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ENUMERATION OF ORIENTED WALKS IN DIGRAPHS USING SYSTEM OF RECURRENCE RELATIONS

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Abstract

An algorithm which enumerates all oriented walks of the length k ($k \in N$) in any digraph (or graph) with the initial and final vertices in the given sets of vertices is presented. It is shown that the recurrence relation for these numbers is obtainable.

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1. Introduction

The number of all oriented walks of the length k $(k \in N)$ in any digraph with a fixed initial vertex and a fixed final vertex or the number of all walks of that lenght, at all, can be counted by raising the adjacency matrix to the k -th power [1] or using the theory of graph spectra [2]. But, sometimes, it is more practical to have the recurrence relation for those numbers.

By $A = [\overline{a}_{i,j}]_{n\times n}$ we denote the adjacency matrix of the given digraph D with the vertex-set $V(D) = \{1, 2, \ldots, n\}$.

Let $\overline{A}_{(n+1)\times(n+1)}$ denote the extended matrix A by a zero row and a zero column in the following way:

- the zero column is the zero vector i.e. $\overline{a}_{i,0} = 0$, $i = 0, \dots, n$;
- the zero row doesn't have entries equal to 0 only in the columns corresponding to the given initial set of vertices I $(I \subseteq V(D))$ i.e.

$$\overline{a}_{0,i} = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases} .$$

We denote by f_p^k $(0 < p \leq n)$ the number of all oriented walks of the length k with the initial vertex p and the final vertex in the given final set of vertices F $(F \subseteq V(D))$ and by f_0^k the number of all oriented walks of the length $(k-1)$ with initial vertices in I and final vertices in F.

Now, the matrix $\overline{A}$ represents the matrix of the following system of recurrence relations with $(n+1)$ unknown functions

$$f_p : N \cup \{0\} \mapsto N \cup \{0\} \ (0 \leq p \leq n)$$

$$\text{(1)} \qquad \begin{bmatrix} f_0^k \\ f_1^k \\ \cdot \\ \cdot \\ \cdot \\ f_n^k \end{bmatrix} = \overline{A} \begin{bmatrix} f_0^{k-1} \\ f_1^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ f_n^{k-1} \end{bmatrix}$$

Our task is to solve this system by the unknown function f_0^k i.e. reducing it to a recurrence relation with only one unknown function f_0^k .

2. The Results

The main results of this paper are the following two statements.

Theorem 1. *The system of recurrence relations (1) (the matrix $\overline{A}_{(n+1)\times(n+1)}$ is an arbitrary real matrix) is solvable by unknown function f_0^k i.e. , we can obtain the recurrence relation for the numbers f_0^k $(k \in N)$, which is of order 2^n , in the worst case.*

Let $L^{t,m}(f_{i_1}, f_{i_2}, \dots, f_{i_p})$; $1 \leq t < m$ denote an arbitrary linear combination of

$$f_{i_1}^{k-t}, f_{i_1}^{k-t-1}, \dots, f_{i_1}^{k-m}, f_{i_2}^{k-t}, f_{i_2}^{k-t-1}, \dots, f_{i_2}^{k-m}, \dots, f_{i_p}^{k-t}, f_{i_p}^{k-t-1}, \dots, f_{i_p}^{k-m}$$

Lemma 1. *If*

(2) $$a_0 f_1^k + L^{1,p}(f_1) = L^{q_1,q_2}(f_2, f_3, \dots, f_v)$$

(3) $$b_0 f_2^k + L^{1,r}(f_2) = L^{s_1,s_2}(f_1, f_3, \dots, f_v)$$

where $s_1 > 0$, $q_1 > 0$, $a_0 \neq 0$ *and* $b_0 \neq 0$,
then

$$a_0 b_0 f_1^k + L^{1,max(p+r,q_2+s_2)}(f_1) = L^{q_1,max(s_2,r)+q_2}(f_3, f_4, \dots, f_v)$$

Proof. We may rewrite the above linear combination as the following sums

$$L^{1,p}(f_1) = \sum_{i=1}^{p} a_i f_1^{k-i}$$

$$L^{1,r}(f_2) = \sum_{j=1}^{r} b_j f_2^{k-j}$$

$$L^{q_1,q_2}(f_2, f_3, \dots, f_v) = \sum_{j=q_1}^{q_2} \sum_{i=2}^{v} a_{i,j} f_i^{k-j}$$

$$L^{s_1,s_2}(f_1, f_3, \dots, f_v) = \sum_{j=s_1}^{s_2} \sum_{\substack{i=1 \\ i \neq 2}}^{v} b_{i,j} f_i^{k-j}$$

From (2) we obtain

$$\varphi(k) \overset{def}{=} a_0 f_1^k + L^{1,p}(f_1) - L^{q_1,q_2}(f_2, f_3, \dots, f_v) = 0$$

and futher

$$b_0 \varphi(k) + \sum_{i=1}^{r} b_i \varphi(k-i) = 0 \quad \text{i.e.}$$

$$b_0 a_0 f_1^k + \sum_{i=1}^{r} b_i \sum_{j=1}^{p} a_j f_1^{k-j-i} - \sum_{l=0}^{r} b_l \sum_{j=q_1}^{q_2} \sum_{i=2}^{v} a_{i,j} f_i^{k-j-l} = 0$$

i.e.

$$b_0a_0f_1^k+\sum_{i=1}^{r}b_i\sum_{j=1}^{p}a_jf_1^{k-j-i}=\sum_{l=0}^{r}b_l\sum_{j=q_1}^{q_2}a_{2,j}f_2^{k-j-l}+\sum_{l=0}^{r}b_l\sum_{j=q_1}^{q_2}\sum_{i=3}^{v}a_{i,j}f_i^{k-j-l}$$

The above expression is equivalent to the following one:

$$b_0a_0f_1^k+\sum_{i=1}^{r}b_i\sum_{j=1}^{p}a_jf_1^{k-j-i}=\sum_{j=q_1}^{q_2}a_{2,j}\sum_{l=0}^{r}b_lf_2^{k-j-l}+\sum_{l=0}^{r}b_l\sum_{j=q_1}^{q_2}\sum_{i=3}^{v}a_{i,j}f_i^{k-j-l}$$
(4)

If we take

$$\psi(k)\overset{def}{=}\sum_{l=0}^{r}b_lf_2^{k-l}$$

then using (4) we obtain:

(5) $$b_0a_0f_1^k+\sum_{i=1}^{r}b_i\sum_{j=1}^{p}a_jf_1^{k-j-i}=$$

$$\sum_{j=q_1}^{q_2}a_{2,j}\psi(k-j)+\sum_{l=0}^{r}b_l\sum_{j=q_1}^{q_2}\sum_{i=3}^{v}a_{i,j}f_i^{k-j-l}$$

From (3) it follows

(6) $$\psi(k)=\sum_{j=s_1}^{s_2}\sum_{\substack{i=1\\ i\neq 2}}^{v}b_{i,j}f_i^{k-j}$$

Substituing (6) into (5) we obtain

$$b_0a_0f_1^k+\sum_{i=1}^{r}\sum_{j=1}^{p}b_ia_jf_1^{k-j-i}=\sum_{j=q_1}^{q_2}a_{2,j}\sum_{l=s_1}^{s_2}\sum_{\substack{i=1\\ i\neq 2}}^{v}b_{i,l}f_i^{k-l-j}+\sum_{l=0}^{r}b_l\sum_{j=q_1}^{q_2}\sum_{i=3}^{v}a_{i,j}f_i^{k-j-l}$$

i.e.

$$b_0a_0f_1^k+\sum_{i=1}^{r}\sum_{j=1}^{p}b_ia_jf_1^{k-j-i}-\sum_{j=q_1}^{q_2}\sum_{l=s_1}^{s_2}a_{2,j}b_{1,l}f_1^{k-l-j}=$$

$$\sum_{l=s_1}^{s_2}\sum_{j=q_1}^{q_2}\sum_{i=3}^{v} a_{2,j}b_{i,l}f_i^{k-l-j} + \sum_{l=0}^{r}\sum_{j=q_1}^{q_2}\sum_{i=3}^{v} b_l a_{i,j} f_i^{k-j-l}$$

i.e.

$$a_0 b_0 f_1^k + L^{1,max(p+r,q_2+s_2)}(f_1) = L^{q_1,max(s_2,r)+q_2}(f_3, f_4, \dots, f_v) \ .$$

Proof of Theorem 1. Using Lemma 1 in the system (1) n times, (the function f_n from Theorem 1 plays the role of f_2 from Lemma 1 and the functions $f_0, f_1, \dots, f_{n-1}$ play the role of f_1 from Lemma 1) the system (1) is transformed into:

$$f_0^k + L^{1,2}(f_0) = L^{1,2}(f_1, f_2, \dots, f_{n-1})$$

$$f_1^k + L^{1,2}(f_1) = L^{1,2}(f_0, f_2, \dots, f_{n-1})$$

$$f_2^k + L^{1,2}(f_2) = L^{1,2}(f_0, f_1, f_3, \dots, f_{n-1})$$

$$\dots$$

$$f_{n-1}^k + L^{1,2}(f_{n-1}) = L^{1,2}(f_0, f_1, \dots, f_{n-1})$$

Using again Lemma 1 $(n-1)$ times and removing now function f_{n-1} as function f_n in the previous step the above system is reduced to:

$$f_0^k + L^{1,4}(f_0) = L^{1,4}(f_1, f_2, \dots, f_{n-2})$$

$$f_1^k + L^{1,4}(f_1) = L^{1,4}(f_0, f_2, \dots, f_{n-2})$$

$$f_2^k + L^{1,4}(f_2) = L^{1,4}(f_0, f_1, f_3, \dots, f_{n-2})$$

$$\dots$$

$$f_{n-2}^k + L^{1,4}(f_{n-2}) = L^{1,4}(f_0, f_1, \dots, f_{n-3})$$

Continuing this proceeding and removing the functions $f_{n-2}, f_{n-3}, \dots, f_2$ our system is reduced to

$$f_0^k + L^{1,2^n}(f_0) = 0 \qquad (7)$$

(see the example 1.) □

Definition 1. *Consider a system of recurrence relations. Let the function* f_j *appears in i-th recurrence relation from* $f_j^{n-t_i^1}$ *to* $f_j^{n-t_i^2}$, $t_i^1 \leq t_i^2$ *(i.e. coefficients of* $f_j^{n-t_i^1}$ *and* $f_j^{n-t_i^2}$ *are not equal to* 0*). Then the value*

$$t = \max_i (t_i^2 - t_i^1)$$

is called the weight of the function f_j *in considering system of recurrence relations.*

We can note that during removing each of the function $f_n, f_{n-1}, \ldots, f_1$ (in the proof of Theorem 1) in every step the weight of each left function was increased at the most two times and because of that the order of recurrence relation (7) is at most 2^n .

If we permit in Lemma 1 q_1 and s_1 to be zero then we obtain the following

Lemma 2. *If*

$$a_0 f_1^k + L^{1,p}(f_1) = \sum_{j=0}^{q_2} \sum_{i=2}^{v} a_{i,j} f_i^{k-j}$$

$$b_0 f_2^k + L^{1,r}(f_2) = \sum_{j=0}^{s_2} \sum_{\substack{i=1 \\ i \neq 2}}^{v} b_{i,j} f_i^{k-j}$$

where $q_2 \geq 0$, $s_2 \geq 0$, $a_0 \neq 0$ *and* $b_0 \neq 0$, *then*

$$(a_0 b_0 - a_{2,0} b_{1,0}) f_1^k + L^{1,max(p+r, q_2+s_2)}(f_1) = L^{0,max(s_2,r)+q_2}(f_3, f_4, \ldots, f_v)$$

Remark 1. Consider the system of recurrence relations (1) with an arbitrary real matrix $\overline{A}_{(n+1)\times(n+1)}$ again. If rank of matrix $\overline{A}$ is less than $(n+1)$ then there is a function f_i , $1 \leq i \leq n$ which we can express as a linear combination of the rest function from the set $\{f_j \mid 0 \leq j \leq n \ , \ j \neq i\}$. Now, we can remove the function f_i from the system (1) without increasing the weight of the rest function.

In this way we get the final recurrence relation which order is at least twice the order of the final recurrence relation which we get without using this property.

Theorem 2. *Consider an arbitrary digraph* D *with* n *vertices. Let* f_0^k *be the number of all oriented walks of the length* $(k-1)$ *with initial vertex in set* I $(I \subseteq V(D))$ *and final vertex in set* F $(F \subseteq V(D))$ *. Then there is a recurrence relation of* f_0^k *with order at most* 2^{n-1} *. The initial conditions are obtained from the system recurrence relations. The value* f_i^1 *is equal to the number of all oriented edges with initial vertices* i *and final vertices in the set* F *.*

Proof. The rank of the matrix $\overline{A}$ which corresponds to the digraph D is less than $(n+1)$. Use the above property and Lemma 2 instead of Lemma 1 as in the proof of Theorem 1 (see the example 2). [1] □

Sometimes, we can make this increasing of the weight of some functions to be smaller using the following statement:

Lemma 3. *If*

$$\sum_{i=0}^{p} a_i f_1^{k-i} = \sum_{j=q_1}^{q_2} a_{2,j} f_2^{k-j} + \sum_{j=r_1}^{r_2} \sum_{i=3}^{v} a_{i,j} f_i^{k-j} \tag{8}$$

$$\sum_{i=0}^{r} b_i f_2^{k-i} = \sum_{j=s_1}^{s_2} b_{1,j} f_1^{k-j} + \sum_{j=p_1}^{p_2} \sum_{i=3}^{v} b_{i,j} f_i^{k-j} \tag{9}$$

where

$$a_0 \neq 0 \ , \ a_p \neq 0 \ , \ b_0 \neq 0 \ , \ b_r \neq 0 \ ,$$

$$a_{2,q_1} \neq 0 \ , \ a_{2,q_2} \neq 0 \ , \ b_{1,s_1} \neq 0 \ , \ b_{1,s_2} \neq 0$$

and the polynomial $\sum_{j=q_1}^{q_2} a_{2,j} x^{q_2-j}$ *is divisible by the polynomial* $\sum_{i=0}^{r} b_i x^{r-i}$ *then*

$$L^{0,max(p,q_2+s_2-r)}(f_1) = L^{min(r_1,q_1+p_1),max(p_2+q_2-r,r_2)}(f_3, f_4, \ldots, f_v)$$

[1]This theorem can be proved as in Proof of Theorem 1 but taking f_0^k for the number of all oriented walks of the length k with initial vertices in set I and final vertices in set F .

Proof. Putting the replacement $\psi(k) = \sum_{i=0}^{r} b_i f_2^{k-i}$ into (8) we obtain

$$(10) \qquad \sum_{i=0}^{p} a_i f_1^{k-i} = \sum_{l=q_1}^{q_2-r} c_l \psi(k-l) + \sum_{j=r_1}^{r_2} \sum_{i=3}^{v} a_{i,j} f_i^{k-j}$$

where the values c_l are the coefficents of the polynomial which is obtained by dividing the polynomial

$$\sum_{j=q_1}^{q_2} a_{2,j} x^{q_2-j} \quad \text{by} \quad \sum_{i=0}^{r} b_i x^{r-i} \quad \text{i.e.}$$

$$\sum_{j=q_1}^{q_2} a_{2,j} x^{q_2-j} = \sum_{i=0}^{r} b_i x^{r-i} \cdot \sum_{l=q_1}^{q_2-r} c_l x^{q_2-r-l}$$

Using the same replacement (9) is transformed into:

$$(11) \qquad \psi(k) = \sum_{j=s_1}^{s_2} b_{1,j} f_1^{k-j} + \sum_{j=p_1}^{p_2} \sum_{i=3}^{v} b_{i,j} f_i^{k-j}$$

Substituing (11) into (10) we obtain:

$$\sum_{i=0}^{p} a_i f_1^{k-i} = \sum_{l=q_1}^{q_2-r} c_l \sum_{j=s_1}^{s_2} b_{1,j} f_1^{k-j-l} + \sum_{l=q_1}^{q_2-r} c_l \sum_{j=p_1}^{p_2} \sum_{i=3}^{v} b_{i,j} f_i^{k-j-l} + \sum_{j=r_1}^{r_2} \sum_{i=3}^{v} a_{i,j} f_i^{k-j}$$

i.e.

$$L^{0,max(p,q_2+s_2-r)}(f_1) = L^{min(r_1,q_1+p_1),max(p_2+q_2-r,r_2)}(f_3, f_4, \ldots, f_v)$$

Remark 2.

If we aplied Lemma 2 to system (8) and (9) we would obtain

$$L^{0,max(r+p,max(s_2,p_2)+max(q_2,r_2))}(f_1) = L^{0,max(r,s_2,p_2)+max(q_2,r_2)}(f_3, f_4, \ldots, f_v)$$

Note that using the above lemma we can save the weight of the increasing at least for value r .

Remark 3.

If the polynomials $\sum_{j=q_1}^{q_2} a_{2,j}x^{q_2-j}$ from (8) and $\sum_{i=0}^{r} b_i x^{r-i}$ from (9) have as the greatist common divisor the polynomial $\sum_{i=0}^{f} d_i x^{f-i}$ $(f < r)$ then using the replacement $\varphi^k = \sum_{i=0}^{f} d_i f_i^{k-i}$ we can save the weight of increasing at least for value f .

As illustration of the method can be of use [3].

Example 1.

Consider the digraph shown in Figure 1 . If we take for the sets I and F the following:
$I = \{1,5\}$
$F = \{3,4,5\}$
then the matrix $\overline{A}$ which corresponds to the graph is the following one

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

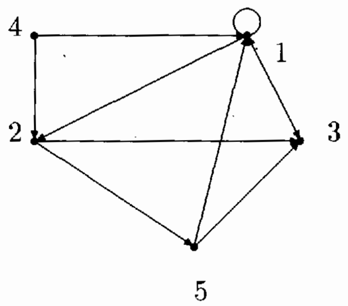

Fig.1.

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The initial conditions:
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```
f0(1) = 2       f0(4) = 31       f0(7) = 391    f0(10) = 5028
f0(2) = 6       f0(5) = 72       f0(8) = 918
f0(3) = 12      f0(6) = 165      f0(9) = 2144
----------------------------------------------------------------
f0(n) =  f1(n-1)  +  f5(n-1)
f1(n) - f1(n-1) =  f2(n-1)  +  f3(n-1)
f2(n) =  f3(n-1)  +  f5(n-1)
f3(n) =  f1(n-1)  +  f4(n-1)
f4(n) =  f1(n-1)  +  f2(n-1)
f5(n) =  f1(n-1)  +  f3(n-1)
-----------------------------------------------------------------
            removing f5(k)
-----------------------------------------------------------------
f0(n) =  f1(n-1)  +  f1(n-2)  +  f3(n-2)
f1(n) - f1(n-1) =  f2(n-1)  +  f3(n-1)
f2(n) =  f1(n-2)  +  f3(n-1)  +  f3(n-2)
f3(n) =  f1(n-1)  +  f4(n-1)
f4(n) =  f1(n-1)  +  f2(n-1)
------------------------------------------------------------------
            removing f4(k)
------------------------------------------------------------------
f0(n) =  f1(n-1)  +  f1(n-2)  +  f3(n-2)
f1(n) - f1(n-1) =  f2(n-1)  +  f3(n-1)
f2(n) =  f1(n-2)  +  f3(n-1)  +  f3(n-2)
f3(n) =  f1(n-1)  +  f1(n-2)  +  f2(n-2)
-----------------------------------------------------------------
            removing f3(k)
-----------------------------------------------------------------
f0(n) =  f1(n-1)  +  f1(n-2)  +  f1(n-3)  +  f1(n-4)  +  f2(n-4)
f1(n) - f1(n-1) - f1(n-2) - f1(n-3) =  f2(n-1)  +  f2(n-3)
f2(n) - f2(n-3) - f2(n-4) = 2 f1(n-2)  + 2 f1(n-3)  +  f1(n-4)
-----------------------------------------------------------------
            removing f2(k)
-----------------------------------------------------------------
f0(n) - f0(n-3) - f0(n-4)=f1(n-1)+ f1(n-2) + f1(n-3) - 2 f1(n-5)
f1(n) - f1(n-1) - f1(n-2) - 4 f1(n-3) - 2 f1(n-4) - f1(n-5) =  0
-----------------------------------------------------------------
            removing f1(k)
-----------------------------------------------------------------
```

```
f0(n) - f0(n-1) - f0(n-2) - 5 f0(n-3)  - 2 f0(n-4)  +  f0(n-5)
  + 5 f0(n-6)  + 6 f0(n-7) + 3 f0(n-8)  +  f0(n-9)  =  0
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```

Example 2. Consider the same digraph as in Example 1. The rank of the matrix $\overline{A}$ is 5 and the following relation is satisfied:

$$f_5 = f_0 + 2f_1 - f_2 - 2f_4 \ .$$

Now, using this we can transform the matrix $\overline{A}$ into the following matrix:

$$\overline{B} = \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

```
-------------------------------------------------------------
f0(n)-f0(n-1) = 3 f1(n-1)  - f2(n-1)  - 2 f4(n-1)
f1(n)-f1(n-1) =  f2(n-1)  +  f3(n-1)
f2(n)+f2(n-1)  =  f0(n-1)  + 2 f1(n-1)  +  f3(n-1) - 2 f4(n-1)
f3(n)=f1(n-1)  +  f4(n-1)
f4(n)=f1(n-1)  +  f2(n-1)
-------------------------------------------------------------
            removing f4(k)
-------------------------------------------------------------
f0(n)-f0(n-1) = 3 f1(n-1)  - 2 f1(n-2)  - f2(n-1)  - 2 f2(n-2)
f1(n)-f1(n-1) =  f2(n-1)  +  f3(n-1)
f2(n)+f2(n-1) + 2 f2(n-2) = f0(n-1)+2 f1(n-1)-2 f1(n-2)+f3(n-1)
f3(n)=f1(n-1)  +  f1(n-2)  +  f2(n-2)
-------------------------------------------------------------
            removing f3(k)
-------------------------------------------------------------
f0(n)-f0(n-1) = 3 f1(n-1)  - 2 f1(n-2)  - f2(n-1)  - 2 f2(n-2)
f1(n)-f1(n-1) - f1(n-2) - f1(n-3) =  f2(n-1)  +  f2(n-3)
f2(n)+f2(n-1) + 2 f2(n-2) - f2(n-3) =
      = f0(n-1) + 2 f1(n-1) - f1(n-2) + f1(n-3)
-------------------------------------------------------------
```

```
            removing f2(k)
----------------------------------------------------------------
f0(n)+2f0(n-2) - f0(n-3) + f0(n-4) =
      = 3 f1(n-1) - f1(n-2) + f1(n-3) - 6 f1(n-4)
f1(n)-2f1(n-2)  - 4 f1(n-3)  - 5 f1(n-4)  =  f0(n-2) + f0(n-4)
----------------------------------------------------------------
            removing f1(k)
----------------------------------------------------------------
f0(n)-8f0(n-3) - 7 f0(n-4)-10 f0(n-5)-f0(n-6)+f0(n-8)  =  0
----------------------------------------------------------------
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