\theta \wedge [y]_\theta = \{0\}$ i.e. $[x \wedge y]_\theta = \{0\}$ and thus $x \wedge y = 0$, i.e. $x \neq y$. Thereby, since $[x]_\theta \vee [y]_\theta = [x \vee y]_\theta = \{1\}$, it follows that $x \vee y = 1$, and y is a complement of x. □

A **partition** in a complete lattice is a collection of its pairwise disjoint elements, supremum of which is 1.

Lemma 2.3. *If the lattice L allows the power, then $\{p_i \mid i \in I\}$ is a partition in L if and only if $\{[p_i]_\theta \mid i \in I\}$ is a partition in L/θ.*

Proof. Let $\{p_i \mid i \in I\}$ be a partition in L. Now, since for $i, j \in I$ $p_i \wedge p_j = 0$, it follows that $[p_i]_\theta \wedge [p_j]_\theta = [p_i \wedge p_j]_\theta = \{0\}$, and

$$\bigvee_{i \in I} [p_i]_\theta = \left[\bigvee_{i \in I} p_i\right] = \{1\}.$$

On the other hand, if $\{[p_i]_\theta \mid i \in I\}$ is a partition in L/θ, then $\{0\} = [0]_\theta = [p_i]_\theta \wedge [p_j]_\theta = [p_i \wedge p_j]_\theta$ implies $p_i \wedge p_j = 0$, and from $\{1\} = [1]_\theta = \bigvee_{i \in I} [p_i]_\theta = [\bigvee_{i \in I} p_i]_\theta$ it follows that $\bigvee_{i \in I} p_i = 1$. □

From the above, if L is distributive, then it is obviously a Boolean lattice. Some nondistributive lattices allowing the power are represented by the Hasse diagrams in Figure 1. For all of them, L/θ is a four-element Boolean algebra.

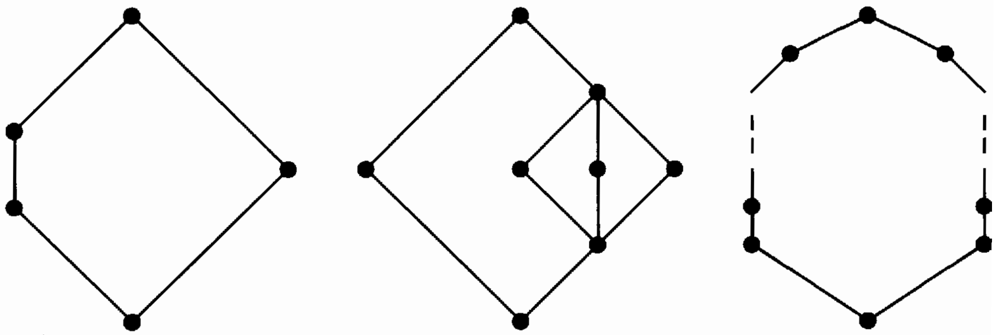

Fig. 1

Lemma 2.4. *If $P = \{p_i \mid i \in I\}$ is a partition in the lattice L allowing the power, then the sublattice L_P of L, generated by P is Boolean, and it has at most one element in each class of L/θ.*

Proof. Every element of L_P is of the form $\bigvee_{j \in J \subseteq I} p_j$. By Lemma 3 these elements determine the classes in the Boolean algebra L/θ. Since θ is a complete congruence on L, the above joins obey the axioms for Boolean algebras, and thus form the sublattice of L, generated by P. In addition, every class in L/θ contains at most one element from L_P, since by Lemma 3 again, the mapping $p_i \mapsto [p_i]$ is an injection from P to L/θ. □

Thus, for every partition $\{P_k \mid k \in K\}$ in L, there is a Boolean sublattice B_k of L, generated by that partition. Hence, every lattice allowing the power has a collection $B_K = \{B_k \mid k \in K\}$ of Boolean sublattices. Every of these sublattices has at most one element in each class of L/θ.

3. Lattice powers

In this section we shall describe a construction of lattice powers and give some properties of these new algebras. Let $\mathcal{A} = (A, F)$ be an algebra, and L a lattice allowing the power. Let $A(L)$ be the collection of all mappings $X : A \to L$, such that

$$X(a) \wedge X(b) = 0, \text{for all } a, b \in L,\ a \neq b; \tag{1}$$

$$\bigvee_{a \in A} X(a) = 1. \tag{2}$$

Operations on $A(L)$ are defined in the following way: if $f \in F_n \subseteq F$, and $X_1, ..., X_n \in A(L)$, then

$$f(X_1, ..., X_n) = Y, \quad \text{where for } a \in A$$

$$Y(a) = \bigvee (X_1(a_1) \wedge ... \wedge X_n(a_n); f(a_1, ..., a_n) = a). \tag{3}$$

The proof that the operations on $A(L)$ are well defined was given in [1]. The algebra $\mathcal{A}(L) = (A(L), F)$ belongs to the same similarity class as $\mathcal{A}$, and is said to be a **lattice power** of $\mathcal{A}$.

Remark 3.1. *$A(L)$ is a special collection of lattice valued (fuzzy) sets on $\mathcal{A}$, since its elements are particular mappings from the algebra $\mathcal{A}$ to a lattice. Therefore, the lattice power was defined in* [1] *to be a* **fuzzy power**.

To describe varieties closed under lattice powers, we need the following theorem.

Theorem 3.1. [1] *Let $\mathcal{A} = (A, F)$ be an algebra, $\mathcal{A}(L)$ its lattice power, and $h \in F_1$, $f, g \in F_n$, $F_1, F_n \subseteq F$. Now, if the equalities*

$$f(x_1, ..., x_n) = g(x_1, ..., x_n), \text{ and} \tag{4}$$

(5) $$g(x_1, ..., x_n) = h(f(x_1, ..., x_n))$$

are true in $\mathcal{A}$ for any $x_1, ..., x_n$, then they also hold in $\mathcal{A}(L)$. □

Corollary 3.1. *The equality*

(6) $$f_{i_1} ... f_{i_m} g(x_1, ..., x_n) = f_{j_1} ... f_{j_p} h(x_1, ..., x_n),$$

where all the f operations are unary and $g, h \in F_n$, holds in $\mathcal{A}(L)$, provided that it is true in $\mathcal{A}$.

Proof. Straightforward, successively applying (5). □

Lemma 3.1. *Let $\mathcal{A}(L)$ be a lattice power of the algebra $\mathcal{A}$, and $X \in A(L)$. Then, $\{p \in L \mid p = X(a)$ for some $a \in A\} \subseteq B_k$ for some Boolean algebra B_k from the collection B_K.*

Proof. If $X \in L(A)$, then all the elements $X(a)$ for $a \in A$ belong to different classes in L/θ. Indeed, by (1), $X(a) \wedge X(b) = 0$ for $a \neq b$, and $[0]_\theta = \{0\}$. Moreover, these elements form a partition in L, by (1) and (2). By the construction of algebras in B_K, there is a Boolean algebra from that family to which all these elements belong. □

Thus, an element from a lattice power as a function from an algebra $\mathcal{A}$ to a lattice, always maps the set A into a Boolean algebra B_k from the collection B_K. The following proposition shows that the set of all such mappings (as a Boolean power) form a subalgebra of the lattice power.

Proposition 3.1. *If B_k is a Boolean lattice from the collection B_K of sublattices of the lattice L allowing the power and $\mathcal{A}=(A, F)$ is an algebra, then the Boolean power $\mathcal{A}(B_k)$ is a subalgebra of the lattice power $\mathcal{A}(L)$.*

Proof. Let f be an n-ary operational symbol from F, and $X_1, ..., X_n \in A(B_k)$. These elements are also in $A(L)$ and by (3), since B_k is a sublattice of L, $f(X_1, ..., X_n)$ as the function from A to L has the same values as the corresponding function from A to B_k. This proves that $A(B_k)$ is, as a subset of $A(L)$, closed under operations from F. □

There is another way to connect lattice powers with the Boolean ones, as shown by the follwing theorem.

Theorem 3.2. *If L is a lattice allowing the power and $\mathcal{A}=(A,F)$ an algebra, then the Boolean power $\mathcal{A}(L/\theta)$ is a homomorphic image of the lattice power $\mathcal{A}(L)$.*

Proof. Let $h : A(L) \to A(L/\theta)$ be defined with

$$h(X) = Y, \text{where for } a \in A, Y(a) = [X(a)]_\theta. \tag{7}$$

To prove that h admits the operations, take an $f \in F_n \subseteq F$, and let $X_1, ..., X_n \in A(B_i)$. Then, for $a \in A$,

$$h(f(X_1, ..., X_n))(a) = [f(X_1, ..., X_n)(a)]_\theta = f([X_1]_\theta, ..., [X_n]_\theta)(a),$$

by (3) and by the fact that θ is a congruence relation on L. □

4. Unars

Theorem 4.1. *Any variety of unary algebras is closed under the formation of lattice powers.*

Proof. Immediately by Corollary 1. □

In order to characterize lattice powers of algebras in the above varieties, we shall introduce a particular union of Boolean powers.

Let L be the lattice allowing the power and $\mathcal{A}=(A,F)$ a unary algebra. Let also $\{\mathcal{A}(B_k) \mid k \in K\}$ be the collection of Boolean powers constructed by means of the Boolean lattices from the family B_K, introduced at the end of section 2.

Lemma 4.1. *The union $\bigcup_{k \in K} A(B_k)$ is a unary algebra from the same similarity class as the algebra $\mathcal{A}$.*

Proof. Indeed, every Boolean power from the collection is a unary algebra, and the union of unary algebras from the same similarity class is again an algebra from the same class. The operation in the union is the one that, restricted to any of these Boolean powers, gives the corresponding fundamantal unary operation. The elements (functions) belonging at the same time to different Boolean powers, have equal values under these unars - exactly those that are determined in $\mathcal{A}(L)$. □

Let

$$\mathcal{A}(B_K) := (\bigcup_{k\in K} A(B_k), F).$$

Theorem 4.2. *If $\mathcal{A}$=(A, F) is a unary algebra and L the lattice allowing the power, then the lattice power $\mathcal{A}(L)$ is isomorphic with the algebra $\mathcal{A}(B_K)$.*

Proof. Let h be the mapping from the lattice power to the union of Boolean powers, defined by: $h(X) = Y$, where for $a \in A$ $Y(a) = X(a)$, and $Y(a)$ belongs to the Boolean algebra determined by the partition being the codomain of X. The proof now follows by the fact that the functions from $A(L)$ are at the same time the functions from the union $A(B_K)$. □

Thus we have proved that the lattice power of a unar is a union of Boolean powers of the same algebra. However, for algebras in varieties of unars, it would be more convenient to characterize lattice powers in terms of H, S and P. And it turns out that the lattice power of an algebra in such variety is embeddable into a Boolean power.

Theorem 4.3. *For any lattice power $\mathcal{A}(L)$ of a unary algebra $\mathcal{A}$, there is a Boolean power $\mathcal{A}(B)$ of the same algebra, having a subalgebra isomorphic with $\mathcal{A}(L)$.*

Proof. Let B be a complete Boolean algebra, with the following property: it contains a collection $C_K = \{C_k \mid k \in K\}$ of different, complete Boolean subalgebras, such that for every $k \in K$ $C_k \cong B_k$, where B_k belongs to the collection B_K of Boolean sublattices of L. Obviously, for every $k \in K$ the Boolean powers $\mathcal{A}(B_k)$ and $\mathcal{A}(C_k)$ are isomorphic. Hence, by Theorem 4, $\mathcal{A}(L)$ is embeddable into $\mathcal{A}(B)$. □

It is known that Boolean powers of finite algebras are special subdirect powers of these algebras.

Corollary 4.1. *A lattice power of a finite algebra $\mathcal{A}$ in a variety of unars is isomorphic with a subdirect power of $\mathcal{A}$.*

Proof. Every Boolean power of $\mathcal{A}$ is a subdirect power of the same algebra, which is obviously the case with their union as well. The proof now follows directly by Theorem 5. □

Example 4.1. *Let $\mathcal{A}=(\{a,b,c\},f)$ be a three-element algebra with one unary operation given by the following table.*

$$f=\begin{pmatrix} a & b & c \\ b & c & c \end{pmatrix}$$

The lattice L allowing the power is represented in Figure 2.

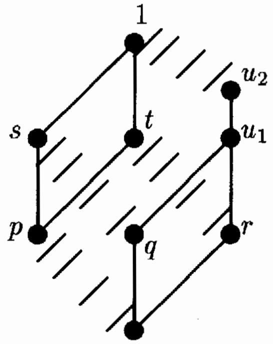

Fig. 2

The lattice power $\mathcal{A}(L)$ is the union of two Boolean powers of $\mathcal{A}$, constructed by means of two Boolean sublattices of L presented in Figure 3. These two Boolean algebras are the only members of the collection B_K.

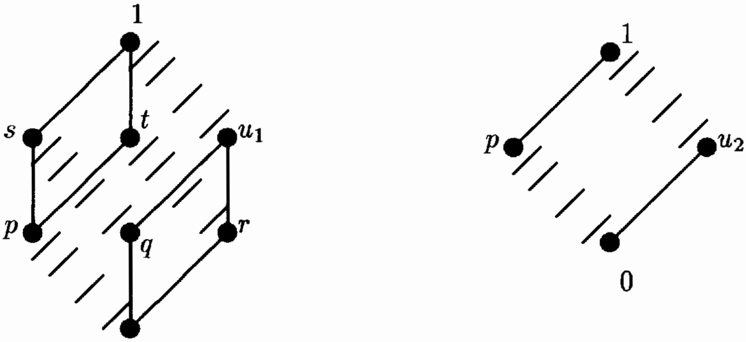

Fig. 3

The number of mappings - elements of $\mathcal{A}(L)$ is 33. (The sets of functions belonging to these two Boolean powers are not disjoint.) Namely, if the mappings are represented by their functional values (pqr instead of $\begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}$ and so on), then all the elements of $\mathcal{A}(L)$ are all permutations of each of the following values:

$$100 \quad pqr \quad pu_1 0 \quad pu_2 0 \quad qt0 \quad rs0$$

By (3), since the operation f is unary, for every X from $A(L)$ and for $x \in \{a, b, c\}$ we have:

$$X(x) = \bigvee_{f(y)=x} X(y).$$

Hence,

$$f(pqr) = 0pu_1, \quad f(010) = 001, \quad F(u_2 p0) = 0u_2 p,$$

and so on.

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