Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 1 (1995), 89-97

Review of Research
Faculty of Science
Mathematics Series

ON THE PARTIAL ALGEBRAS OF WEAK CONGRUENCES

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Abstract

In [9] the notion of weak partial congruence algebra is introduced and, among other things, three problems are posed. In the present paper we give the complete solution for the first problem and some partial answers for the second one. Also, we prove that there are modular algebraic lattices which are not representable as the lattice of weak congruences of any congruence permutable algebra. This fact solves Conjecture from [10] in the negative.

AMS Mathematics Subject Classification (1991): 080A30
Key words and phrases: weak congruence, lattice

1. Introduction

Let $\mathcal{A}$ be an algebra of the type $\mathcal{F}$. With $\mathcal{F}_n$ we denote the set of all n-ary functional symbols. A *weak congruence relation* ρ on $\mathcal{A}$ (see [11]) is a symmetric, transitive binary relation on A, which satisfies the usual substitution property and the weak reflexivity: for any $c \in \mathcal{F}_0$, $c^{\mathcal{A}} \rho c^{\mathcal{A}}$. We denote by $C_w(\mathcal{A})$ the set of all weak congruences on the algebra $\mathcal{A}$. It is not hard to see that $C_w(\mathcal{A})$ is the set of all congruences of all subalgebras of $\mathcal{A}$ or, equivalently, the set of all symmetric and transitive subalgebras of $\mathcal{A}^2$. So, $\mathcal{C}_w(\mathcal{A}) = \langle C_w(\mathcal{A}), \subseteq \rangle$ is an algebraic lattice. Denote by $\vee$ and $\wedge$ the corresponding lattice operations. This lattice contains the congruence lattice $\mathcal{C}(\mathcal{A})$ as its sublattice (namely, this is the filter generated by the diagonal relation $\triangle_A = \{(x,x)|x \in A\}$). Also, the lattice $\mathcal{S}(\mathcal{A})$ of all subalgebras of $\mathcal{A}$ can be embedded into $\mathcal{C}_w(\mathcal{A})$ in a natural way: to any subalgebra $\mathcal{B}$ we assign the diagonal relation $\triangle_B$.

It is not hard to see that if $\mathcal{A}$ has at least one nontrivial subalgebra, there are relations $\rho, \sigma \in C_w(\mathcal{A})$ which are not permutable. As the relation composition of two weak congruences is again a weak congruence if and only if they commute, we conclude that the relation composition $\circ$ on $C_w(\mathcal{A})$ is only a partial operation. In [9] the *weak partial congruence algebra* is defined as the partial algebra

$$\mathcal{K}_w(\mathcal{A}) = (C_w(\mathcal{A}), \vee, \wedge, \circ, ^{-1}, \triangle_A, \sigma, A^2),$$

where $^{-1}$ is the inversion of the relations and σ is the diagonal relation of the least subalgebra of $\mathcal{A}$. Here, we slightly modify this notion and obtain the notion of *partial algebra of weak congruences* simply by omitting the operation $^{-1}$ from the set of basic operations (because $^{-1}$ is always an identity function on the set of weak congruences).

2. What does $\mathcal{K}_w$ "know"?

It was shown in [9] that for any natural number $n = pq$, where p and q are distinct primes, there exist algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ such that they have isomorphic lattices of weak congruences, but the corresponding partial algebras of weak congruences are not isomorphic. So, the least algebra which can be constructed in this way have six elements. Problem A in [9] asks *whether there exist two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of the same type and same cardinality less than six such that*

$$\mathcal{C}_w(\mathcal{A}_1) \cong \mathcal{C}_w(\mathcal{A}_2),$$

but

$$\mathcal{K}_w(\mathcal{A}_1) \ncong \mathcal{K}_w(\mathcal{A}_2).$$

In the sequel we will prove that the least algebra of that kind (which show that $\mathcal{K}_w$ "knows more about algebras than $\mathcal{C}_w$") has three elements. It is easy to see that if two algebras of the same type, with two elements, have isomorphic lattice of weak congruences, then the corresponding partial algebras of weak congruences are also isomorphic.

Theorem 1. *The least algebras of the same type and same universe such that they have isomorphic lattices of weak congruences, but not isomorphic partial algebras of weak congruences, have three elements.*

Proof. Let the universe of the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ be $A = \{0,1,2\}$, and let both of them have the type $\mathcal{F} = \{f, g, a\}$, where f and g are unary and a is a nullary functional symbol. Let the interpretation of a in both algebras be 0, and let

$$f^{\mathcal{A}_1} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } g^{\mathcal{A}_1} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix},$$

and

$$f^{\mathcal{A}_2} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \text{ and } g^{\mathcal{A}_2} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

It can be proved that the weak congruences of the algebra $\mathcal{A}_1$ are $A^2, \triangle_A$ and $\{(0,0)\}$. So, the lattice of weak congruences of $\mathcal{A}_1$ is the three-element chain. On the other hand, the weak congues of the algebra $\mathcal{A}_2$ are $A^2, \triangle_A$ and the equivalence relation with the classes $\{1,2\}, \{0\}$. So, the lattice of weak congruences of $\mathcal{A}_2$ is also the three-element chain. The corresponding partial algebras of weak congruences are not isomorphic because the only possible isomorphism between these chains does not map the diagonal relation to the diagonal relation. Note, that if the diagonal relation is not taken to be a nullary operation in the definition of the partial algebra of weak congruences, even in this case we could prove that $\mathcal{K}_w(\mathcal{A}_1)$ and $\mathcal{K}_w(\mathcal{A}_2)$ are not isomorphic. Namely, the only possible isomorphism φ between these two chains does not commute with the operation $\circ$, because

$$\varphi(\{(0,0)\} \circ \triangle_A) \neq \varphi(\{(0,0)\}) \circ \varphi(\triangle_A).$$

□

Problem B from [9] asks for a description of such classes of algebras in which any two algebras are uniquely determined by its partial algebras of weak congruences. We can prove the following:

Proposition 1. *Let the type $\mathcal{F}$ of algebras contain at least one constant symbol, and at least one non-nullary functional symbol. Then there are algebras of the type $\mathcal{F}$, with the same universe, which are not isomorphic, but their partial algebas of weak congruences are isomorphic.*

Proof. Let $c \in \mathcal{F}_0$ and $\mathcal{A}_1$ be the corresponding constant algebra, and $\mathcal{A}_2$ an algebra with the same universe, in which every non-nullary operation is the first projection. If the universe contains more than one element, then the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are not isomorphic. But they have the same weak congruences. □

Proposition 2. *Let the type $\mathcal{F}$ of algebras contain at least one operation symbol of arity at least two. Then, there are algebras of the type $\mathcal{F}$, with the same universe which are not isomorphic, but their partial algebas of weak congruences are isomorphic.*

Proof. It is enough to consider two algebras with the same (non-trivial) universe such that in the firt of them the non-nullary operations are the first projections, and in the second one, all the operations are the second projections (of suitable arities). These algebras are not isomorphic, but have the same weak congruences. □

3. Permutability of congruences and modularity

It is proved in [10] that the partial algebra of weak congruences of an algebra is a (full) algebra if and only if the algebra is congruence permutable and has no non-trivial subalgebras. It is well known that every congruence permutable algebra is congruence modular. In [10] the following Conjecture was stated:

For arbitrary modular algebraic lattice $\mathcal{L}$, there exists an algebra $\mathcal{A}$ with permutable congruences such that

$$\mathcal{C}_w(\mathcal{A}) \cong \mathcal{L}.$$

Here we will prove that this Conjecture is false. First of all we need some definitions and theorems. The counterexample will be constructed as the lattice of subspaces of some projective geometry. As the geometry we will use has dimension two, we will recall here only the definition of two dimensional projective geometries.

Definition 1. *A* projective plane *is a pair $\pi = (P, \Lambda)$ such that P is a nonvoid set (the set of* points*), Λ is a collection of subsets of P (the elements of Λ are called* lines*), and the following axioms are satisfied:*

1. *Any two distinct points belong to one and only one line.*

2. *Any two distinct lines contain precisely one point in common.*

3. *Every line has at least three points.*

4. *There exists three distinct points such that no line contains all of them.*

If π is a projective plane, it can be proved that there is a cardinal $\kappa \geq 2$ (called the *order* of π) such that every line of π contains precisely $\kappa + 1$ points and every point of π lies on precisely $\kappa + 1$ lines.

To any projective plane π we will associate a lattice $\mathcal{L}^\pi$, the lattice of *subspaces* of π. Namely, if $\pi = (P, \Lambda)$ is a projective plane, let

$$L^\pi = \{\emptyset\} \cup \{\{p\} : p \in P\} \cup \Lambda \cup \{P\}.$$

It is easy to see that L^π is a closed set system. We define the lattice of subspaces $\mathcal{L}^\pi = (L^\pi, \wedge, \vee)$ as the corresponding lattice of closed sets. If p, q are points, then instead of $\{p\} \vee \{q\}$ we write simply $p \vee q$. It is well known, that the following representation theorem can be proved (for proof see for example [7]):

Theorem 2. *Let $\mathcal{L}$ be a lattice. $\mathcal{L}$ is isomorphic to $\mathcal{L}^\pi$ for some (necessarily unique up to isomorphism) projective plane π iff $\mathcal{L}$ is a complemented modular lattice of height three and every coatom of $\mathcal{L}$ contains at least three atoms.*

For the set $X \subseteq P$ of points we say that they are *collinear* iff $X \subseteq l$ for some line $l \in \Lambda$. A triplet (a_0, a_1, a_2) of non-collinear points is a *triangle.* Two triangles (a_0, a_1, a_2) and (b_0, b_1, b_2) are *perspective with respect to a point p* iff $a_i \neq b_i$, lines $a_i \vee a_j$ and $b_i \vee b_j$, $0 \leq i, j < 3$, are distinct, and the points p, a_i, b_i are collinear for $i = 0, 1, 2$. They are *perspective with respect to a line l* iff $c_{01}, c_{12}, c_{20} \subseteq l$, where c_{ij} is the intersection of lines $a_i \vee a_j$ and $b_i \vee b_j$. We say that Desargues' theorem holds in the projective plane π if any two triangles which are prspective with respect to a point, are also perspective with respect to a line. M.P. Schützenberger (1945) first showed that this property (formulated in the same way for projective geometries of higher dimension) can be expressed as a lattice identity holding in the associated lattice of subspaces. (Recall that any lattice inclusion is equivalent to a lattice identity, since $x \leq y$ iff $x \vee y = y$.)

Definition 2. *Let $\mathcal{L}$ be a lattice. Consider six elements $a_i, b_i \in L$, $(i = 0, 1, 2)$, form the elements*

$$c_0 = (a_1 \vee a_2) \wedge (b_1 \vee b_2)$$

and cyclically, and let

$$d = c_0 \wedge (c_1 \vee c_2).$$

The inclusion

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq (a_1 \wedge (d \vee a_2)) \vee b_1$$

is called the Arguesian identity, *and the lattice in which this identity holds is said to be* Arguesian.

For the proof of the following theorem (which holds also for projective geometries of higher dimension too) see for example [2].

Theorem 3. *Let π be a projective plane. Then Desargues' theorem holds in π iff the associated lattice $\mathcal{L}^\pi$ of subspaces satisfies the Arguesian law.*

It is well known (see for example [8], [1]) that there are projective planes in which Desargues' theorem does not hold. For example, there is such a projective plane of order 9.

Now, we can turn back to congruence lattices. The proof of the following theorem can be found in [3].

Theorem 4. (B. Jónsson, 1953)
If $\mathcal{A}$ is an algebra whose congruences permute, then $\mathcal{C}(\mathcal{A})$ is Arguesian.

Theorem 5. *There is a finite modular lattice which is not isomorphic to the lattice of weak congruences of any congruence permutable algebra.*

Proof. Let π be a finite projective plane in which Desargues' theorem does not hold, and let $\mathcal{M}_0$ be the associated lattice of subspaces. Because of T2. and T3., $\mathcal{M}_0$ is a modular lattice in which the Arguesian identity does not hold. Let us construct a new lattice $\mathcal{M}$ by adding three new elements a, b, c to the lattice $\mathcal{M}_0$ in such a way that in $\mathcal{M}$ we have $a < b < c < 0$, where 0 is the least element of the lattice $\mathcal{M}_0$. Then $\mathcal{M}$ is a modular lattice with $\mathcal{M}_0$ as a sublattice. Suppose that there is an algebra $\mathcal{A}$ with permutable congruences such that $C_w(\mathcal{A}) \cong \mathcal{M}$, and let $\varphi : \mathcal{M} \to C_w(\mathcal{A})$ be the corresponding isomorphism. Denote by d the element $\varphi^{-1}(\triangle_A)$ and prove that $d \notin M_0 \setminus \{0\}$.

Suppose that $d \in M_0 \setminus \{0\}$ and show that we get a contradiction. Let $\alpha = \varphi(a), \beta = \varphi(b), \gamma = \varphi(c), \delta = \varphi(0)$. As $d \in M_0 \setminus \{0\}$, then $\alpha, \beta, \gamma \subseteq \triangle_A$. Further, because of $\alpha \subset \beta \subset \gamma$, there are distinct elements x and y such that $(x,x) \in \beta \setminus \alpha$ and $(y,y) \in \gamma \setminus \beta$. Let

$$\rho = \{(u,v) : (u,u) \in \gamma \text{ and } (v,v) \in \gamma\},$$

and prove that ρ is a weak congruence of the algebra $\mathcal{A}$. Trivially, ρ is symmetric and transitive. As $\gamma \subseteq \rho$, we conclude that ρ satisfies the weak reflexivity. It is easy to see that ρ satisfies the substitution property. Namely, if $f^{\mathcal{A}}$ is a n-ary operation of the algebra $\mathcal{A}$, and $(u_i, v_i) \in \rho$, $i = 1, 2, \ldots, n$, then $(u_i, u_i) \in \gamma$ and $(v_i, v_i) \in \gamma$, so that

$$(f^{\mathcal{A}}(u_1, \ldots, u_n), f^{\mathcal{A}}(u_1, \ldots, u_n)) \in \gamma,$$

$$(f^{\mathcal{A}}(v_1, \ldots, v_n), f^{\mathcal{A}}(v_1, \ldots, v_n)) \in \gamma,$$

so by the definition of ρ we get

$$(f^{\mathcal{A}}(u_1, \ldots, u_n), f^{\mathcal{A}}(v_1, \ldots, v_n)) \in \rho.$$

Now, we have $\triangle_A \cap \rho = \gamma$. On the other hand, $\rho \not\subseteq \triangle_A$, because for distinct elements x and y, $(x,y) \in \rho$. So, $\rho \in \varphi(M_0) \setminus \{\delta\}$, and $\triangle_A \in \varphi(M_0) \setminus \{\delta\}$. As $\varphi(\mathcal{M}_0)$ is sublattice of $\mathcal{M}$, we get

$$\triangle_A \cap \rho \in \varphi(M_0),$$

i.e. $\gamma \in \varphi(M_0)$, and finally $c \in M_0$, which is a contradiction.

We conclude that $d \notin M_0 \setminus \{0\}$, and so $d \in \{0, a, b, c\}$. In this way, the filter generated by the element d in $\mathcal{M}$ has $\mathcal{M}_0$ as its sublattice, which means that $\varphi(\mathcal{M}_0)$ is a sublattice of $Con(\mathcal{A})$. But in this way the congruence lattice of the algebra $\mathcal{A}$ would not satisfy the Arguesian identity, which is a contradiction with T4.

□

Remark 1. The first example of a non-representable relation algebra (in the sense of Tarski) was constructed by Lyndon. Jónsson later used non-Desarguesian projective planes to construct a non-representable integral relation algebra. Also, projective planes were used in the proof of Monk's theorem that the class of representable relation algebras is not finitely axiomatizable (see [5]).

Remark 2. It can be proved that every Arguesian lattice is modular. So, the Arguesian identity is stronger than modularity. It is interesting to note that if we consider these two identities as identities of congruence varieties, they have the same "strenght". Namely, Freese and Jónsson proved that if the congruence variety of some variety V (i.e. the variety generated by $\{Con(\mathcal{A}) : \mathcal{A} \in V\}$) is a variety of modular lattices, then it satisfies the Arguesian identity too (for the proof see for example [3]).

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Received by the editors April 13, 1994.