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GENERALIZED PSEUDO-BOOLEAN FUNCTIONAL EQUATIONS WITH GENERAL COEFICIENTS AND n-VARIABLES

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Abstract

The necessary and sufficient condition is given for the solution of a generalized pseudo-Boolean functional equation with n-variables and general coeficients.

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Let $(P, \oplus, \cdot)$ be a commutative ring with identity element 1 without divisors of zero, and let L be a finite set. A generalized pseudo-Boolean function (GPB function) is every mapping f of L^n into P, i.e., $f : L^n \to P$, where L^n is a direct power of L.

The definition of partial derivatives of GPB functions and some properties of these partial derivatives were given by Gilezan in [4].

Definition 1. *A partial derivative of a GPB function $f : L^n \to P$ with the variables $x_i (i = 1, 2, \ldots, n)$ are GPB functions*

$$\frac{\partial f_a}{\partial x_i} : L^n \to P$$

defined by

$$\frac{\partial f_a}{\partial x_i}(X) = f(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_n) - f(X), a \in L \tag{1}$$

See [4] for details.

A partial derivative of higher order of a GPB function is

$$\frac{\partial^m f_{a_{i_1}\dots a_{i_m}}}{\partial x_{i_1}\dots\partial x_{i_m}} = \frac{\partial}{\partial x_{i_m}}\cdots\left(\frac{\partial}{\partial x_{i_2}}\left(\frac{\partial f_{a_{i_1}}}{\partial x_{i_1}}\right)_{a_{i_2}}\right)\cdots)_{a_{i_m}} \tag{2}$$

$$a_{i_j} \in L, j = 1,2,\dots,m \ \ (m \geq 1).$$

However, the relation

$$F(g_1,g_2,\dots,g_k,f,\frac{\partial f_{\alpha_1}}{\partial x_1},\dots,\frac{\partial f_{\alpha_n}}{\partial x_n},\frac{\partial^2 f_{\alpha_1\alpha_2}}{\partial x_1\partial x_2},\dots$$
$$,\frac{\partial^2 f_{\alpha_{n-1}\alpha_n}}{\partial x_{n-1}\partial x_n},\frac{\partial^3 f_{\alpha_1\alpha_2\alpha_3}}{\partial x_1\partial x_2\partial x_3},\dots,\frac{\partial^3 f_{\alpha_{n-2}\alpha_{n-1}\alpha_n}}{\partial x_{n-2}\partial x_{n-1}\partial x_n},\dots,\frac{\partial^n f_{\alpha_1\dots\alpha_n}}{\partial x_1\dots\partial x_n}) = 0$$
$$\alpha_1,\dots,\alpha_n \in L$$

is a generalized pseudo-Boolean functional equation, where g_1, $g_2,\dots$, g_n are known functions, an unknown function f and some of its partial derivatives take place. Hence, the solution of this functional equation has to be found.

Lemma 1. *A functional equation with an unknown GPB function f*

$$\frac{\partial f_{\alpha_i}}{\partial x_i} = g(X), \quad \textit{where} \ \ \alpha_i \in L, i = 1,2,\dots,n$$

has a solution if and only if

$$g(x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n) = 0.$$

The solutions f are determined by the following formula

$$f(X) = C(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) - g(X), \quad \text{or else}$$

$$\int_{\alpha_i} g(X)dx_i = C(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) - g(X),$$

where C is an arbitrary function of the variables $x_1, \ldots, x_n$. The proof of this lemma was given in [5].

Let us introduce the following notation

$$(\tilde{x}_i) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

and

$$(\tilde{\tilde{\alpha}}_i) = (x_1, \ldots, x_{i-1}, \alpha_i, x_{i+1}, \ldots, x_n)$$

Theorem 1. *A system of GPB functional equations*

$$\frac{\partial f_{\alpha_i}}{\partial x_i} = P_i(X) \;\; i = 1, 2, \ldots, n, \; X = (x_1, \ldots, x_n) \tag{3}$$

has a solution if and only if

$$P_i(\tilde{\tilde{\alpha}}_i) = 0 \;\; i = 1, 2, \ldots, n \tag{4}$$

$$\frac{\partial P_{j\,\alpha_i}}{\partial x_i} = \frac{\partial P_{i\,\alpha_j}}{\partial x_j}, \;\; i, j = 1, 2, \ldots, n \;\; i \neq j. \tag{5}$$

The solutions f are determined by the following formula (they are equivalent to each other)

$$f(X) = C - \sum_{k=1}^{n} P_{i_k}(\tilde{\tilde{\alpha}}_{i_{k-1}}, \tilde{\tilde{\alpha}}_{i_{k-1}}, \ldots, \tilde{\tilde{\alpha}}_{i_n}) - P_{i_n}(X) \tag{6}$$

where $i_1 i_2 \ldots i_n$ are permutations of the set $\{1, 2, \ldots, n\}$ and c is an arbitrary constant from P. The proof of this theorem was given in [5].

Let us consider an equation of the following form

$$F: \; a(x_1, x_2)\frac{\partial f_\alpha}{\partial x_1} \oplus b(x_1, x_2)\frac{\partial f_\beta}{\partial x_2} \oplus c(x_1, x_2)\frac{\partial^2 f_{\alpha\beta}}{\partial x_1 \partial x_2} = d \cdot g(x_1, x_2), \tag{7}$$

where $f : L^2 \to P$ is an unknown GPB function, while $a : L^2 \to P$, $b : L^2 \to P$, $c : L^2 \to P$ and $g : L^2 \to P$ are known.

Theorem 2. *The functional equation* (7) *has a solution if and only if*

$$\frac{\partial^2 g_{\alpha_1\alpha_2}}{\partial x_1 \partial x_2} \oplus \frac{\partial g_{\alpha_1}}{\partial x_1} \oplus \frac{\partial g_{\alpha_2}}{\partial x_2} \oplus g \;\; = \;\; 0 \tag{8}$$

$$\left(\frac{\partial a_{\alpha_2}}{\partial x_2} \oplus a\right)\left(\frac{\partial b_{\alpha_1}}{\partial x_1} \oplus b\right)(a \oplus b - c) \;\; = \;\; d. \tag{9}$$

The solutions f are determined by the following formula

$$f(x,y)=C-X(x_1,x_2)-Y(x_1,x_2)-\frac{\partial X_{\alpha 2}}{\partial x_2}, \tag{10}$$

Where C is an arbitrary constant from P, and

$$\begin{aligned} X &= g\left[\left(\frac{\partial a_{\alpha 2}}{\partial x_2}\oplus a\ oplusb-c\right)\frac{\partial b_{\alpha 1}}{\partial x_1}\oplus b(b-c)\right]\oplus \\ &\oplus(b-c)\left(\frac{\partial b_{\alpha 1}}{\partial x_1}\oplus b\right)\frac{\partial g_{\alpha 2}}{\partial x_2}-b\left(\frac{\partial a_{\alpha 2}}{\partial x_2}\oplus a\right)\frac{\partial g_{\alpha 1}}{\partial x_1}, \\ Y &= g\left[\left(\frac{\partial b_{\alpha 1}}{\partial x_1}\oplus a\oplus b-c\right)\frac{\partial a_{\alpha 2}}{\partial x_2}\oplus a(a-c)\right]\oplus \\ &\oplus(a-c)\left(\frac{\partial a_{\alpha 2}}{\partial x_2}\oplus a\right)\frac{\partial g_{\alpha 1}}{\partial x_1}-a\left(\frac{\partial b_{\alpha 1}}{\partial x_1}\oplus b\right)\frac{\partial g_{\alpha 2}}{\partial x_2}. \end{aligned}$$

The proof of Theorem 2 was given in [6].

Here we shall give a generalization of Theorem 2. First, let us introduce the following notations:

$$m(\beta)=\beta_1+\beta_2+\ldots+\beta_n,\ \beta_i\in\{0,1\}$$

(+ is the usual addition in the set of real numbers).

$$M_n=\{\beta\mid\beta=(\beta_1,\beta_2,\ldots,\beta_n),\ \beta_i\in\{0,1\},m(\beta)\geq 1\},$$

$$k(M_n)=2^n-1\ (k\ \text{ is the cardinality of }\ M_n).$$

Let us consider the n-tuple $\alpha=\alpha_{(k)}$, where 1 is on the k-th place, and $m(\alpha_{(k)})=1$, while $\alpha\in M_n$, $\alpha_{(i)}\dot{+}\alpha_{(j)}$ is the n-tuple obtained by the modulo 2 addition of the corresponding components. Let $\dot{+}$ denote the addition modulo 2.

$\alpha^i_{(1)}\alpha^i_{(2)}\cdots\alpha^i_{(k)}$ is the i-th combination of k-elements from the set $\{\alpha_{(1)},\ldots,\alpha_{(n)}\}$, where $m(\alpha^i_{(k)})=1$ and $2\leq k\leq n$

$$a_{\alpha^i_{(1)}\dot{+}\alpha^i_{(2)}\dot{+}\ldots\dot{+}\alpha^i_{(k)}}=a_{\sum_{t=1}^{k}\,\dot{+}\,\alpha^i_{(t)}}.$$

Further let us denote $\bar{\alpha}_{(k)}=k$. Let a_β be GPB-functions, where $\beta\in M_n$, and

$$\frac{\partial(a_\beta)_{b_k}}{\partial x_k}=a_\beta(x_1,\ldots,x_{k-1},b_k,x_{k+1},\ldots,x_n)-a_\beta(x_1,\ldots,x_k,\ldots,x_n)$$

Is the partial derivative of the GPB-function a_β with the variable x_k.

$$\partial x^\beta = \partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}, \quad \beta \in M_n$$

$$b_i^{[0]} = 1,\ b_i^{[1]} = b_i,\ i = 1, \dots, n,\ b_i \in L,$$

$$\frac{\partial^0 f_{b_i}}{\partial x_i^0} = 1.$$

Let us consider the following functional equation

$$F: \quad \sum_{\beta \in M_n \ \oplus} a_\beta \frac{\partial^n}{\partial x} \tag{11}$$

Where $f : L^n \to P$ is an unknown GPB-function, $a_\beta : L^n \to P$ and $R : L^n \to P$ are known GPB functions, $d \in P$, $\beta \in M_n$ and $k(M_n) = 2^n - 1$.

Theorem 3. *The functional equation* (11) *has a solution if and only if*

$$R \oplus \sum_{\beta \in M_n \ \oplus} \frac{\partial^{m(\beta)} R_{b_1^{[\beta_1]} \dots b_n^{[n]}}}{\partial x_1^{\beta_1} \dots x_n^{\beta_n}} = 0, \tag{12}$$

and

$$d = \prod_{i=1}^{n} \left(a_{\alpha_{(i)}} \oplus \sum_{k=1 \ \oplus}^{n-i} \left(\sum_{i_1 \dots i_k \ \oplus}^{\{1,\dots,n\} \setminus \{\bar{\alpha}_{(i)}\}} \frac{\partial^k (a_{(i)})_{b_{i_1} \dots b_{i_k}}}{\partial x_{i_1} \dots \partial x_{i_k}} \right) \right) \tag{13}$$

$$\prod_{k=2}^{n} (\prod_{i=1}^{\binom{n}{n-k}} (\sum_{t=1 \ \oplus}^{k} a_{\alpha^i_{(t)}} \oplus (-1)^{k+1} a_{\sum_+ \alpha^i_{(t)}} \oplus \sum_{s=1 \ \oplus}^{n-k} (\sum_{i_1 \dots i_s \ \oplus}^{\{1,\dots,n\} \setminus \{\bar{\alpha}^i_{(1)}, \dots, \bar{\alpha}^i_{(k)}\}}$$
$$(\frac{\partial^s (a^i_{\alpha_{(t)}})_{b_{i_1} \dots b_{i_s}}}{\partial x_{i_1} \dots \partial x_{i_s}} \oplus (-1)^{k+1} \frac{\partial^s (a_{\sum_{t=1}^k + \alpha^i_{(t)}})_{b_{i_1} \dots b_{i_s}}}{\partial x_{i_1} \dots \partial x_{i_s}})))) \neq 0.$$

Proof. If we find the partial derivatives of the functional equation (11), i.e.

$$\frac{\partial^{m(\gamma)} F_{b_1^{[\gamma_1]} \dots b_2^{[\gamma_2]} \dots b_n^{[\gamma_n]}}}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}, \quad \gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in M_n,$$

Then we obtain the following system of functional equations

$$(14)\ \sum_{\beta\in M_n\ \oplus} \frac{\partial^{m(\gamma)}}{\partial x^\gamma}\left(a_\beta \frac{\partial^{m(\beta)} f_{b_1^{[\beta_1]}\dots b_n^{[\beta_n]}}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}\right)_{b_1^{[\gamma_1]}\dots b_n^{[\gamma_n]}} = d\frac{\partial^{m(\gamma)}(R)_{b_1^{[\gamma_1]}\dots b_n^{[\gamma_n]}}}{\partial x_1^{\gamma_1}\dots\partial x_n^{\gamma_n}},$$

$$\gamma\in M_n\ \ \partial x^\gamma = \partial x_1^{\gamma_1}\dots\partial x_n^{\gamma_n}.$$

The system (14) has a unique solution $\frac{\partial f_{b_1}}{\partial x_1},\dots,\frac{\partial f_{b_n}}{\partial x_n}$ if and only if the rank of the augmented matrix of the system (14) is 2^n-1.

For $n=2$ the augmented matrix of the system (14) is equivalent to the matrix A_2'.

$$A_2' = \begin{bmatrix} a_{10} & a_{01} & a_{11}-a_{10}-a_{01} & R \\ 0 & \frac{\partial(a_{01})_{b_1}}{\partial x_1}\oplus a_{01} & 0 & \frac{\partial R_{b_1}}{\partial x_1}\oplus R \\ \frac{\partial(a_{10})_{b_1}}{\partial x_2} & 0 & 0 & \frac{\partial R_{b_2}}{\partial x_2}\oplus R \\ 0 & 0 & 0 & R\oplus\frac{\partial^2 R_{b_1b_2}}{\partial x_1\partial x_2}\oplus\frac{\partial R_{b_2}}{\partial x_2}\oplus\frac{\partial R_{b_1}}{\partial x_1} \end{bmatrix}$$

Therefore rank $A_2'=3$ if and only if

$$R\oplus\frac{\partial R_{b_1}}{\partial x_1}\oplus\frac{\partial R_{b_2}}{\partial x_2}\oplus\frac{\partial^2 R_{b_1b_2}}{\partial x_1\partial x_2}=0,$$

and

$$d=(a_{10}\oplus\frac{\partial(a_{10})_{b_2}}{\partial x_2})(a_{01}\oplus\frac{\partial(a_{01})_{b_1}}{\partial x_1})(a_{10}\oplus a_{01}-a_{11})\neq 0.$$

This has been proved in Theorem 2. According to (13) for $n=2$ it follows that

$$(15)\quad d = \prod_{i=1}^{2}\left(a_{\alpha_{(i)}}\oplus\sum_{i_k\ \oplus}^{\{1,2\}\backslash\{\bar\alpha_{(i)}\}}\frac{\partial(a_{\alpha_{(i)}})_{b_{i_k}}}{\partial x_k}\right)$$

$$\prod_{k=2}^{2}\left(\prod_{i=1}^{\binom{2}{2-k}}\left(\sum_{t=1\ \oplus}^{k} a_{\alpha^i_{(t)}} - a_{\alpha^i_{(1)}+\alpha^i_{(2)}}\right)\right)$$

$$= (a_{10}\oplus\frac{\partial(a_{10})_{b_2}}{\partial x_2})(a_{01}\oplus\frac{\partial(a_{01})_{b_1}}{\partial x_1})(a_{10}\oplus a_{01}-a_{11})\neq 0.$$

For $n = 3$ according to (12) and (13) it follows that

$$R \oplus \frac{\partial R_{b_1}}{\partial x_1} \oplus \frac{\partial R_{b_2}}{\partial x_2} \oplus \frac{\partial R_{b_3}}{\partial x_3} \oplus \frac{\partial^2 R_{b_1 b_2}}{\partial x_1 \partial x_2} \oplus \frac{\partial^2 R_{b_1 b_3}}{\partial x_1 \partial x_3} \oplus \frac{\partial^2 R_{b_2 b_3}}{\partial x_2 \partial x_3} \oplus \frac{\partial^3 R_{b_1 b_2 b_3}}{\partial x_1 \partial x_2 \partial x_3} = 0,$$

$$(16)\quad d = \prod_{i=1}^{3} \left(a_{\alpha_{(i)}} \oplus \sum_{k=1\ \oplus}^{2} \left(\sum_{i_1 \dots i_k}^{\{1,2,3\} \setminus \{\bar{\alpha}_{(i)}\}} \frac{\partial^k (a_{\alpha_{(i)}})_{b_{i_1} \dots b_{i_k}}}{\partial x_{i_1} \dots \partial x_{i_k}} \right) \right) \cdot$$

$$\prod_{k=2}^{3} \Big(\prod_{i=1}^{\binom{3}{(3-k)}} \Big(\sum_{t=1\ \oplus}^{k} a_{\alpha^i_{(t)}} \oplus (-1)^{k+1} a_{\sum_{t=1}^{k} +\ \alpha^i_{(t)}}$$

$$\oplus \sum_{s=1\ \oplus}^{3-k} \Big(\sum_{i_1 \dots i_s\ \oplus}^{\{1,2,3\} \setminus \{\bar{\alpha}^i_{(1)}, \dots, \alpha^i_{(k)}\}}$$

$$\Big(\frac{\partial^s (a_{\alpha^i_{(t)}})_{b_{i_1} \dots b_{i_s}}}{\partial x_{i_1} \dots \partial x_{i_s}} \oplus (-1)^{k+1} \frac{\partial^s (a_{\sum_{t=1}^{k} +\ \alpha^i_{(t)}})_{b_{i_1} \dots b_{i_s}}}{\partial x_{i_1} \dots \partial x_{i_s}} \Big) \Big) \Big) \Big) =$$

$$= \Big(a_{100} \oplus \frac{\partial (a_{100})_{b_2}}{\partial x_2} \oplus \frac{\partial (a_{100})_{b_3}}{\partial x_3} \oplus \frac{\partial^2 (a_{100})_{b_2 b_3}}{\partial x_2 \partial x_3} \Big) \cdot$$

$$\Big(a_{010} \oplus \frac{\partial (a_{010})_{b_1}}{\partial x_1} \oplus \frac{\partial (a_{010})_{b_3}}{\partial x_3} \oplus \frac{\partial^2 (a_{010})_{b_1 b_3}}{\partial x_1 \partial x_3} \Big) \cdot$$

$$\Big(a_{001} \oplus \frac{\partial (a_{001})_{b_1}}{\partial x_1} \oplus \frac{\partial (a_{001})_{b_2}}{\partial x_2} \oplus \frac{\partial^2 (a_{001})_{b_1 b_2}}{\partial x_1 \partial x_2} \Big) \cdot$$

$$\Big(a_{100} \oplus \frac{\partial (a_{100})_{b_3}}{\partial x_3} \oplus a_{010} \oplus \frac{\partial (a_{010})_{b_3}}{\partial x_3} - a_{110} - \frac{\partial (a_{110})_{b_3}}{\partial x_3} \Big) \cdot$$

$$\Big(a_{100} \oplus \frac{\partial (a_{100})_{b_2}}{\partial x_2} \oplus a_{001} \oplus \frac{\partial (a_{001})_{b_2}}{\partial x_2} - a_{101} - \frac{\partial (a_{101})_{b_2}}{\partial x_2} \Big) \cdot$$

$$\Big(a_{010} \oplus \frac{\partial (a_{010})_{b_1}}{\partial x_1} \oplus a_{001} \oplus \frac{\partial (a_{001})_{b_1}}{\partial x_1} - a_{011} - \frac{\partial (a_{011})_{b_1}}{\partial x_1} \Big).$$

The condition (13) can be proved by mathematical induction using (15), (16) and some properties of partial derivatives of GPB-functions □.

If we denote with $\frac{\partial f_{b_i}}{\partial x_i} = P_i(X)$, $(i = 1, 2, \dots, n)$ and if the following conditions are satisfied

$$(17)\qquad P_i(\tilde{\tilde{b}}_i) = 0 \quad i = 1, 2, \dots, n$$

$$\frac{\partial P_{j\ b_i}}{\partial x_i} = \frac{\partial P_{i\ b_j}}{\partial x_j}, \quad i,j = 1,2,\ldots,n \quad i \neq j, \tag{18}$$

Then by Theorem 1 the solution of the functional equation (11) is of the form

$$f(X) = C - \sum_{k=1\ \oplus}^{n-1} P_{i_k}(\tilde{\tilde{b}}_{i_{k+1}}, \tilde{\tilde{b}}_{i_{k+2}}, \ldots, \tilde{\tilde{b}}_{i_n}) - P_{i_n}(X), \tag{19}$$

Where $i_1, \ldots, i_n$ is a permutation of the set $\{1, 2, \ldots, n\}$, C is an arbitrary constant from P, $X = (x_1, \ldots, x_n)$ and
$(\tilde{\tilde{b}}_{i_k}) = (x_i, \ldots, x_{i_{k-1}}, b_{i_k}, x_{i_{k+1}}, \ldots, x_{i_n})$.

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