Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 1 (1995), 179-199

Review of Research
Faculty of Science
Mathematics Series

A NOTE ON MANIFOLD WHOSE PRODUCT CONFORMAL CURVATURE TENSOR IS SEMI-SYMMETRIC

Mileva Prvanović
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper we investigate the manifold whose product - conformal curvature tensor is semi-symmetric and which admits product concircular transformation. We show that curvature tensor of such manifold has the form (4.10).

AMS Mathematics Subject Classification (1991): 53B20.
Key words and phrases: locally decomposable Riemannian space, product - conformal curvature tensor, product conformal transformation, product concircular transformation.

1. Product concircular transformation of locally decomposable Riemannian space

An n–dimensional differentiable manifold M_n of class C^∞ is called a locally decomposable Riemannian space [5] if in M_n a tensor field $F_j^i \neq \delta_j^i$ and a positive definite Riemannian metric $ds^2 = g_{ij}(x^k)dx^i dx^j$ are given satisfying the conditions

$$(1.1) \qquad F_j^i F_j^k = \delta_j^i, \quad g_{ij} F_k^i F_h^j = g_{kh}, \quad \nabla_k F_j^i = 0,$$

where ∇ is the operator of the covariant derivative with respect to the Riemannian metric. If we put

$$F_{ij} = F_i^k g_{kj}, \quad \text{then} \quad F_{ij} = F_{ji},$$

and the condition $\nabla_k F_j^i = 0$ is equivalent to the condition $\nabla_k F_{ij} = 0$.

Locally decomposable space can be covered by a separating coordinate system, that is by such a system of coordinate neighbourhoods (x^i) that in any intersection of the two coordinate neighbourhoods (x^i) and $(x^{i'})$ we get

$$x^{a'} = x^{a'}(x^a), \quad x^{y'} = x^{y'}(x^y),$$

where the indices a, b, c run over the range $1, 2, ..., p$ and the indices x, y, z run over the range $p+1, ..., p+q = n$.

With respect to a separating coordinate system, the metric of the space has the form

$$ds^2 = g_{ab}(x^c)dx^a dx^b + g_{yz}(x^x)dx^y dx^z,$$

while

$$(F_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & -g_{xy} \end{pmatrix}, \quad (F_j^i) = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_y^x \end{pmatrix}.$$

Therefore $F_i^i = \varphi = p - q = n - 2q$.

In the following, we suppose $p > 2, \ q > 2$.

Suppose that locally decomposable Riemannian space M_n also admits the metric $\overline{g}$ such that

$$\bar{g}_{ij} = \mu g_{ij} + \nu F_{ij}, \tag{1.2}$$

where μ and ν are scalar functions satisfying

$$\sigma_i^* = F_i^j \sigma_j, \quad \sigma_i = \frac{\partial \mu}{\partial x^i}, \quad \sigma_i^* = \frac{\partial \nu}{\partial x^i}, \quad \mu^2 - \nu^2 \neq 0. \tag{1.3}$$

The relation (1.2) is said to be product - conformal transformation [3] of M_n.

With respect to a separating coordinate system, (1.2) is written as follows

$$\bar{g}_{ab} = (\mu + \nu)g_{ab}, \quad \bar{g}_{ax} = 0, \quad \bar{g}_{xy} = (\mu - \nu)g_{xy}.$$

On the other hand, (1.3) gives

$$\sigma_a^* = \sigma_a, \quad \sigma_x^* = -\sigma_x.$$

Therefore

$$\frac{\partial}{\partial x^y}(\mu - \nu) = \sigma_y + \sigma_y^* = 0$$

$$\frac{\partial}{\partial x^a}(\mu - \nu) = \sigma_a - \sigma_a^* = 0.$$

This means that $\mu+\nu$ is a function of coordinates x^a only and $\mu-\nu$ is a function of coordinates x^y only. Hence the, product conformal transformation (1.2) implies the conformal change of both metrics g_{ab} and g_{xy}.

In the sequel we shall use the notations

$$\sigma = \sigma_i \sigma^i = \sigma_i \sigma_j g^{ij}, \quad \overset{*}{\sigma} = \sigma_i^* \sigma^i = \sigma_i \sigma_j^* g^{ij}. \tag{1.4}$$

With respect to a separating coordinate system, we have

$$\sigma = g^{ab}\sigma_a\sigma_b + g^{xy}\sigma_x\sigma_y, \quad \overset{*}{\sigma} = g^{ab}\sigma_a\sigma_b - g^{xy}\sigma_x\sigma_y.$$

Therefore

$$\sigma^2 - \overset{*}{\sigma}{}^2 = 4(g^{ab}\sigma_a\sigma_b)(g^{xy}\sigma_x\sigma_y) \neq 0. \tag{1.5}$$

Now, we suppose that the function μ satisfies the condition

$$\nabla_j \sigma_i - \frac{1}{2}\sigma_i\sigma_j - \frac{1}{2}\sigma_i^*\sigma_j^* = f g_{ij} + h F_{ij}, \tag{1.6}$$

where f and h are scalar functions such that

$$h_i = F_i^j f_j, \quad f_i = \frac{\partial f}{\partial x^i}, \quad h_i = \frac{\partial h}{\partial x^i}. \tag{1.7}$$

Transvecting (1.6) with F_k^i, we get

$$\nabla_j \sigma_k^* - \frac{1}{2}\sigma_k^*\sigma_j - \frac{1}{2}\sigma_k\sigma_j^* = f F_{jk} + h g_{jk}. \tag{1.8}$$

As a consequence of (1.6) and (1.8), we find

$$F_h^t \nabla_j \sigma_t = F_j^t \nabla_t \sigma_h, \quad F_h^t \nabla_j \sigma_t^* = F_j^t \nabla_t \sigma_h^*.$$

This means that σ_i and σ_i^* are decomposable vector fields, i.e. $\sigma_a(x)$ are functions of x^b only and $\sigma_y(x)$ are functions of x^z only.

With respect to a separating coordinate system, (1.6) and (1.7) can be written in the forms

$$\nabla_a \sigma_b - \sigma_a\sigma_b = (f+h)g_{ab}, \quad \nabla_x\sigma_y - \sigma_x\sigma_y = (f-h)g_{xy}, \tag{1.9}$$

$$h_a = f_a, \qquad h_x = -f_x,$$

respectively. Therefore,

$$\frac{\partial}{\partial x^y}(f+h) = f_y + h_y = 0, \quad \frac{\partial}{\partial x^a}(f-h) = f_a - h_a = 0,$$

i.e. $f+h$ is a function of the coordinates x^a only and $f-h$ is a function of the coordinates x^y only. On the other hand, (1.9) can be written as follows

$$\nabla_a \sigma_b - \sigma_a \sigma_b + \frac{1}{2} g_{ab} \sigma_c \sigma^c = (\frac{1}{2} \sigma_c \sigma^c + f + h) g_{ab}$$

or, taking into account that $\sigma_c \sigma^c = g^{ab} \sigma_a \sigma_b$ is a function of x^a only, as follows

$$\nabla_a \sigma_b - \sigma_a \sigma_b + \frac{1}{2} g_{ab} \sigma_c \sigma^c = \theta(x^c) g_{ab}.$$

This shows that $\bar{g}_{ab} = \frac{\mu+\nu}{2} g_{ab}$ is the concircular change [4] of metric g_{ab}. In the similar way we can show that $\bar{g}_{xy} = \frac{\mu-\nu}{2} g_{xy}$ is a concircular change of metric g_{xy}.

Definition 1.1. *The product conformal transformation (1.2) is said to be product concircular one if the function μ satisfies the conditions (1.6) and (1.7).*

Remark. In [2] the product conformal transformation (1.2) is said to be product - concircular one if

$$\lambda \cdot \nabla_b \lambda_a - 2\lambda_a \lambda_b + \frac{1}{2} g_{ab} \lambda_c \lambda^c = \theta_1(x^c) g_{ab},$$

$$\eta \cdot \nabla_y \eta_x - 2\eta_x \eta_y + \frac{1}{2} g_{xy} \eta_z \eta^z = \theta_2 g_{xy},$$

where

$$\lambda^2 = \mu + \nu, \quad \eta^2 = \mu - \nu, \quad \lambda_i = \frac{\partial \lambda}{\partial x^i}, \quad \eta_i = \frac{\partial \eta}{\partial \lambda^i}.$$

Let R^h_{ijk}, R_{ij} and $R = g^{ij} R_{ij}$ be the Riemann curvature tensor, the Ricci tensor and the scalar curvature of the Riemannian space respectively. Let us put

$$R^*_{ij} = F^t_i R_{tj}, \quad \overset{*}{R} = R^{*i}_i + F^{ij} R_{ij},$$

and let us consider the tensor

$$C_{hijk} = R_{hijk} + \alpha_2 s_{hijk} + \beta_2 s^*_{hijk}$$
$$-2[(\alpha_1\alpha_2 + \beta_1\beta_2)R + (\alpha_1\beta_2 + \alpha_2\beta_1)\overset{*}{R}]r_{hijk}$$
$$-2[(\alpha_1\beta_2 + \alpha_2\beta_1)R + (\alpha_1\beta_2 + \beta_1\beta_2)\overset{*}{R}]r^*_{hijk}, \tag{1.10}$$

where

$$r_{hijk} = g_{hj}g_{ik} - g_{hk}g_{ij} + F_{hj}F_{ik} - F_{hk}F_{ij}$$
$$r^*_{hijk} = F^t_h r_{tijk} \tag{1.11}$$

$$s_{hijk} = R_{hj}g_{ik} - g_{hj}R_{ik} - R_{hk}F_{ik} - R_{hk}g_{ij}R_{ij}$$
$$+R^*_{hj}F_{ik} + F_{hj}R^*_{ik} - R^*_{hk}F_{ij} - F_{hk}F^*_{ij} \tag{1.12}$$

$$s^*_{hijk} = F^t_h s_{tijk}$$

$$\alpha_1 = \frac{n-2}{2[(n-2)^2-\varphi^2]}, \quad \beta_1 = -\frac{\varphi}{2[(n-2)^2-\varphi^2]},$$
$$\alpha_2 = \frac{n-4}{(n-4)^2-\varphi^2}, \quad \beta_2 = -\frac{\varphi}{(n-4)^2-\varphi^2}. \tag{1.13}$$

It is easy to see that $(n-2)^2 - \varphi^2 \neq 0$, $(n-4)^2 - \varphi^2 \neq 0$ because of $p > 2$, $q > 2$.

It can be proved that the tensor $C^h_{ijk} = g^{th}C_{tijk}$ is invariant with respect to the product conformal transformation (1.2). It is said to be product - conformal curvature tensor [3].

In this paper we investigate the manifolds whose product - conformal curvature tensor is semi - symmetric:

$$\nabla_r\nabla_s C_{hijk} - \nabla_s\nabla_r C_{hijk} = 0 \tag{1.14}$$

and which admits product concircular transformation, i.e. which admits a function μ such that $\sigma_i = \frac{\partial\mu}{\partial x^i}$ satisfies (1.3), (1.6) and (1.7).

In §2 we calculate the expression

$$\sigma_t(\nabla_m\nabla_\ell R^t_{ijk} - \nabla_\ell\nabla_m R^t_{ijk}) \tag{1.15}$$

using the conditionss (1.6) and (1.7), while in §3 we calculate the same expression, this time using the condition (1.14). In §4, comparing two obtained relations, we show that the curvature tensor of considered manifold has, under some conditions, the form (4.10). In §5 we show that in the special case when (1.14) reduces to $\nabla_r\nabla_s R_{hijk} - \nabla_s\nabla_r R_{hijk} = 0$, the manaifold is of almost constant curvature.

Our result is the enlargement of the results obtained in [1] to the locally decomposable Riemannian space.

2. The form of the expression (1.15), obtained from the conditions (1.6), (1.7)

Applying the operator ∇_k to (1.6) and then substituting (1.6), we find

$$\begin{aligned}\nabla_k\nabla_j\sigma_i &= \tfrac{1}{2}(\sigma_k\sigma_i\sigma_j + \sigma_k\sigma_i^*\sigma_j^* + \sigma_k^*\sigma_i^*\sigma_j + \sigma_k^*\sigma_i\sigma_j^*)\\ &+ \tfrac{1}{2}g_{ki}(f\sigma_j + h\sigma_j^*) + \tfrac{1}{2}F_{ki}(h\sigma_j + f\sigma_j^*)\\ &+ \tfrac{1}{2}g_{kj}(f\sigma_i + h\sigma_i^*) + \tfrac{1}{2}F_{kj}(h\sigma_i + f\sigma_i^*) + f_k g_{ij} + h_k F_{ij}\end{aligned} \tag{2.1}$$

$$\begin{aligned}\nabla_k\nabla_j\sigma_i - \nabla_j\nabla_k\sigma_i &= \tfrac{1}{2}f(g_{ik}\sigma_j - g_{ij}\sigma_k + F_{ik}\sigma_j^* - F_{ij}\sigma_k^*)\\ &+ \tfrac{1}{2}h(g_{ik}\sigma_j^* - g_{ij}\sigma_k^* + F_{ik}\sigma_j - F_{ij}\sigma_k)\\ &+ f_k g_{ij} - f_j g_{ik} + h_k F_{ij} - h_j F_{ik}\end{aligned} \tag{2.2}$$

from which, using the Ricci identity, we obtain

$$\begin{aligned}\sigma_t R^t_{ijk} &= \tfrac{1}{2}f(g_{ik}\sigma_i - g_{ij}\sigma_k + F_{ik}\sigma_j^* - F_{ij}\sigma_k^*)\\ &+ \tfrac{1}{2}h(g_{ik}\sigma_j^* - g_{ij}\sigma_k^* + F_{ik}\sigma_j - F_{ij}\sigma_k)\\ &+ f_k g_{ij} = f_j g_{ik} + h_k F_{ij} - h_j F_{ik}.\end{aligned} \tag{2.3}$$

Transvecting (2.1) with F_h^i and using Ricci identity, we have

$$\begin{aligned}\sigma_t^* R^t_{ijk} &= \frac{1}{2}f(F_{ik}\sigma_j - F_{ij}\sigma_k + g_{ik}\sigma_j^* - g_{ij}\sigma_k^*)\\ &+\frac{1}{2}h(F_{ik}\sigma_j^* - F_{ij}\sigma_k^* + g_{ik}\sigma_j - g_{ij}\sigma_k)\\ &+f_k F_{ij} - f_j F_{ik} + h_k g_{ij} - h_j g_{ik}.\end{aligned} \tag{2.4}$$

Covariant differentiation of (2.3) and substitution from (1.6), (1.7) and (2.3) give us

$$\sigma_t\nabla_\ell R^t_{ijk} + fR_{\ell ijk} + hF^t_\ell R_{tijk} = \tag{2.5}$$

$$= -\frac{1}{2}\sigma_\ell(f_k g_{ij} - f_j g_{ik} + h_k F_{ij} - h_j F_{ik})$$
$$-\frac{1}{2}\sigma^*_\ell(f_k F_{ij} - f_j F_{ik} + h_k g_{ij} - h_j g_{ik}$$
$$+\frac{1}{2}f_\ell(g_{ik}\sigma_j - g_{ij}\sigma_k + F_{ik}\sigma^*_j - F_{ij}\sigma^*_k)$$
$$+\frac{1}{2}h_\ell(g_{ik}\sigma^*_j - g_{ij}\sigma^*_k + F_{ik}\sigma_j - F_{ij}\sigma_k)$$
$$+g_{ij}\nabla_\ell f_k - g_{ik}\nabla_\ell f_j + F_{ij}\nabla_\ell h_k - F_{ik}\nabla_\ell h_j$$
$$+\frac{1}{2}(f^2 + h^2)r_{\ell ijk} + fhr^*_{\ell ijk}.$$

Similary

$$\sigma^*_t\nabla_\ell R^t_{ijk} + hR_{\ell ijk} + fF^t_\ell R_{tijk} = \tag{2.6}$$
$$= -\frac{1}{2}\sigma^*_\ell(f_k g_{ij} - f_j g_{ik} + h_k F_{ij} - h_j F_{ik})$$
$$-\frac{1}{2}\sigma_\ell(f_k F_{ij} - f_j F_{ik} + h_k g_{ij} - h_j g_{ik})$$
$$+\frac{1}{2}f_\ell(g_{ik}\sigma^*_j - g_{ij}\sigma^*_k + F_{ik}\sigma_j - F_{ij}\sigma_k)$$
$$+\frac{1}{2}h_\ell(g_{ik}\sigma_j - g_{ij}\sigma_k + F_{ik}\sigma^*_j - F_{ij}\sigma^*_k)$$
$$+g_{ij}\nabla_\ell h_k - g_{ik}\nabla_\ell h_j + F_{ij}\nabla_\ell f_k - F_{ik}\nabla_\ell f_j$$
$$+\frac{1}{2}(f^2 + h^2)r^*_{\ell ijk} + fhr_{\ell ijk}.$$

Here the notations (1.11) have been used.

Applying the operator ∇_m to (2.5) substituting (1.6) and (1.8), then interchanging the indices m an ℓ and subtracting, we obtain

$$\sigma_t(\nabla_m\nabla_\ell R^t_{ijk} - \nabla_\ell\nabla_m R^t_{ijk}) = -(\nabla_m\sigma_t)\nabla_\ell R^t_{ijk} \tag{2.7}$$
$$-f_m R_{\ell ijk} - h_m F^t_\ell R_{tijk} - f\nabla_m R_{\ell ijk} - hF^t_\ell\nabla_m R_{tijk}$$
$$+\frac{1}{2}(ff_m + hh_m)r_{\ell ijk} + \frac{1}{2}(hf_m + fh_m)r^*_{hjk}$$
$$+g_{ij}\nabla_m\nabla_\ell f_k - g_{ik}\nabla_m\nabla_\ell f_j + F_{ij}\nabla_m\nabla_\ell h_k - F_{ik}\nabla_m\nabla_\ell h_j$$

$$-\frac{1}{2}g_{ij}\sigma_\ell\nabla_m f_k+\frac{1}{2}g_{ik}\sigma_\ell\nabla_m f_j$$
$$-\frac{1}{2}F_{ij}\sigma_\ell\nabla_m h_k+\frac{1}{2}F_{ik}\sigma_\ell\nabla_m h_j$$
$$-\frac{1}{2}F_{ij}\sigma^*_\ell\nabla_m f_k+\frac{1}{2}F_{ik}\sigma^*_\ell\nabla_m f_j$$
$$-\frac{1}{2}g_{ij}\sigma^*_\ell\nabla_m h_k+\frac{1}{2}g_{ik}\sigma^*_\ell\nabla_m h_j$$
$$+\frac{1}{4}f_\ell[g_{ik}(\sigma_m\sigma_j+\sigma^*_m\sigma^*_j)-g_{ij}(\sigma_m\sigma_k+\sigma^*_m\sigma^*_k)$$
$$+F_{ik}(\sigma_m\sigma^*_j+\sigma^*_m\sigma_j)-F_{ij}(\sigma_m\sigma^*_k+\sigma_k\sigma^*_m)]$$
$$+\frac{1}{4}h_\ell[g_{ik}(\sigma_m\sigma^*_j+\sigma^*_m\sigma_j)-g_{ij}(\sigma_m\sigma^*_k+\sigma^*_m\sigma_k)$$
$$+F_{ik}(\sigma_m\sigma_j+\sigma^*_m\sigma^*_j)-F_{ij}(\sigma_m\sigma_k+\sigma^*_m\sigma^*_k)]$$
$$-[b,m]$$

where $[b,m]$ indicates the total preceding expression with interchanged position of the indices ℓ and m.

Taking into account (1.6), we have

$$(\nabla_m\sigma_t)\nabla_\ell R^t_{ijk}+$$
$$=(\frac{1}{2}\sigma_m\sigma_t+\frac{1}{2}\sigma^*_m\sigma^*_t+fg_{mt}+hF_{mt})\nabla_\ell R^t_{ijk}.$$

Using (2.5) and (2.6), we can find the expression

$$(\nabla_m\sigma_t)(\nabla_\ell R^t_{ijk})-(\nabla_\ell\sigma_t)(\nabla_m R^t_{ijk}).$$

Substituting it into (2.7), we get

$$\sigma_t(\nabla_m\nabla_\ell R^t_{ijk}-\nabla_\ell\nabla_m R^t_{ijk})= \tag{2.8}$$
$$=(\frac{1}{2}f\sigma_m+\frac{1}{2}h\sigma^*_m-f_m)R_{\ell ijk}+(\frac{1}{2}f\sigma^*_m+\frac{1}{2}h\sigma_m-h_m)F^t_\ell R_{tijk}$$
$$+[\frac{1}{2}(ff_m+hh_m)-\frac{1}{4}(f^2+h^2)\sigma_m-\frac{1}{2}fh\sigma^*_m]r_{\ell ijk}$$
$$+[\frac{1}{2}(hf_m+fh_m)-\frac{1}{2}fh\sigma_m-\frac{1}{4}(f^2+h^2)\sigma^*_m]r^*_{\ell ijk}$$

$$+g_{ij}\nabla_m\nabla_\ell f_k - g_{ik}\nabla_m\nabla_\ell f_j + F_{ij}\nabla_m\nabla_\ell h_k - F_{ik}\nabla_m\nabla_\ell j_j$$
$$-[\ell, m].$$

Now, we have to calculate the expressions

$$\nabla_m\nabla_\ell f_k - \nabla_\ell\nabla_m f_k \quad \text{and} \quad \nabla_m\nabla_\ell h_k - \nabla_\ell\nabla_m h_k.$$

Because of $\sigma^i\sigma^t R_{tijk} = 0$, we have from (2.3)

$$f_k\sigma_j - f_j\sigma_k + h_k\sigma^*_j - h_j\sigma^*_k = 0.$$

Transvecting this with σ^j and $\sigma^{*\,j}$ and taking into account notations (1.4) and the fact that

$$\sigma^*_j\sigma^{*\,j} = \sigma$$

and

$$h_j\sigma^{*\,j} = f_j\sigma^j, \qquad h_j\sigma^j = f_j\sigma^{*\,j}$$

because of (1.7), we get

$$f_k\sigma + h_k\sigma^* = \sigma_k f_j\sigma^j + \sigma^*_k f_j\sigma^{*\,j},$$

$$f_k\sigma^* + h_k\sigma = \sigma_k f_j\sigma^{*\,j} + \sigma^*_k f_j\sigma^j,$$

from

$$(\sigma^2 - \sigma^{*\,2})f_k = [\sigma(f_j\sigma^j) - \sigma^*(f_j\sigma^{*\,j})]\sigma_k + [\sigma f_j\sigma^{*\,j} - \sigma^* f_j\sigma^j]\sigma^*_k.$$

Because of (1.5), this can be written in the form

$$f_k = v\sigma_k + w\sigma^*_k, \tag{2.9}$$

where v and w are some scalar functions. Consequently

$$h_k = f_t F^t_k = w\sigma_k + v\sigma^*_k. \tag{2.10}$$

Thus, we have

$$\nabla_m\nabla_\ell f_k - \nabla_\ell\nabla_m f_k = v(\nabla_m\nabla_\ell\sigma_k - \nabla_\ell\nabla_m\sigma_k) + w(\nabla_m\nabla_\ell\sigma^*_k - \nabla_\ell\nabla_m\sigma^*_k),$$

which, because of (2.2), (2.9) and (2.10) can be written in the form

$$\nabla_m\nabla_\ell f_k - \nabla_\ell\nabla_m f_k =$$

$$= g_{km}\{[\frac{1}{2}(vf+wh)-(v^2+w^2)]\sigma_\ell+[\frac{1}{2}(vh+wf)-2vw]\sigma^*_\ell\}$$

$$+F_{km}\{[\frac{1}{2}(vh+wf)-2vw]\sigma_\ell+[\frac{1}{2}(vf+wf)-(v^2+w^2)]\sigma^*_\ell\}$$

$$-[\ell,m].$$

Similarly, we can find the expression $\nabla_m\nabla_\ell h_k-\nabla_\ell\nabla_m h_k$. Substituting them, (2.9) and (2.10) into (2.8), we get

$$\sigma_t(\nabla_m\nabla_\ell R^t_{ijk}-\nabla_\ell\nabla_m R^t_{ijk})= \tag{2.11}$$

$$=[PR_{\ell ijk}+QF^t_\ell R_{tijk}-(P^2+Q^2)r_{\ell ijk}-2PQr^*_{\ell ijk}]\sigma_m$$

$$+[QR_{\ell ijk}+PF^t_\ell R_{tijk}-2PQr_{\ell ijk}-(P^2+Q^2)r^*_{\ell ijk}]\sigma^*_m$$

$$-[\ell,m]$$

where

$$P=\frac{1}{2}\,f-v,\qquad Q=\frac{1}{2}h-w. \tag{2.12}$$

Using (2.9), (2.10) and (2.12), we can rewrite (2.3) and (2.3) as follows

$$\sigma_t R^t_{ijk}=(Pg_{ik}+QF_{ik})\sigma_j-(Pg_{ij}+QF_{ij})\sigma_k \tag{2.13}$$

$$+(Qg_{ik}+PF_{ik})\sigma^*_j-(Qg_{ij}+PF_{ij})\sigma^*_k,$$

$$\sigma^*_t R^t_{ijk}=(Qg_{ik}+PF_{ik})\sigma_j-(Qg_{ij}+PF_{ij})\sigma_k \tag{2.14}$$

$$+(Pg_{ik}+QF_{ik})\sigma^*_j-(Pg_{ij}+QF_{ij})\sigma^*_k.$$

Transvecting (2.13) and (2.14) with g^{ij} we obtain

$$\sigma_t R^t_k=\sigma^*_t R^{*\,t}_k=[(2-n)P-\varphi Q]\sigma_k+[(2-n)Q-\varphi P]\sigma^*_k, \tag{2.15}$$

$$\sigma_t R^{*\,t}_k=\sigma^*_t R^t_k=[(2-n)Q-\varphi P]\sigma_k+[(2-n)P-\varphi Q]\sigma^*_k. \tag{2.16}$$

3. The form of the expression (1.15), obtained from the condition (1.14)

Taking into account (1.10), (1.11), and (1.13), as well as (1.1), the condition (1.14) reduces to

$$\nabla_m \nabla_\ell R_{hijk} - \nabla_\ell \nabla_m R_{hijk} =$$

$$= -\alpha_2(\nabla_m \nabla_\ell s_{hijk} - \nabla_\ell \nabla_m s_{hijk}) - \beta_2(\nabla_m \nabla_\ell s^*_{hijk} - \nabla_\ell \nabla_m s^*_{hijk}).$$

Therefore

$$\sigma_t(\nabla_m \nabla_\ell R^t_{ijk} - \nabla_\ell \nabla_m R^t_{ijk}) = \tag{3.1}$$

$$= -\alpha_2 \sigma_t(\nabla_m \nabla_\ell s^t_{ijk} - \nabla_\ell \nabla_m s^t_{ijk}) - \beta_2 \sigma_t(\nabla_m \nabla_\ell s^{*\,t}_{ijk} - \nabla_\ell \nabla_m s^{*\,t}_{ijk}).$$

Let us put

$$\rho_{hijk} = R_{hj} g_{ik} - R_{hk} g_{ij} + R^*_{hj} \cdot F_{ik} - R^*_{hk} F_{ij}, \tag{3.2}$$

$$\rho^*_{hijk} = F^t_h \rho_{tijk} = R^*_{hj} g_{ik} + R^*_{hk} g_{ij} + R_{hj} \cdot F_{ik} - R_{hk} F_{ij}, \tag{3.3}$$

Then

$$s_{hijk} = \rho_{hijk} - \rho_{ihkj}$$

and

$$\nabla_m \nabla_\ell s_{hijk} - \nabla_\ell \nabla_m s_{hijk} = \tag{3.4}$$

$$= (\nabla_m \nabla_\ell \rho_{hijk} - \nabla_\ell \nabla_m \rho_{hijk}) - (\nabla_m \nabla_\ell \rho_{ihkj} - \nabla_\ell \nabla_m \rho_{ihkj})$$

Using the Ricci identity, we get from (3.2)

$$\nabla_m \nabla_\ell \rho_{hijk} - \nabla_\ell \nabla_m \rho_{hijk} =$$

$$= (R_{tj} R^t_{h\ell m} + R_{ht} R^t_{j\ell m}) g_{ik} - (R_{tk} R^t_{h\ell m} + R_{ht} R^t_{k\ell m}) g_{ij}$$

$$+ (R^*_{tj} R^t_{h\ell m} + R^*_{ht} R^t_{j\ell m}) F_{ik} - (R^*_{tk} R^t_{h\ell m} + R^*_{ht} R^t_{k\ell m}) F_{ij},$$

or

$$\sigma_p(\nabla_m \nabla_\ell \rho^p_{ijk} - \nabla_\ell \nabla_m \rho^p_{ijk}) = (-\sigma_p R^p_{t\ell m} R^t_j + \tag{3.5}$$

$$+ \sigma_p R^p_t R^t_{j\ell m}) g_{ik} + (\sigma_p R^p_{r\ell m} R^t_k - \sigma_p R^p_t R^t_{k\ell m}) g_{ij}$$

$$+(-\sigma_p R^p_{t\ell m} R^{*\,t}_j \sigma_p R^{*\,p}_t R^t_{j\ell m}) F_{ik} + (\sigma_p R^p_{t\ell m} R^{*\,t}_k - \sigma_p R^{*\,p}_t R^t_{k\ell m}) F_{ij}.$$

Similarly

$$\sigma_p(\nabla_m \nabla_\ell g^p_{ikj} - \nabla_\ell \nabla_m \rho^p_{ijk}) = \tag{3.6}$$

$$= (R_{tj} R^t_{i\ell m} + R_{it} R^t_{j\ell m})\sigma_k - (R_{tk} R^t_{i\ell m} + R_{it} R^t_{k\ell m})\sigma_j +$$
$$R^*_{tj} R^t_{i\ell m} + R^*_{it} R^t_{j\ell m})\sigma^*_k - (R^*_{tk} R^t_{i\ell m} + R^*_{it} R^t_{k\ell m})\sigma^*_j.$$

Substituting (2.13), (2.15) and (2.16) into (3.5), we find

$$\sigma_p(\nabla_m \nabla_\ell \rho^p_{ijk} - \nabla_\ell \nabla_m \rho^p_{ijk}) = \tag{3.7}$$

$$= (Lr_{mijk} + Nr^*_{mijk} - P\rho_{mijk} - Q\rho^*_{mijk})\sigma_\ell$$
$$+(Nr_{mijk} + Lr^*_{mijk} - P\rho^*_{mijk} - Q\rho_{mijk})\sigma^*_\ell$$
$$-[\ell, m],$$

where

$$L = (2-n)(P^2 + Q^2) - 2\varphi PQ, \quad N = 2(2-n)PQ - \varphi(P^2 + Q^2). \tag{3.8}$$

From (3.4), (3.6) and (3.7), we obtain

$$\sigma_t(\nabla_m \nabla_\ell s^t_{ijk} - \nabla_\ell \nabla_m s^t_{ijk}) =$$

$$= -(R_{tj} R^t_{i\ell m} + R_{it} R^t_{j\ell m})\sigma_k + (R_{tk} R^t_{i\ell m} + R_{it} R^t_{k\ell m})\sigma_j$$
$$= (R^*_{tj} R^t_{i\ell m} + R^*_{it} R^t_{j\ell m})\sigma^*_k + (R^*_{tk} R^t_{i\ell m} + R^*_{it} R^t_{k\ell m})\sigma^*_j$$
$$+(Lr_{mijk} + N^*_{mijk} - P\rho_{mijk} - Q\rho^*_{mijk})\sigma_\ell$$
$$-(Lr_{\ell ijk} + Nr^*_{\ell ijk} - P\rho_{\ell ijk} - Q\rho^*_{\ell ijk})\sigma_m$$
$$+(Nr_{mijk} + Lr^*_{mijk} - Q\rho_{mijk} - P\rho^*_{mijk})\sigma^*_\ell$$
$$-(Nr_{\ell ijk} + Lr^*_{\ell ijk} - Q\rho_{\ell ijk} - P\rho^*_{\ell ijk})\sigma^*_m.$$

In a similar way we find the expression for

$$\sigma_t(\nabla_m \nabla_\ell s^{*\,t}_{ijk} - \nabla_\ell \nabla_m s^{*\,t}_{ijk}).$$

Therefore, (3.1) can be written in the form

$$\sigma_t(\nabla_m \nabla_\ell R^t_{ijk} - \nabla_\ell \nabla_m R^t_{ijk}) = \tag{3.9}$$

3. The form of the expression (1.15), obtained from the condition (1.14)

Taking into account (1.10), (1.11), and (1.13), as well as (1.1), the condition (1.14) reduces to

$$\nabla_m \nabla_\ell R_{hijk} - \nabla_\ell \nabla_m R_{hijk} =$$
$$= -\alpha_2(\nabla_m \nabla_\ell s_{hijk} - \nabla_\ell \nabla_m s_{hijk}) - \beta_2(\nabla_m \nabla_\ell s^*_{hijk} - \nabla_\ell \nabla_m s^*_{hijk}).$$

Therefore

(3.1) $$\sigma_t(\nabla_m \nabla_\ell R^t_{ijk} - \nabla_\ell \nabla_m R^t_{ijk}) =$$
$$= -\alpha_2 \sigma_t(\nabla_m \nabla_\ell s^t_{ijk} - \nabla_\ell \nabla_m s^t_{ijk}) - \beta_2 \sigma_t(\nabla_m \nabla_\ell s^{*\,t}_{ijk} - \nabla_\ell \nabla_m s^{*\,t}_{ijk}).$$

Let us put

(3.2) $$\rho_{hijk} = R_{hj} g_{ik} - R_{hk} g_{ij} + R^*_{hj} \cdot F_{ik} - R^*_{hk} F_{ij},$$

(3.3) $$\rho^*_{hijk} = F^t_h \rho_{tijk} = R^*_{hj} g_{ik} + R^*_{hk} g_{ij} + R_{hj} \cdot F_{ik} - R_{hk} F_{ij},$$

Then

$$s_{hijk} = \rho_{hijk} - \rho_{ihkj}$$

and

(3.4) $$\nabla_m \nabla_\ell s_{hijk} - \nabla_\ell \nabla_m s_{hijk} =$$
$$= (\nabla_m \nabla_\ell \rho_{hijk} - \nabla_\ell \nabla_m \rho_{hijk}) - (\nabla_m \nabla_\ell \rho_{ihkj} - \nabla_\ell \nabla_m \rho_{ihkj})$$

Using the Ricci identity, we get from (3.2)

$$\nabla_m \nabla_\ell \rho_{hijk} - \nabla_\ell \nabla_m \rho_{hijk} =$$
$$= (R_{tj} R^t_{h\ell m} + R_{ht} R^t_{j\ell m}) g_{ik} - (R_{tk} R^t_{h\ell m} + R_{ht} R^t_{k\ell m}) g_{ij}$$
$$+ (R^*_{tj} R^t_{h\ell m} + R^*_{ht} R^t_{j\ell m}) F_{ik} - (R^*_{tk} R^t_{h\ell m} + R^*_{ht} R^t_{k\ell m}) F_{ij},$$

or

(3.5) $$\sigma_p(\nabla_m \nabla_\ell \rho^p_{ijk} - \nabla_\ell \nabla_m \rho^p_{ijk}) = (-\sigma_p R^p_{t\ell m} R^t_j +$$
$$+ \sigma_p R^p_t R^t_{j\ell m}) g_{ik} + (\sigma_p R^p_{r\ell m} R^t_k - \sigma_p R^p_t R^t_{k\ell m}) g_{ij}$$

$$+(-\sigma_p R^p_{t\ell m} R^{*\,t}_j \sigma_p R^{*\,p}_t R^t_{j\ell m})F_{ik} + (\sigma_p R^p_{t\ell m} R^{*\,t}_k - \sigma_p R^{*\,p}_t R^t_{k\ell m})F_{ij}.$$

Similarly

$$(3.6)\qquad \sigma_p(\nabla_m \nabla_\ell g^p_{ikj} - \nabla_\ell \nabla_m \rho^p_{ijk}) =$$

$$= (R_{tj} R^t_{i\ell m} + R_{it} R^t_{j\ell m})\sigma_k - (R_{tk} R^t_{i\ell m} + R_{it} R^t_{k\ell m})\sigma_j +$$
$$R^*_{tj} R^t_{i\ell m} + R^*_{it} R^t_{j\ell m})\sigma^*_k - (R^*_{tk} R^t_{i\ell m} + R^*_{it} R^t_{k\ell m})\sigma^*_j.$$

Substituting (2.13), (2.15) and (2.16) into (3.5), we find

$$(3.7)\qquad \sigma_p(\nabla_m \nabla_\ell \rho^p_{ijk} - \nabla_\ell \nabla_m \rho^p_{ijk}) =$$

$$= (Lr_{mijk} + Nr^*_{mijk} - P\rho_{mijk} - Q\rho^*_{mijk})\sigma_\ell$$
$$+(Nr_{mijk} + Lr^*_{mijk} - P\rho^*_{mijk} - Q\rho_{mijk})\sigma^*_\ell$$
$$-[\ell, m],$$

where

$$(3.8)\quad L = (2-n)(P^2+Q^2) - 2\varphi PQ, \quad N = 2(2-n)PQ - \varphi(P^2+Q^2).$$

From (3.4), (3.6) and (3.7), we obtain

$$\sigma_t(\nabla_m \nabla_\ell s^t_{ijk} - \nabla_\ell \nabla_m s^t_{ijk}) =$$

$$= -(R_{tj} R^t_{i\ell m} + R_{it} R^t_{j\ell m})\sigma_k + (R_{tk} R^t_{i\ell m} + R_{it} R^t_{k\ell m})\sigma_j$$
$$= (R^*_{tj} R^t_{i\ell m} + R^*_{it} R^t_{j\ell m})\sigma^*_k + (R^*_{tk} R^t_{i\ell m} + R^*_{it} R^t_{k\ell m})\sigma^*_j$$
$$+(Lr_{mijk} + N^*_{mijk} - P\rho_{mijk} - Q\rho^*_{mijk})\sigma_\ell$$
$$-(Lr_{\ell ijk} + Nr^*_{\ell ijk} - P\rho_{\ell ijk} - Q\rho^*_{\ell ijk})\sigma_m$$
$$+(Nr_{mijk} + Lr^*_{mijk} - Q\rho_{mijk} - P\rho^*_{mijk})\sigma^*_\ell$$
$$-(Nr_{\ell ijk} + Lr^*_{\ell ijk} - Q\rho_{\ell ijk} - P\rho^*_{\ell ijk})\sigma^*_m.$$

In a similar way we find the expression for

$$\sigma_t(\nabla_m \nabla_\ell s^{*\,t}_{ijk} - \nabla_\ell \nabla_m s^{*\,t}_{ijk}).$$

Therefore, (3.1) can be written in the form

$$(3.9)\qquad \sigma_t(\nabla_m \nabla_\ell R^t_{ijk} - \nabla_\ell \nabla_m R^t_{ijk}) =$$

$$= [(\alpha_2 R_{tj} + \beta_2 R^*_{tj}) R^t_{i\ell m} + (\alpha_2 R_{it} + \beta_2 R^*_{it}) R^t_{j\ell m}]\sigma_k$$

$$= [(\alpha_2 R_{tk} + \beta_2 R^*_{tk}) R^t_{i\ell m} + (\alpha_2 R_{it} + \beta_2 R^*_{it}) R^t_{k\ell m}]\sigma_j$$

$$+[(\alpha_2 R^*_{tj} + \beta_2 R_{tj}) R^t_{i\ell m} + (\alpha_2 R^*_{it} + \beta_2 R_{it}) R^t_{j\ell m}]\sigma^*_k$$

$$-[(\alpha_2 R^*_{tk} + \beta_2 R_{tk}) R^t_{i\ell m} + (\alpha_2 R^*_{it} + \beta_2 R_{it}) R^t_{k\ell m}]\sigma^*_j$$

$$-[(\alpha_2 P + \beta_2 Q)\rho_{mijk} + (\beta_2 P + \alpha_2 Q)\rho^*_{mijk} - (\alpha_2 L + \beta_2 N) r_{mijk}$$

$$-(\beta_2 L + \alpha_2 N) r^*_{mijk}]\sigma_\ell$$

$$-[(\alpha_2 P + \beta_2 Q)\rho_{\ell ijk} + (\beta_2 P + \alpha_2 Q)\rho^*_{\ell ijk} - (\alpha_2 L + \beta_2 N) r_{\ell ijk}$$

$$-(\beta_2 L + \alpha_2 N) r^*_{\ell ijk}]\sigma_m$$

$$+[(\beta_2 P + \alpha_2 Q)\rho_{mijk} + (\alpha_2 P + \beta_2 Q)\rho^*_{mijk} - (\beta_2 L + \alpha_2 N) r_{mijk}$$

$$-[(\alpha_2 L + \beta_2 N) r^*_{mijk}]\sigma^*_\ell$$

$$-[(\beta_2 P + \alpha_2 Q)\rho_{\ell ijk} + (\alpha_2 P + \beta_2 Q)\rho^*_{\ell ijk} - (\beta_2 L + \alpha_2 N) r_{\ell ijk}$$

$$-(\alpha_2 L + \beta_2 N) r^*_{\ell ijk}]\sigma^*_m.$$

4. The form of the curvature tensor

Substituting (3.9) into (2.11), we find

$$[(\alpha_2 R_{tj} + \beta_2 R^*_{tj}) R^t_{i\ell m} + (\alpha_2 R_{it} + \beta_2 R^*_{it}) R^t_{j\ell m}]\sigma_k$$

$$+[(\alpha_2 R^*_{tj} + \beta_2 R_{tj}) R^t_{i\ell m} + (\alpha_2 R^*_{it} + \beta_2 R_{it}) R^t_{j\ell m}]\sigma^*_k$$

$$-[k, j] =$$

$$= [P R_{\ell ijk} + Q F^t_\ell R_{tijk} - L_1 r_{\ell ijk} - N_1 r^*_{\ell ijk} + P_1 \rho_{\ell ijk} + Q_1 \rho^*_{\ell ijk}]\sigma_m$$

$$+[Q R_{\ell ijk} + P F^t_\ell R_{tijk} - N_1 r_{\ell ijk} - L_1 r^*_{\ell ijk} + Q_1 \rho_{\ell ijk} + P_1 \rho^*_{\ell ijk}]\sigma^*_m$$

$$-[\ell, m],$$

where

$$P_1 = \alpha_2 P + \beta_2 Q, \quad Q_1 = \beta_2 P + \alpha_2 Q, \tag{4.1}$$

$$L_1 = \alpha_2 L + \beta_2 N + P^2 + Q^2, \quad N_1 = \beta_2 L + \alpha_2 N + 2PQ. \tag{4.2}$$

Transvecting the preceding equation with σ^ℓ and using (1.4), we have

$$(4.3)\qquad [(\alpha_2 R_{tj}+\beta_2 R^*_{tj})\sigma^\ell R^t_{i\ell m}+(\alpha_2 R_{it}+\beta_2 R^*_{it})\sigma^\ell R^t_{j\ell m}]\sigma_k$$

$$+[(\alpha_2 R^*_{tj}+\beta_2 R_{tj})\sigma^\ell R^t_{i\ell m}+(\alpha_2 R^*_{it}+\beta_2 R_{it})\sigma^\ell R^t_{j\ell m}]\sigma^*_k$$

$$-[k,j]=$$

$$=-\sigma[PR_{mijk}+QF^t_m R_{tijk}-L_1 r_{mijk}-N_1 r^*_{mijk}+P_1\rho_{mijk}+Q_1\rho^*_{mijk}]$$

$$-\sigma^*[QR_{mijk}+PF^t_m R_{tijk}-N_1 r_{mijk}-L_1 r^*_{mijk}+Q_1\rho_{mijk}+P_1\rho^*_{mijk}]$$

$$+[P\sigma^\ell R_{\ell ijk}+Q\sigma^{*\,\ell} R_{\ell ijk}-L_1\sigma^\ell r_{\ell ijk}-N_1\sigma^\ell r^*_{\ell ijk}+P_1\sigma^\ell\rho_{\ell ijk}+Q_1\sigma^\ell\rho^*_{\ell ijk}]\sigma_m$$

$$+[Q\sigma^\ell R_{\ell ijk}+P\sigma^{*\,\ell} R_{\ell ijk}-N_1\sigma^\ell r_{\ell ijk}-L_1\sigma^\ell r^*_{\ell ijk}+Q_1\sigma^\ell\rho_{\ell ijk}+P_1\sigma^\ell\rho^*_{\ell ijk}]\sigma^*_m.$$

Using (2.13) and (2.14), we can see that

$$P\sigma^\ell R_{\ell ijk}+Q\sigma^{*\,\ell}R_{\ell ijk}=$$

$$=[(P^2+Q^2)g_{ik}+2PQF_{ik}]\sigma_j+[2PQg_{ik}+(P^2+Q^2)F_{ik}]\sigma^*_j-[j,k],$$

and

$$Q\sigma^\ell R_{\ell ijk}+P\sigma^{*\,\ell}R_{\ell ijk}+$$

$$=[2PQg_{ik}+(P^2+Q^2)F_{ik}]\sigma_j+[(P^2+Q^2)g_{ik}+2PQF_{ik}]\sigma^*_j-[j,k].$$

Similarly, using (3.2), (3.3), (2.15) and 2.16), we can show that

$$P_1\sigma^\ell g_{\ell ijk}+Q_1\sigma^\ell\rho^*_{\ell ijk}=$$

$$=\{[(2-n)(PP_1+QQ_1)-\varphi(P_1Q+Q_1P)]\sigma_j$$

$$+[(2-n)(P_1Q+Q_1P)-\varphi(PP_1+QQ_1)]\sigma^*_j\}g_{ik}$$

$$+\{[(2-n)(P_1Q+Q_1P)-\varphi(PP_1+QQ_1P)]\sigma_j$$

$$+[(2-n)(PP_1+QQ_1P)-\varphi(P_1Q+Q_1P)]\sigma^*_j\}F_{ik}$$

$$-[j,k],$$

and

$$Q_1\sigma^\ell\rho_{\ell ijk}+P_1\sigma^\ell\rho^*_{\ell ijk}=$$

$$=\{[(2-n)(PQ_1+QP_1)-\varphi(PP_1+QQ_1)]\sigma_j$$

$$+[(2-n)(PP_1+QQ_1)-\varphi(PQ_1+QP_1)]\sigma^*_j\}g_{ik}$$

$$+\{[(2-n)(PP_1+QQ_1)-\varphi(P_1Q+Q_1P)]\sigma_j$$
$$+[(2-n)(P_1Q+Q_1P)-\varphi(PP_1+QQ_1)]\sigma_j^*\}F_{ik}$$
$$=[j,k].$$

Also

$$L_1\sigma^\ell r_{\ell ijk}+N_1\sigma^\ell r^*_{\ell ijk}=$$
$$=(L_1g_{ik}+N_1F_{ik})\sigma_j+(N_1g_{ik}+L_1F_{ik})\sigma_j^*-[j,k]$$

and

$$N_1\sigma^\ell r_{\ell ijk}+L_1\sigma^\ell r^*_{\ell ijk}=$$
$$=(N_1g_{ik}+L_1F_{ik})\sigma_j+(L_1g_{ik}+N_1F_{ik})\sigma_j^*-[j,k].$$

Therefore

$$P\sigma^\ell R_{\ell ijk}+Q\sigma^{*\,\ell}R_{\ell ijk}-L_1\sigma^\ell r_{\ell ijk}-N_1\sigma^\ell r^*_{\ell ijk}+P_1\sigma^\ell\rho_{\ell ijk}+Q_1\sigma^\ell\rho^*_{\ell ijk}=$$

$$=[(2-n)(PP_1+QQ_1)-\varphi(P_1Q+Q_1P)+P^2+Q^2-L_1]\sigma_jg_{ik}$$
$$=[(2-n)(P_1Q+PQ_1)-\varphi(PP_1+QQ_1)+2PQ-N_1]\sigma_jF_{ik}$$
$$=[(2-n)(P_1Q+PQ_1)-\varphi(PP_1+QQ_1)+2PQ-N_1]\sigma_j^*g_{ik}$$
$$=[(2-n)(P_1Q+PQ_1)-\varphi(PP_1+QQ_1)+2PQ-N_1]\sigma_j^*g_{ik}$$
$$=[(2-n)(PP_1+QQ_1)-\varphi(P_1Q+Q_1P)+P^2+Q^2-L_1]\sigma_jF_{ik}$$
$$=[j,k].$$

But

$$(2-n)(PP_1+QQ_1)-\varphi(P_1Q+PQ_1)+P^2+Q^2=L_1,$$
$$(2-n)(P_1Q+PQ_1)-\varphi(PP_1+QQ_1)+2PQ=N_1,$$

because of (4.1) and (4.2). Therefore

$$P\sigma^\ell R_{\ell ijk}+Q\sigma^{*\,\ell}R_{\ell ijk}-L_1\sigma^\ell r_{\ell ijk}-N_1\sigma^\ell r^*_{\ell ijk}+P_1\sigma^\ell\rho_{\ell ijk}+Q_1\sigma^\ell\rho^*_{\ell ijk}=0$$

and (4.3) reduces to

$$(P\sigma+Q\sigma^*)R_{mijk}+(Q\sigma+P\sigma^*)F^t_mR_{tijk}$$
$$-\sigma(L_1r_{mijk}+N_1r^*_{mijk}-P_1\rho_{mijk}-Q_1\rho^*_{mijk})$$
$$+\sigma^*(N_1r_{mijk}+L_1r^*_{mijk}-Q_1\rho_{mijk}-P_1\rho^*_{mijk})=$$

$$= [(\alpha_2 R_{tk} + \beta_2 R^*_{tk})\sigma^\ell R^t_{i\ell m} + (\alpha_2 R_{it} + \beta_2 R^*_{it})\sigma^\ell R^t_{k\ell m}]\sigma_j$$
$$+[(\alpha_2 R^*_{tk} + \beta_2 R_{tk})\sigma^\ell R^t_{i\ell m} + (\alpha_2 R^*_{it} + \beta_2 R_{it})\sigma^\ell R^t_{k\ell m}]\sigma^*_j$$
$$-[j,k].$$

Using first (2.13) and then (2.15) and (2.16) as well as (3.8), (4.1) and (4.2), we can rewrite the preceding relation in the form

$$(P\sigma + Q\sigma^*)R_{mijk} + (Q\sigma + P\sigma^*)F^t_m R_{tijk} = T_{mijk} \tag{4.4}$$

where

$$T_{mijk} = (L_1\sigma + N_1\sigma^*)r_{mijk} + (N_1\sigma + L_1\sigma^*)r^*_{mijk}$$
$$-(P_1\sigma + Q_1\sigma^*)\rho_{mijk} - (Q_1\sigma + P_1\sigma^*)\rho^*_{mijk}$$
$$-(L_0 g_{mj} + N_0 F_{mj} - P_1 R_{mj} - Q_1 R^*_{mj})(\sigma_i\sigma_k + \sigma^*_i\sigma^*_k)$$
$$-(N_0 g_{mj} + L_0 F_{mj} - Q_1 R_{mj} - P_1 R^*_{mj})(\sigma^*_i\sigma_k + \sigma_i\sigma^*_k)$$
$$+(L_0 g_{mk} + N_0 F_{mk} - P_1 R_{mk} - Q_1 R^*_{mk})(\sigma_i\sigma_j + \sigma^*_i\sigma^*_j)$$
$$+(N_0 g_{mk} + L_0 F_{mk} - Q_1 R_{mk} - P_1 R^*_{mk})(\sigma^*_i\sigma_j + \sigma_i\sigma^*_j)$$

and

$$L_0 = \alpha_2 L + \beta_2 N; \quad N_0 = \alpha_2 N + \beta_2 L.$$

Puting in (4.4) s instead of m and transvecting with F^s_m, we find

$$(Q\sigma + P\sigma^*)R_{mijk} + (P\sigma + Q\sigma^*)F^t_m R_{tijk} = F^s_m T_{sijk}. \tag{4.5}$$

The equations (4.4) and (4.5) give us

$$(P^2 - Q^2)(\sigma^2 - \sigma^{*2})R_{mijk} =$$
$$= (P\sigma + Q\sigma^*)T_{mijk} - (Q\sigma - P\sigma^*)F^s_m T_{sijk},$$

which, after some calculation, can be expressed in the form

$$(P^2 - Q^2)(\sigma^2 - \sigma^{*2})R_{mijk} = (P^2 - Q^2)\{(\sigma^2 - \sigma^{*2})$$
$$\cdot[(P + P_3)r_{mijk} + (Q + Q_3)r^*_{mijk} - \alpha_2\rho_{mijk} - \beta_2\rho^*_{mijk}]$$
$$+[-(P_3\sigma - Q_3\sigma^*)g_{mj} + (P_3\sigma^* - Q_3\sigma)F_{mj} + (\alpha_2\sigma - \beta_2\sigma^*)R_{mj}$$
$$-(\alpha_2\sigma^* - \beta_2\sigma)R^*_{mj}](\sigma_i\sigma_k + \sigma^*_i\sigma^*_k)$$

$$+[(P_3\sigma^* - Q_3\sigma)g_{mj} - (P_3\sigma - Q_3\sigma^*)F_{mj} - (\alpha_2\sigma^* - \beta_2\sigma)R_{mj}$$
$$+(\alpha_2\sigma - \beta_2\sigma^*)R^*_{mj}](\sigma^*_i\sigma_k + \sigma_i\sigma^*_k)$$
$$+[(P_3\sigma - Q_3\sigma^*)g_{km} - (P_3\sigma^* - Q_3\sigma)F_{mk} - (\alpha_2\sigma - \beta_2\sigma^*)R_{mk}$$
$$+(\alpha_2\sigma^* - \beta_2\sigma)R^*_{mk}](\sigma_i\sigma_j + \sigma^*_i\sigma^*_j)$$
$$+[-(P_3\sigma^* - Q_3\sigma)g_{mk} + (P_3\sigma - Q_3\sigma^*)F_{mk} + (\alpha_2\sigma^* - \beta_2\sigma)R_{mk}$$
$$-(\alpha_2\sigma - \beta_2\sigma^*)R^*_{mk}](\sigma^*_i\sigma_j + \sigma_i\sigma^*)\},$$

where

$$P_3 = [(2-n)\alpha_2 - \varphi\beta_2]P - (\alpha_2\varphi - (2-n)\beta_2]Q,$$
$$Q_3 = -[\alpha_2\varphi - (2-n)\beta_2]P + [(2-n)\alpha_2 - \varphi\beta_2]Q.$$

Thus, if $P^2 - Q^2 \neq 0$, we have

(4.6) $$R_{mijk} = (P + P_3r_{mijk} + (Q + Q_3)r^*_{mijk} - \alpha_2\rho_{mijk} - \beta_2\rho^*_{mijk}$$
$$+\frac{1}{\sigma^2 - \sigma^{*2}}\{[-(P_3\sigma - Q_3\sigma^*)g_{mj} + (P_3\sigma^* - Q_3\sigma)F_{mj} + (\alpha_2\sigma - \beta_2\sigma^*)R_{mj}$$
$$-(\alpha_2\sigma^* - \beta_2\sigma)R^*_{mj}](\sigma_i\sigma_k + \sigma^*_i\sigma^*_k)$$
$$+[(P_3\sigma^* - Q_3\sigma)g_{mj} - (P_3 - Q_3\sigma^*)F_{mj} - (\alpha_2\sigma^* - \beta_2\sigma)R_{mj}$$
$$+(\alpha_2 - \beta_2\sigma^*)R^*_{mj}](\sigma^*_i\sigma_k - \sigma_i\sigma^*_k)$$
$$+[(P_3\sigma - Q_3\sigma^*)g_{mk} - (P_3\sigma^* - Q_3\sigma)F_{mk} - (\alpha_2\sigma - \beta_2\sigma^*)R_{mk}$$
$$+(\alpha_2\sigma^* - \beta_2\sigma)R^*_{mk}](\sigma_i\sigma_j - \sigma^*_i\sigma^*_j)$$
$$+[(P_3\sigma^* - Q_3\sigma)g_{mk} + (P_3\sigma - Q_3\sigma^*)F_{mk}$$
$$+(\alpha_2\sigma^* - \beta_2\sigma)R_{mk}$$
$$+(\alpha_2 - \beta_2\sigma^*)R^*_{mj}](\sigma^*_i\sigma_j + \sigma_i\sigma^*_j)\}$$

because of (1.5).

Now, we have to calculate R_{ij} and R^*_{ij}.

Transvecting (4.6) with g^{mk} and using (2.15), (2.16), (1.11), (3.2) and (3.3), we find

(4.7) $$(1 + 2\alpha_2)R_{ij} + 2\beta_2R^*_{ij} = E(\sigma_i\sigma_j + \sigma^*_i\sigma^*_j) + H(\sigma^*_i\sigma_j + \sigma_i\sigma^*_j)$$
$$+[\alpha_2R + \beta_2R^* + (2-n)(P + P_3) - \varphi(Q + Q_3)]g_{ij}$$

$$+[\alpha_2 R^* + \beta_2 R - \varphi(P+P_3) + (2-n_/(Q+Q_3)]\bar{r}_{ij}$$

where

$$E = \frac{1}{\sigma^2 - \sigma^{*\,2}}[(\alpha_4 P + \beta_4 Q)\sigma - (\beta_4 P + \alpha_4 Q)\sigma^*$$
$$-(\alpha_2\sigma - \beta_2\sigma^*)R + (\alpha_2\sigma^* - \beta_2\sigma)R^*],$$
$$H = \frac{1}{\sigma^2 - \sigma^{*\,2}}[(\beta_4 P + \alpha_4 Q)\sigma - (\alpha_4 P + \beta_4 Q)\sigma^*$$
$$-(\alpha_2\sigma - \beta_2\sigma)R - (\alpha_2\sigma - \beta_2\sigma^*)R^*],$$
$$\alpha_4 = -[n(n-2) + \varphi^2]\alpha_2 - 2\varphi(n-1)\beta_2$$
$$\beta_4 = -2\varphi(n-1)\alpha_2 - [n(n-2) + \varphi^2]\beta_2.$$

Putting in (4.7) s instead of i and transvecting with F_i^s, we get

(4.8) $$2\beta_2 R_{ij} + (1+2\alpha_2)R^*_{ij} = H(\sigma_i\sigma_j + \sigma_i^*\sigma_j^*) + E(\sigma_i^*\sigma_j + \sigma_i\sigma_j^*)$$
$$+[\alpha_2 R^* + \beta_2 R - \varphi(P+P_3) + (2-n)(Q+Q_3)]g_{ij}$$
$$+[\alpha_2 R + \beta_2 R^* + (2-n)(P+P_3) - \varphi(Q+Q_3)]F_{ij}.$$

From (4.7) and (4.8), we have

$$[(1+2\alpha_2)^2 - 4\beta_2^2]R_{ij} =$$
$$= [(1+2\alpha_2)E - 2\beta_2 H](\sigma_i\sigma_j + \sigma_i^*\sigma_j^*) + [(1+2\alpha_2)H - 2\beta_2 E)(\sigma_i^*\sigma_j + \sigma_i\sigma_j^*)$$
$$+\{(1+2\alpha_2)[\alpha_2 R + \beta_2 R^* + (2-n)(P+P_3) - \varphi(Q+Q_3)]$$
$$-2\beta_2[\alpha_2 R^* + \beta_2 R - \varphi(P+P_3) + (2-n)(Q+Q_3)]\}g_{ij}$$
$$\{(1+2\alpha_2)[\alpha_2 R^* + \beta_2 R - \varphi(P+P_3) + (2-n)(Q+Q_3)]$$
$$-2\beta_2[\alpha_2 R + \beta_2 R^* + (2-n)(P+P_3) - \varphi(Q+Q_3)]\}F_{ij}.$$

Because of the assumption $p > 2,\ q > 2,\ [(1_2\alpha_2)^2 - 4\beta_2^2] \neq 0$. Thus, we have the expressions for R_{ij} and $R^*_{ij} = R_{sj}F_i^s$. Substituting them into (3.2) and (3.3), we obtain the expressions for ρ_{hijk} and ρ^*_{hijk} such that, after some calculations, (4.6) becomes

(4.9) $$R_{mijk} = (P-a)r_{mijk} + (Q-b)r^*_{mijk}$$
$$+E_i\{(\sigma_m\sigma_k + \sigma_m^*\sigma_k^*)g_{ij} + (\sigma_m\sigma_k^* + \sigma_m^*\sigma_k)F_{ij}$$

$$-(\sigma_i\sigma_k + \sigma_i^*\sigma_k^*)g_{mj} - (\sigma_i^*\sigma_k + \sigma_i\sigma_k^*)F_{mj} - [j,k]\}$$
$$+H_1\{(\sigma_m\sigma_k^* + \sigma_m^*\sigma_k)g_{ij} + (\sigma_m\sigma_k + \sigma_m^*\sigma_k^*)F_{ij}$$
$$-(\sigma_i^*\sigma_k + \sigma_i\sigma_k^*)g_{mj} - (\sigma_i\sigma_k + \sigma_i^*\sigma_k^*)F_{mj} - [j,k]\}$$

where

$$E_1 = \frac{1}{\sigma^2 - \sigma^{*\,2}}(-a\sigma + b\sigma^*), \quad H_1 = \frac{1}{\sigma^2 - \sigma^{*\,2}}(-b\sigma + a\sigma^*),$$
$$a = (n\alpha_2 + \varphi\beta_2)P + (\varphi\alpha_2 + n\beta_2)Q + (\alpha_1\alpha_2 + \beta_1\beta_2)R$$
$$+(\alpha_1\beta_2 + \beta_1\alpha_2)R^*,$$
$$b = (\varphi\alpha_2 + n\beta_2)P + (n\alpha_2 + \varphi\beta_2)Q + (\alpha_1\beta_2 + \beta_1\alpha_2)R$$
$$+(\alpha_1\alpha_2 + \beta_1\beta_2)R^*,$$

P, Q being given by (2.12) and α_2, β_2 by (1.13).

Thus, we have

Theorem 4.1. *If a locally decomposable Riemannian space admits product - concircular transformation (1.6) and its product conformal curvature tensor is semi - symmetric, then its curvature tensor has the form (4.9) or* $P^2 - Q^2 = 0$.

5. A semi - symmetric locally decomposable Riemannian space

In this section, we consider a locally decomposable Riemannian space, satisfying (1.6), (1.7) and

$$\nabla_r\nabla_s R_{hijk} - \nabla_s\nabla_r R_{hijk} = 0 \tag{5.1}$$

We proved in §2 that (1.6) and (1.7) yield (2.11). Therefore, (1.6), (1.7) and (5.1) imply

$$[PR_{\ell ijk} + QF_\ell^t R_{tijk} - (P^2 + Q^2)r_{\ell ijk} - 2PQr^*_{\ell ijk}]\sigma_m$$
$$[QR_{\ell ijk} + PF_\ell^t R_{tijk} - 2PQr_{\ell ijk} - (P^2 + Q^2)r^*_{\ell ijk}]\sigma^*_m$$
$$-[\ell, m] = 0$$

Transvecting this relation with σ^ℓ, we find

$$(5.2)\qquad (\sigma P+\sigma^* Q)R_{mijk}+(\sigma Q+\sigma^* P)F^t_m R_{tijk}+V_{mijk},$$

where

$$V_{mijk}=[\sigma(P^2+Q^2)+2\sigma^* PQ]r_{mijk}+[2\sigma PQ+\sigma^*(P^2+Q^2)]r^*_{mijk}$$
$$+[P\sigma_t R^t_{ijk}+Q\sigma^*_t R^t_{ijk}-(P^2+Q^2)\sigma^t r_{tijk}-2PQ\sigma^t_{tijk}]\sigma_m$$
$$+[Q\sigma_t R^t_{ijk}+P\sigma^*_t R^t_{ijk}-2PQ\sigma^t r_{tijk}-(P^2+Q^2)\sigma^t r^*_{tijk}]\sigma^*_m.$$

But, in view of (1.11), (2.13) and (2.14), we have

$$P\sigma_t R^t_{ijk}+Q\sigma^*_t R^t_{ijk}-(P^2+Q^2)\sigma_t r^t_{ijk}-2PQ\sigma_t r^{*\,t}_{ijk}=0$$

and

$$Q\sigma_t R^t_{ijk}+P\sigma^*_t R^t_{ijk}-PQ\sigma_t r_{tijk}-(P^2+Q^2)\sigma^t r^*_{tijk}=0$$

This means that

$$(5.3)\quad V_{mijk}=[\sigma(P^2+Q^2)+2\sigma^* PQ]r_{mijk}+[2\sigma PQ+\sigma^*(P^2+Q^2)]r^*_{mijk}$$

On the other hand, we get from (5.2)

$$(P^2-Q^2)(\sigma^2-\sigma^{*\,2})R_{mijk}=(P\sigma+Q\sigma^*)V_{mijk}-(Q\sigma+P\sigma^*)F^t_m V_{tijk},$$

or, in view of (5.3) and (1.5).

$$(P^2-Q^2)R_{mijk}=(P^2-Q^2)(Pr_{mijk}+Qr^*_{mijk}).$$

Thus, we have

Theorem 5.1. *If a locally decomposable Riemannian space is semi - symmetric and admits an almost product - concircular transformation (1.6), (1.7), then* $P^2=Q^2$, *or the curvature tensor of the manifold has the form*

$$R_{mijk}=Pr_{mijk}+Qr^*_{mijk},$$

i.e. the manifold is a space of almost constant curvature [5].

References

[1] Luo Cjongshan, On concircular transformations in Riemannian spaces, J. Austral. Math. Soc. (Series A) 40 (1986), 218-225.

[2] Petrović, V., Product - concircular curvature tensor, Pub. Inst. Math. 25 (39) , 1979, 131-137.

[3] Shun-ichi Tachibana, Some theorems on locally product Riemannian space, Tohoku Math. J., 12 (1960), 281-292.

[4] Yano, K., Concircular geometry I. Concircular transformations, Proc. Imper. Acad. Tokyo, 16 (1940), 195-200.

[5] Yano, K., Differential geometry on complex and almost complex spaces, Pergamon press, 1965.

Received by the editors May 13, 1993.