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# ON HOSSZÚ-GLUSHKIN ALGEBRAS CORRESPONDING TO THE SAME $n$-GROUP

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**Abstract**

Let $\cdot$, $\varphi$ and $b$ be a binary operation in $Q$, a unary operation in $Q$, and a constant in $Q$, respectively. Let, also, $n \in N \setminus \{1,2\}$. Then, in the present article, the algebra $(Q, \{\cdot, \varphi, b\})$ is said to be a Hosszú-Glushkin algebra of order $n$ (briefly: $nHG$-algebra) iff the following hold: 1. $(Q, \cdot)$ is a group; 2. $\varphi \in Aut(Q, \cdot)$; 3. $\varphi(b) = b$ ; and 4. $\varphi^{n-1}(x) \cdot b = b \cdot x$ for every $x \in Q$. Under this condition the Hosszú-Glushkin Theorem (:[2-3]) can be formulated in the following way: If $(Q, A)$ is an $n$-group and $n \in N \setminus \{1,2\}$, then there is an $nHG$-algebra $(Q, \{\cdot, \varphi, b\})$ such that $A(x_1, ..., x_n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n$ for every $x_1, ..., x_n \in Q$. Then, we say that this $nHG$-algebra is a *corresponding* $nHG$-algebra for the $n$-group $(Q, A)$. The main result of the paper is a description of all $nHG$-algebras *corresponding* to an $n$-group $(Q, A)$, by means of one of them (:Theorem 5.1).

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# 1. Preliminaries

## 1.1. About the expression $a_p^q$

Let $p \in N$, $q \in N \cup \{0\}$ and let $a$ be a mapping of the set $\{i | i \in N \wedge i \geq p \wedge i \leq q\}$ into the set $S$ ; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, ..., a_q \ ; & p < q \\ a_p \ ; & p = q \\ \text{empty sequence } (= \emptyset) \ ; & p > q. \end{cases}$$

For example:

$$A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1}),\ j \in \{1, ..., n\},\ n \in N \backslash \{1, 2\},\ \text{for } j = n$$

stands for

$$A(a_1, ..., a_{n-1}, A(a_n, ..., a_{2n-1})).$$

Besides, in some situations *instead of* $a_p^q$ *we write* $(a_i)_{i=p}^q$ (briefly: $(a_i)_p^q$).

For example:

$$(\forall x_i \in Q)_1^q$$

for $q > 1$ stands for

$$\forall x_1 \in Q ... \forall x_q \in Q$$

[usually, we write: $(\forall x_1 \in Q)...(\forall x_q \in Q)$],
for $q = 1$ stands for

$$\forall x_1 \in Q$$

[usually, we write: $(\forall x_1 \in Q)$],
and for $q = 0$ it stands for an empty sequence $(= \emptyset)$.

In *some cases*, instead of $a_p^q$ only, we write: sequence $a_p^q$ (sequence $a_p^q$ over a set $S$). For example: ... for every sequence $a_p^q$ over a set $S$ ... . And if $p \leq q$, we usually write: $a_p^q \in S$.

If $a_p^q$ is a sequence over a set $S$, $p \leq q$ and the equalities $a_p = \ldots = a_q = b\ (\in S)$ are satisfied, then

$$a_p^q \text{ is denoted by } \overset{q-p+1}{b}\ .$$

In connection with this, if $q - p + 1 = r$ (when we assume that there is no misunderstanding),

$$\text{instead of } \overset{q-p+1}{b} \quad \text{we write } \overset{r}{b}\ .$$

In some situations,

$$\text{instead of } \overset{q-p+1}{b} \ (\text{ or } \overset{r}{b}\ ) \text{ we write } \overset{q-p+1}{\overline{b}|} \text{ or } \overset{r}{\overline{b}|}\ ).$$

For example, *instead of*

$$\overset{q-p+1}{\mathbf{e}(c_1^{n-2})} \quad (\text{ or } \overset{r}{\mathbf{e}(c_1^{n-2})}\ )$$

*we write*

$$\overset{q-p+1}{\overline{\mathbf{e}\ (c_1^{n-2})}|} \quad (\text{ or } \overset{r}{\overline{\mathbf{e}\ (c_1^{n-2})}|}\ )$$

In addition, we denote *the empty sequence over* $S$ with $\overset{0}{b}$ , where $b$ is an arbitrary element from $S$.

## 1.2. About $n$-groups

Let $n \in N \setminus \{1\}$ and let $A$ be the mapping of the set $Q^n$ into the set $Q$ . $(Q,\ A)$ is said to be an $n$-semigroup iff for every $i \in \{2, \ldots, n\}$ and for all $x_1^{2n-1} \in Q$ the following equality holds:

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

$(Q,\ A)$ is an $n$-quasigroup iff for every $i \in \{1, \ldots, n\}$ and for all $a_1^n \in Q$

there is exactly one $x_i \in Q$ such that the equality

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds. $(Q, A)$ is said to be a Dörnte $n$-group (briefly: an $n$-group) iff $(Q, A)$ is both, $n$-semigroup and $n$-quasigroup. For $n = 2$ it is a group.[1] The notion of an $n$-group has been introduced in [1]. The following proposition holds:

1.2.1 [7]: *Let* $(Q, A)$ *be an* $n$*-quasigroup and* $n \in N \setminus \{1\}$. *Then:* $(Q, A)$ *is an* $n$*-group iff there is an* $i \in \{1, ..., n-1\}$ *such that the following law holds:*

$$A(x_i^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^i, A(x_{i+1}^{i+n}), x_{i+n+1}^{2n-1}).$$

## 1.3. On a $\{1, n\}$-neutral operation in an $n$-groupoid

Let $(Q, A)$ be an $n$-groupoid and $n \in N \setminus \{1\}$. Let also $\mathbf{e}$ be an $(n-2)$-ary operation in $Q$; for $n = 2$ this is a nullary operation. We say that $\mathbf{e}$ is a $\{1, n\}$-*neutral operation* of this $n$-groupoid $(Q, A)$ iff the following holds:

$$(1) \quad (\forall a_i \in Q)_1^{n-2} (\forall x \in Q) \; (A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x \wedge A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x).$$

For $n = 2$, $\mathbf{e}(a_1^0) (= \mathbf{e}(\emptyset)) = e \in Q$ is a neutral element of the groupoid $(Q, A)$. The notion of an $\{i, j\}$-neutral operation of an $n$-groupoid (: $n \in N \setminus \{1\}$, $(i, j) \in \{1, ..., n\}^2, i < j$) has been introduced in [8]. The following propositions hold:

1.3.1 [8]: *In an* $n$*-groupoid (* $n \in N \setminus \{1\}$ *) there is at most one* $\{1, n\}$*-neutral operation;*

1.3.2 [8]: *In every* $n$*-group there is a* $\{1, n\}$*-neutral operation;*

1.3.3 [8]: *For* $n \geq 3$, *an* $n$*-semigroup* $(Q, A)$ *is an* $n$*-group iff* $(Q, A)$ *has a* $\{1, n\}$*-neutral operation*[2]

and

1.3.4 : *Let* $(Q, A)$ *be an* $n$*-group,* $\mathbf{e}$ *its* $\{1, n\}$*-neutral operation and*

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[1] Menger's $n$-quasigroup for $n = 2$ is also a group (see for example [5]).

[2] Theorem 10 from [9], for $m = 1$ reduces to Proposition 1.3.3.

$n \in N \setminus \{1,2\}$. *Then, the following formulas are satisfied:*

*(2)* $(\forall a_j \in Q)_1^{n-2}(\forall b_j \in Q)_1^{n-2}(\forall x \in Q)$
$$A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x);$$
*and*

*(3)* $(\forall a_j \in Q)_1^{n-2}(\forall b_j \in Q)_1^{n-2}(\forall x \in Q)$
$$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) = A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2}))$$
*for every* $i \in \{1, ..., n-1\}$.

The sketch of the proof:

1) $F(x, b_1^{n-2}) \overset{def}{=} A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) \Longrightarrow$
$A(F(x, b_1^{n-2}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) =$
$A(A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) \Longrightarrow$
$A(F(x, b_1^{n-2}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) =$
$A(x, b_i^{n-2}, A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{i-1}) \Longrightarrow$
$A(F(x, b_1^{n-2}), b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) \Longrightarrow$
$F(x, b_1^{n-2}) = x \Longrightarrow$
$A(x, b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}) = A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x);$

2) $F(x, b_1^{n-2}) \overset{def}{=} A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) \Longrightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, F(x, b_1^{n-2})) =$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x)) \Longrightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, F(x, b_1^{n-2})) =$
$A(b_i^{n-2}, A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{i-1}, x) \Longrightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, F(x, b_1^{n-2})) = A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x)) \Longrightarrow$
$F(b_1^{n-2}, x) = x \Longrightarrow$
$A(b_i^{n-2}, \mathbf{e}(b_1^{n-2}), b_1^{i-1}, x) = A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})).$

## 1.4. On the inversing operation in an $n$-group

The following proposition holds:

1.4.1 [10]: *Let* $(Q, A)$ *be an* $n$*-semigroup and* $n \in N \setminus \{1\}$. *Then:*

*a) There is at most one* $(n-1)$*-ary operation* $f$ *in* $Q$ *such that the following formulas hold*

(1) $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)$

$$A(f(a_1^{n-2},a),a_1^{n-2},A(a,a_1^{n-2},x))=x$$

*and*

(2) $$(\forall a_i\in Q)_1^{n-2}(\forall a\in Q)(\forall x\in Q)$$
$$A(A(x,a_1^{n-2},a),a_1^{n-2},f(a_1^{n-2},a))=x\ ;$$

*b) If there is an* $(n-1)$*-ary operation* $f$ *in* $Q$ *such that the formulas* (1) *and* (2) *are satisfied, then* $(Q, A)$ *is an* $n$*-group; and*

*c) If* $(Q, A)$ *is an* $n$*-group , then there is an* $(n-1)$*-ary operation* $f$ *in* $Q$ *such that the formulas* (1) *and* (2) *hold.*[3]

As for the case $n=2$ we say that the operation $f$ is an *inversing operation in the* $n$*-group* $(Q,A)$; [10].

The following propositions hold:

1.4.2 [10]: *Let* $(Q,A)$ *be an* $n$*-group,* $\mathbf{e}$ *its* $\{1,n\}$*-neutral operation,* $f$ *its inversing operation and* $n\in N\backslash\{1\}$ *. Then, the following formula holds:*

$$(\forall a_i\in Q)_1^{n-2}(\forall a\in Q)(A(f(a_1^{n-2},a),a_1^{n-2},a)=$$
$$\mathbf{e}(a_1^{n-2})\wedge A(a,a_1^{n-2},f(a_1^{n-2},a))=\mathbf{e}(a_1^{n-2}));$$

and

1.4.3 [10] *Let* $(Q,A)$ *be an* $n$*-group,* $\mathbf{e}$ *its* $\{1,n\}$*-neutral operation,* $f$ *its inversing operation and* $n\in N\backslash\{1\}$ *. Then the formula:*

$$(\forall x\in Q)(\forall y\in Q)(\forall a_i\in Q)_1^{n-2}(\forall b_i\in Q)_1^{n-2}$$
$$A(x,b_1^{n-2},y)=A(A(x,a_1^{n-2},f(a_1^{n-2},\mathbf{e}(b_1^{n-2}))),a_1^{n-2},y)$$ [4] holds.

[3] $f(a_1^{n-2},a)\overset{def}{=}\mathsf{E}(a_1^{n-2},a,a_1^{n-2})$, where $\mathsf{E}$ is a $\{1,2n-1\}$-neutral operation of a $(2n-1)$-group $(Q,\overset{2}{A});\overset{2}{A}\,(x_1^{2n-1})\overset{def}{=}A(A(x_1^n),x_{n+1}^{2n-1})$. We note that for $n=2$, this is the inversing in a group.

[4] For $n=2:(\forall x\in Q)(\forall y\in Q)A(x,y)=A(x,y)$.

### 1.5. On Hosszú-Glushkin algebras

1.5.1: *Let* $\cdot$ *be a binary and* $\varphi$ *a unary operation in* $Q$. *Let also* $b$ *be a (fixed) element of the set* $Q$ , *and* $n$ *a (fixed) element of the set* $N \setminus \{1,2\}$. **We shall** *say that* $(Q, \{\cdot, \varphi, b\})$ *is a* **Hosszú-Glushkin algebra of order** $n$ *(briefly:* $nHG$*-algebra) iff the following hold:*

*(1)* $(Q, \cdot)$ *is a group;*

*(2)* $\varphi \in Aut(Q, \cdot)$ ;

*(3)* $\varphi^{n-1}(x) \cdot b = b \cdot x$ *for every* $x \in Q$ ; *and*

*(4)* $\varphi(b) = b$ .

1.5.2: *Hosszú-Glushkin* Theorem [2-3]: *Let* $(Q, A)$ *be an* $n$*-group and* $n \in N \setminus \{1,2\}$. *Then, there is an* $nHG$*-algebra* $(Q, \{\cdot, \varphi, b\})$ *such that for each* $x_1^n \in Q$ *the equality*

*(5)* $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$

*holds.*

By a simple verification (briefly, if the *Theorem of E.I. Sokolov* 1.2.1 is used) we conclude that the following proposition also holds:

1.5.3: *Let* $(Q, \{\cdot, \varphi, b\})$ $nHG$*-algebra* ($n \in N \setminus \{1,2\}$). *Let also*

$$A(x_1^n) \stackrel{def}{=} x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$$

*for all* $x_1^n \in Q$. *Then* $(Q, A)$ *is an* $n$*-group.*

1.5.4: *We shall say that an* $nHG$*-algebra* $(Q, \{\cdot, \varphi, b\})$ **corresponds** *to the* $n$*-group* $(Q, A)$ *iff the equality (5) holds for all* $x_1^n \in Q$.

## 2. Hosszú-Glushkin theorem from the point of view of $\{1, n\}$-neutral operation

In this part, the Hosszú-Glushkin theorem (1.5.2) is proved in the following (more specified) formulation [5]:

**Theorem 2.1.** (*Hosszú-Glushkin*) *Let* $(Q, A)$ *be an* $n$*-group,* **e** *its* $\{1, n\}$*-neutral operation and* $n \in N \setminus \{1,2\}$. *Let also,* $c_1^{n-2}$ *be* an arbitrary

[5] The formulation and the proof of the theorem follow the idea of E. I. Sokolov from [7].

*sequence over a set* $Q$*, and let*

(1) $x \cdot y \overset{def}{=} A(x, c_1^{n-2}, y)$;

(2) $\varphi(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})$; *and*

(3) $b \overset{def}{=} A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n})$

*for all* $x, y \in Q$ . *Then* $(Q, \{\cdot, \varphi, b\})$ *is an* $nHG$*-algebra (:1.5.1) such that*

(4) $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$

*for all* $x_1^n \in Q$ .

*Proof.*

1) Since $(Q, A)$ is an $n$-semigroup (:1.2), by definition (1), we conclude that for all $x, y, z \in Q$ the following sequence of equalities hold

$$(x \cdot y) \cdot z = A(A(x, c_1^{n-2}, y), c_1^{n-2}, z) =$$
$$= A(x, c_1^{n-2}, A(y, c_1^{n-2}, z)) =$$
$$= x \cdot (y \cdot z),$$

and hence we conclude that $(Q, \cdot)$ is a semigroup. Further, since $(Q, A)$ is an $n$-quasigroup (:1.3), by definition (1), we conclude that for all $a, b \in Q$ there is exactly one $x \in Q$ and exactly one $y \in Q$ , such that the equalities

$$a \cdot x = b \text{ and } y \cdot a = b$$

hold, thus we have that $(Q, \cdot)$ is a quasigroup. Hence, $(Q, \cdot)$ is a group.

2) By definitions (1) and (2), using the assumption that $(Q, A)$ is an $n$-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation, $n \in N \setminus \{1, 2\}$, and by Proposition 1.3.4, we conclude that for every $x, y \in Q$ the following sequence of equalities hold

$$\varphi(x \cdot y) = A(\mathbf{e}(c_1^{n-2}), A(x, c_1^{n-2}, y), c_1^{n-2}) =$$
$$= A(A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}), y, c_1^{n-2}) =$$
$$= A(\varphi(x), y, c_1^{n-2}) =$$
$$= A(\varphi(x), A(c_1^{n-2}, \mathbf{e}(c_1^{n-2}), y), c_1^{n-2}) =$$
$$= A(\varphi(x), c_1^{n-2}, A(\mathbf{e}(c_1^{n-2}), y, c_1^{n-2})) =$$
$$= A(\varphi(x), c_1^{n-2}, \varphi(y)) = \varphi(x) \cdot \varphi(y),$$

and hence we conclude that $\varphi \in Aut(Q, \cdot)$.

3) By definitions (2) and (3), using the assumption that $(Q, A)$ is an $n$-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation and $n \in N \setminus \{1, 2\}$, and by Proposition 1.3.4, we conclude that the following sequence of equalities hold

$$\varphi(b) = \varphi A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}\,) =$$

$$= A(\mathbf{e}(c_1^{n-2}), A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}\,), c_1^{n-2}) =$$

$$= A(A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}\,), \mathbf{e}(c_1^{n-2}), c_1^{n-2}) =$$

$$= A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}\,) =$$

$$= b,$$

hence we conclude that $\varphi(b) = b$.

4) By definitions (1) - (3), using the asumption that $(Q, A)$ is an $n$-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation and $n \in N \setminus \{1, 2\}$, we conclude that for every $x \in Q$ the following sequence of equalities hold

$$b \cdot x = A(A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}\,), c_1^{n-2}, x) =$$

$$= A(\overset{n-1}{\overline{\mathbf{e}(c_1^{n-2})}}\,, A(\mathbf{e}(c_1^{n-2}), c_1^{n-2}, x)) =$$

$$= A(\overset{n-1}{\overline{\mathbf{e}(c_1^{n-2})}}\,, A(x, c_1^{n-2}, \mathbf{e}(c_1^{n-2}))) =$$

$$= A(\overset{n-2}{\overline{\mathbf{e}(c_1^{n-2})}}\,, A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}), \mathbf{e}(c_1^{n-2})) =$$

$$= A(\overset{n-2}{\overline{\mathbf{e}(c_1^{n-2})}}\,, \varphi(x), \mathbf{e}(c_1^{n-2})) =$$

$$= A(\overset{n-2}{\overline{\mathbf{e}(c_1^{n-2})}}\,, A(\varphi(x), c_1^{n-2}, \mathbf{e}(c_1^{n-2})), \mathbf{e}(c_1^{n-2})) =$$

$$= A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n-3}, A(\mathbf{e}(c_1^{n-2}), \varphi(x), c_1^{n-2}), \overbrace{\mathbf{e}(c_1^{n-2})}^{2}) =$$

$$= A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n-3}, \varphi^2(x), \overbrace{\mathbf{e}(c_1^{n-2})}^{2}) =$$

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$$= A(\varphi^{n-1}(x), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}) =$$

$$= A(A(\varphi^{n-1}(x), c_1^{n-2}, \mathbf{e}(c_1^{n-2})), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}) =$$

$$= A(\varphi^{n-1}(x), c_1^{n-2}, A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n})) =$$

$$= \varphi^{n-1}(x) \cdot b,$$

hence we conclude that for every $x \in Q$ the equality

$$\varphi^{n-1}x \cdot b = b \cdot x$$

holds.

5) By definitions (1)-(3), using the asumption that $(Q, A)$ is an $n$-group, $\mathbf{e}$ its $\{1, n\}$- neutral operation and $n \in N \setminus \{1, 2\}$, and by Proposition 1.3.4, we conclude that for every $x_1^n \in Q$ the following sequence of equalities hold.

$A(x_1^n) = A(x_1^{n-1}, A(c_1^{n-2}, \mathbf{e}(c_1^{n-2}), A(x_n, c_1^{n-2}, \mathbf{e}(c_1^{n-2})))) =$
$= A(x_1^{n-1}, A(c_1^{n-2}, A(\mathbf{e}(c_1^{n-2}), x_n, c_1^{n-2}), \mathbf{e}(c_1^{n-2}))) =$
$= A(x_1^{n-1}, A(c_1^{n-2}, \varphi(x_n), \mathbf{e}(c_1^{n-2}))) =$
$= A(x_1^{n-2}, A(x_{n-1}, c_1^{n-2}, \varphi(x_n)), \mathbf{e}(c_1^{n-2})) =$
$= A(x_1^{n-2}, x_{n-1} \cdot \varphi(x_n), \mathbf{e}(c_1^{n-2})) =$
$= A(x_1^{n-2}, A(c_1^{n-2}, \mathbf{e}(c_1^{n-2}), A(x_{n-1} \cdot \varphi(x_n), c_1^{n-2}, \mathbf{e}(c_1^{n-2}))), \mathbf{e}(c_1^{n-2})) =$
$= A(x_1^{n-2}, A(c_1^{n-2}, A(\mathbf{e}(c_1^{n-2}), x_{n-1} \cdot \varphi(x_n), c_1^{n-2}), \mathbf{e}(c_1^{n-2})), \mathbf{e}(c_1^{n-2})) =$
$= A(x_1^{n-2}, A(c_1^{n-2}, \varphi(x_{n-1} \cdot \varphi(x_n)), \mathbf{e}(c_1^{n-2})), \mathbf{e}(c_1^{n-2})) =$
$= A(x_1^{n-3}, A(x_{n-2}, c_1^{n-2}, \varphi(x_{n-1} \cdot \varphi(x_n))), \mathbf{e}(c_1^{n-2}), \mathbf{e}(c_1^{n-2})) =$
$= A(x_1^{n-3}, x_{n-2} \cdot \varphi(x_{n-1} \cdot \varphi(x_n)), \mathbf{e}(c_1^{n-2}), \mathbf{e}(c_1^{n-2})) =$

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$$= A(x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}) =$$

$$= A(A(x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n), c_1^{n-2}, \mathbf{e}(c_1^{n-2})), \overbrace{\mathbf{e}(c_1^{n-2})}^{n-1}) =$$

$$= A(x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n), c_1^{n-2}, A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n})) =$$

$= x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b,$
hence we conclude that for every $x_1^n \in Q$ the following equality holds:

$$A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b.$$

## 3. A description of all $nHG$-algebras corresponding to the same $n$-group

**Theorem 3.1.** *Let $(Q, A)$ be an arbitrary $n$-group, $\mathbf{e}$ its $\{1, n\}$-neutral operation, $n \in N \setminus \{1,2\}$, $c_1^{n-2}$ a sequence over a set $Q$ and for all $x, y \in Q$*

$$B_{(c_1^{n-2})}(x, y) \overset{def}{=} A(x, c_1^{n-2}, y);$$

$$\varphi_{(c_1^{n-2})}(x) \overset{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}); \text{ and}$$

$$b_{(c_1^{n-2})} \overset{def}{=} A(\overbrace{\mathbf{e}(c_1^{n-2})}^{n}).$$

*Let also*

$$C_A \overset{def}{=} \{(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}\}) | c_1^{n-2} \in Q\}.$$

*Then, for every $nHG$-algebra $(Q, \{\cdot, \varphi, b\})$ the following equivalence holds*

$$(Q, \{\cdot, \varphi, b\}) \in \mathrm{C}_A \Longleftrightarrow (\forall x_i \in Q)_1^n A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b.$$

*Proof.*

1) By Theorem 2.1, we conclude that for every $nHG$-algebra $(Q, \{\cdot, \varphi, b\})$ the following implication holds:

$$(Q, \{\cdot, \varphi, b\}) \in \mathrm{C}_A \Longrightarrow (\forall x_i \in Q)_1^n A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b.$$

2) Let $(Q,A)$ be an $n$-group, $n \in N \setminus \{1,2\}$, $(Q,\{\cdot,\varphi,b\})$ an $nHG$-algebra, $e$ a neutral element of the group $(Q,\cdot)$ and $^{-1}$ the inversing operation in $(Q,\cdot)$. Let also for every $x_i^n \in Q$ the following equality holds:

$$A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_n) \cdot b \quad (:1.5).$$

If in the above equality we put $x_2^{n-2} = \overset{n-3}{e}$ (:1.1) and $x_{n-1} = b^{-1}$, since $\varphi(b) = b$ (:1.5), and thus also $\varphi(b^{-1}) = b^{-1}$, we conclude that for every $x_1, x_n \in Q$ the following equality holds:

$$A(x_1, \overset{n-3}{e}, b^{-1}, x_n) = x_1 \cdot x_n,$$

and hence we conclude that for all $x, y \in Q$ the equality

$$x \cdot y = B_{(\overset{n-3}{e}, b^{-1})}(x,y)$$

also holds.

3) Let $(Q,\{\cdot,\varphi,b\})$ and $(Q,\{\cdot,\bar{\varphi},\bar{b}\})$ be two $nHG$-algebras, $e$ a neutral element of the group $(Q,\cdot)$ and $^{-1}$ the inversing operation in $(Q,\cdot)$. Let, also, for every $x_1^n \in Q$ the following equality holds:

$$x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n = x_1 \cdot \bar{\varphi}(x_2) \cdot \ldots \cdot \bar{\varphi}^{n-2}(x_{n-1}) \cdot \bar{b} \cdot x_n \quad (:1.5).$$

If in the above equality we put $x_1 = \ldots = x_n = e$, we conclude that

$$b = \bar{b},$$

which means that for every $x_1^n \in Q$ the following equality holds:

$$x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n = x_1 \cdot \bar{\varphi}(x_2) \cdot \ldots \cdot \bar{\varphi}^{n-2}(x_{n-1}) \cdot b \cdot x_n,$$

and hence, by similar argument, we conclude that

$$\varphi = \bar{\varphi}.$$

4) By Theorem 2.1, Proposition 1.5.3 and by that the arguments from 2) and 3), we conclude that for every $nHG$-algebra $(Q,\{\cdot,\varphi,b\})$ the following implication holds:

$$(\forall x_i \in Q)_1^n A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b \implies (Q,\{\cdot,\varphi,b\}) \in \mathrm{C}_A.$$

□

A consequence of Theorem 3.1 and Proposition 1.4.3 is the following proposition:

**Proposition 3.2.** *Let* $(Q, A)$ *be an* $n$*-group,* $\mathbf{e}$ *its* $\{1, n\}$*-neutral operation,* $f$ *its inversing operation and* $n \in N \setminus \{1,2\}$*. Let also* $(Q, \{\cdot, \varphi, b\})$ *be an* $nHG$*-algebra* corresponding *to the* $n$*-group* $(Q, A)$ *(:1.5.4), e the neutral element of the group* $(Q, \cdot)$ *and* $^{-1}$ *the inversing operation in* $(Q, \cdot)$*. Then, for every* $b_1^{n-2} \in Q$ *the following equality holds:*

$$\mathbf{e}(b_1^{n-2}) = (\varphi(b_1) \cdot \ldots \cdot \varphi^{n-2}(b_{n-2}) \cdot b)^{-1}.$$

*Proof.*

By Theorem 3.1, there is a sequence $a_1^{n-2}$ over $Q$ such that for all $x, y \in Q$ the equality

(a) $x \cdot y = A(x, a_1^{n-2}, y)$

holds. The following also hold:

(b) $e = \mathbf{e}(a_1^{n-2})$

and

(c) $(\forall a \in Q)\ a^{-1} = f(a_1^{n-2}, a)$ (:1.3, 1.4) .

Let also $b_1^{n-2}$ be an arbitrary sequence over $Q$. Then, by Proposition 1.4.3, for all $x, y \in Q$ the following equality holds

$$A(x, b_1^{n-2}, y) = A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(b_1^{n-2}))), a_1^{n-2}, y),$$

i.e., by (a) and (c), also the equality

$$A(x, b_1^{n-2}, y) = x \cdot (\mathbf{e}(b_1^{n-2}))^{-1} \cdot y\ .$$

Hence, since $(Q, \{\cdot, \varphi, b\})$ is an $nHG$-algebra *corresponding* to the $n$-group $(Q, A)$, we conclude that for all $x, y \in Q$ the following equality holds

$$x \cdot \varphi(b_1) \cdot \ldots \cdot \varphi^{n-2}(b_{n-2}) \cdot b \cdot y = x \cdot (\mathbf{e}(b_1^{n-2}))^{-1} \cdot y.$$

Hence, we conclude that the proposition holds. □

## 4. About equations $f(a_1^{n-2}, x) = a_{n-1}$ and $\mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}) = a_{n-2}$

**Proposition 4.1.** *Let* $(Q, A)$ *be an* $n$*-group,* $f$ *its inversing operation and* $n \in N \setminus \{1,2\}$*. Then for every sequence* $a_1^{n-1}$ *over* $Q$ *there is exactly one* $x \in Q$ *such that the equality*

$$f(a_1^{n-2}, x) = a_{n-1}$$

*holds.*

*Proof.*

1) Let $\mathbf{e}$ be a $\{1, n\}$-neutral operation of the $n$-group $(Q, A)$. Then, by

Proposition 1.4.2, for every sequence $a_1^{n-2}$ over $Q$ and for every $x \in Q$ the equalities

$$A(f(a_1^{n-2},x),a_1^{n-2},x) = \mathbf{e}(a_1^{n-2}) \text{ and}$$
$$A(f(a_1^{n-2},x),f(a_1^{n-2},f(a_1^{n-2},x))) = \mathbf{e}(a_1^{n-2})$$

hold. Hence, since $(Q,A)$ is an $n$-quasigroup, we conclude that the formula

(1) $\quad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q) f(a_1^{n-2},f(a_1^{n-2},x)) = x$

holds.

2) By the monotonicity of $f$ and by formula (1), we conclude that for all $x,y \in Q$, and for every sequence $a_1^{n-2}$ over $Q$ the following sequence of implications holds

$$f(a_1^{n-2},x) = f(a_1^{n-2},y) \Longrightarrow$$
$$f(a_1^{n-2},f(a_1^{n-2},x)) = f(a_1^{n-2},f(a_1^{n-2},y)) \Longrightarrow$$
$$x = y$$

and hence we conclude that the formula

(2) $\quad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(\forall y \in Q)(f(a_1^{n-2},x) = f(a_1^{n-2},y) \iff x = y)$

also holds.

3) By formulas (2) and (1), we have that for every sequence $a_1^{n-1}$ over $Q$ and for every $x \in Q$ the following sequence of equivalences hold

$$f(a_1^{n-2},x) = a_{n-1} \iff$$
$$f(a_1^{n-2},f(a_1^{n-2},x)) = f(a_1^{n-1}) \iff$$
$$x = f(a_1^{n-1})$$

and hence we conclude that for every $a_1^{n-1}, x \in Q$ the equivalence

$$f(a_1^{n-2},x) = a_{n-1} \iff x = f(a_1^{n-1})$$

holds. □

**Proposition 4.2.** *Let $(Q,A)$ be an $n$-group, $\mathbf{e}$ its $\{1,n\}$-neutral operation and $n \in N \setminus \{1,2\}$. Then, for every sequence $a_1^{n-2}$ over $Q$, and for every $i \in \{1,\dots,n-2\}$ there is exactly one $x_i \in Q$ such that the equality*

$$\mathbf{e}(a_1^{i-1},x_i,a_i^{n-3}) = a_{n-2}$$

*hold.*

*Proof.*

Let $f$ be the inversing operation of the $n$-group $(Q,A)$. Then, by Proposition 4.1 and Proposition 1.4.3, we conclude that for every $i \in \{1,\dots,n-2\}$, for every sequence $b_1^{n-2}$ over $Q$ and for every $x_i \in Q$, the following sequence of equivalences holds

$$\mathbf{e}(b_1^{i-1},x_i,b_i^{n-3}) = b_{n-2} \iff$$

$$f(b_1^{n-2}, \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})) = f(b_1^{n-2}, b_{n-2}) \iff$$
$$A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, f(b_1^{n-2}, \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))) =$$

$$A(\mathbf{e}(b_1^{n-2}), b_1^{n-2} f(b_1^{n-2}, b_{n-2})) \iff$$
$$A(A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, f(b_1^{n-2}, \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))), b_1^{n-2}, \mathbf{e}(b_1^{n-2})) =$$

$$A(A(\mathbf{e}(b_1^{n-2}), b_1^{n-2} f(b_1^{n-2}, b_{n-2})), b_1^{n-2}, \mathbf{e}(b_1^{n-2})) \iff$$
$$A(\mathbf{e}(b_1^{n-2}), b_1^{i-1}, x_i, b_i^{n-3}, \mathbf{e}(b_1^{n-2})) = f(b_1^{n-2}, b_{n-2}),$$

and hence we conclude that the equivalence

$$\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) = b_{n-2} \iff$$

$$A(\mathbf{e}(b_1^{n-2}), b_1^{i-1}, x_i, b_i^{n-3}, \mathbf{e}(b_1^{n-2})) = f(b_1^{n-2}, b_{n-2}),$$

also holds for every $i \in \{1, ..., n-2\}$, and for every $b_1^{n-2}, x_i \in Q$. Hence, since $(Q, A)$ is an $n$-quasigroup, we conclude that the proposition is satisfied. □

## 5. Main results

**Theorem 5.1.** *Let $(Q, A)$ be an $n$-group, $n \in N \setminus \{1, 2\}$, $(Q, \{\cdot, \varphi, b\})$ an arbitrary $nHG$-algebra* **corresponding** *to the $n$-group $(Q, A)$ (:1.5.4), ${}^{-1}$ the inversing operation in $(Q, \cdot)$, $k \in Q$ and for every $x, y \in Q$*

*(1)* $x \cdot_k y \overset{def}{=} x \cdot k \cdot y;$

*(2)* $\varphi_k(x) \overset{def}{=} k^{-1} \cdot \varphi(x) \cdot \varphi(k);$ *and*

*(3)* $b_k \overset{def}{=} k^{-1} \cdot \varphi(k^{-1}) \cdot ... \cdot \varphi^{n-1}(k^{-1}) \cdot b.$

*Let also*

*(4)* $\hat{C}_A \overset{def}{=} \{(Q, \{\cdot_k, \varphi_k, b_k\}) | \ k \in Q\}.$

*Then, $\hat{C}_A$ is a set of all $nHG$-algebras* **corresponding** *to the $n$-group $(Q, A)$.*[6]

*Proof.*

1) Let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG$-algebra *corresponding* to the $n$-group $(Q, A)$. By Theorem 3.1

$$(Q, \{\cdot, \varphi, b\}) \in \mathrm{C}_A$$

i.e. there is a sequence $a_1^{n-2}$ over $Q$ such that

$$x \cdot y = A(x, a_1^{n-2}, y);$$

[6]i.e.: Then $\hat{\mathrm{C}}_A = \mathrm{C}_A$ (: 3.1).

$\varphi(x) = A(\mathbf{e}(a_1^{n-2}), x, a_1^{n-2});$ and

$b = A(\overbrace{\mathbf{e}(a_1^{n-2})}^{n}).$

In addition, by definitions (1)-(4) from the formulation of Theorem 5.1, we conclude that $\cdot = \cdot_e, \varphi = \varphi_e$ and $b = b_e$, where $e$ is a neutral element of the group $(Q, \cdot)$, and hence also

$(Q, \{\cdot, \varphi, b\}) \in \hat{C}_A$

Thus, it also holds

$(Q, \{\cdot, \varphi, b\}) \in C_A \cap \hat{C}_A.$

2) Let also $(Q, \{\Box, \varphi_\Box, b_\Box\})$ be an arbitrary $nHG$-algebra *corresponding* to the $n$-group $(Q, A)$. By Theorem 3.1

$(Q, \{\Box, \varphi_\Box, b_\Box\}) \in C_A,$

i.e. there is a sequence $b_1^{n-2}$ over $Q$ such that

$x \Box y = A(x, b_1^{n-2}, y);$

$\varphi_\Box(x) = A(\mathbf{e}(b_1^{n-2}), x, b_1^{n-2});$ and

$b_\Box = A(\overbrace{\mathbf{e}(b_1^{n-2})}^{n}).$

By Proposition 1.4.3, for all $x, y \in Q$ the equality

$A(x, b_1^{n-2}, y) = A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(b_1^{n-2}))), a_1^{n-2}, y)$

holds. Since $a_1^{n-2}, b_1^{n-2}$ are fixed elements of the set $Q$, there is $k \in Q$ such that

$f(a_1^{n-2}, \mathbf{e}(b_1^{n-2})) = k$ .

Hence, for all $x, y \in Q$, the equality

$x \Box y = x \cdot k \cdot y$

holds. In addition,

$\mathbf{e}(b_1^{n-2}) = k^{-1};$

for the inversing operation $^{-1}$ in the group $(Q, \cdot)$, namely, the following holds: $a^{-1} = f(a_1^{n-2}, a)$ for every $a \in Q$ (:1.3,1.4). Thereby, and by the assumption that $(Q, \{\cdot, \varphi, b\})$ is an $nHG$-algebra *corresponding* to the $n$-group $(Q, A)$, by Proposition 3.2, we conclude that for every $x \in Q$ the following sequence of equalities hold

$$\begin{aligned}\varphi_\Box(x) &= A(\mathbf{e}(b_1^{n-2}), x, b_1^{n-2}) = \\ &= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi^2(b_1) \cdot ... \varphi^{n-1}(b_{n-2}) \cdot b = \\ &= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi^2(b_1) \cdot ... \varphi^{n-1}(b_{n-2}) \cdot \varphi(b) = \\ &= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi(\varphi(b_1) \cdot ... \varphi^{n-2}(b_{n-2}) \cdot b) =\end{aligned}$$

$$= \mathbf{e}(b_1^{n-2}) \cdot \varphi(x) \cdot \varphi((\mathbf{e}(b_1^{n-2}))^{-1}) =$$
$$= k^{-1} \cdot \varphi(x) \cdot \varphi(k),$$

hence we conclude that for every $x \in Q$ the equality

$$\varphi_\Box(x) = k^{-1} \cdot \varphi(x) \cdot \varphi(k)$$

holds. Similarly, we conclude that the following sequence of equalities hold, too.

$$b_\Box = A( \overbrace{\mathbf{e}(b_1^{n-2})}^{n} ) =$$

$$= \mathbf{e}(b_1^{n-2}) \cdot \varphi(\mathbf{e}(b_1^{n-2})) \cdot ... \cdot \varphi^{n-1}(\mathbf{e}(b_1^{n-2})) \cdot b =$$
$$= k^{-1} \cdot \varphi(k^{-1}) \cdot ... \cdot \varphi^{n-1}(k^{-1}) \cdot b,$$

and hence we conclude that

$$b_\Box = k^{-1} \cdot \varphi(k^{-1}) \cdot ... \cdot \varphi^{n-1}(k^{-1}) \cdot b.$$

Thus, it also holds

$$\mathrm{C}_A \subseteq \hat{\mathrm{C}}_A.$$

3) Finally, let $(Q, \{\Box, \varphi_\Box, b_\Box\})$ be an arbitrary element from the set $\hat{\mathrm{C}}$. Then, by definitions (1)-(4) from the formulation of Theorem 5.1, there is $k \in Q$ such that for all $x, y \in Q$ the equalities

$$x \Box y = x \cdot k \cdot y \ ;$$
$$\varphi_\Box(x) = k^{-1} \cdot \varphi(x) \cdot \varphi(k) \ ; \text{ and}$$
$$b_\Box = k^{-1} \cdot \varphi(k^{-1}) \cdot ... \cdot \varphi^{n-1}(k^{-1}) \cdot b.$$

hold. In addition, by Proposition 4.1 and Proposition 4.2, we conclude that for every sequence $b_1^{n-3}$ over $Q$ and for every $i \in \{1, ..., n-2\}$, there is exactly one $x_i \in Q$ such that

$$f(a_1^{n-2}, \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})) = k \ .$$

Hence, firstly, by Proposition 1.4.3, we conclude that for all $x, y \in Q$ the following sequence of equalities hold

$$x \Box y = x \cdot k \cdot y$$
$$= A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))), a_1^{n-2}, y) =$$
$$= A(x, b_1^{i-1}, x_i, b_i^{n-3}, y),$$

i.e. that for all $x, y \in Q$ also the equality

$$x \Box y = A(x, b_1^{i-1}, x_i, b_i^{n-3}, y),$$

holds. Further, by a similar argument as in 2), we conclude that for every $x \in Q$ the following sequence of equalities hold

$$\varphi_\Box(x) = k^{-1} \cdot \varphi(x) \cdot \varphi(k) =$$
$$= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(x) \cdot \varphi((\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))^{-1}) =$$
$$= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(x) \cdot \varphi(\varphi(b_1) \cdot ... \cdot \varphi^i(x_i) \cdot ... \cdot \varphi^{n-2}(b_{n-3}) \cdot b) =$$
$$= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}) \cdot \varphi(x) \cdot \varphi^2(b_1) \cdot .. \cdot \varphi^{i+1}(x_i) \cdot ... \cdot$$

$$\cdot\varphi^{n-1}(b_{n-3})\cdot\varphi(b) =$$
$$= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})\cdot\varphi(x)\cdot\varphi^2(b_1)\cdot\ldots\cdot\varphi^{i+1}(x_i)\cdot\ldots\cdot\varphi^{n-1}(b_{n-3})\cdot b =$$
$$= A(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}), x, b_1^{i-1}, x_i, b_i^{n-3}),$$

i.e. that for every $x \in Q$ the equality

$$\varphi_\square(x) = A(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}), x, b_1^{i-1}, x_i, b_i^{n-3})$$

is satisfied. Finally, since $k^{-1} = \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})$ and by the assumption that $(Q, \{\cdot, \varphi, b\})$ $nHG$-algebra *corresponding* to the $n$-group $(Q, A)$, we conclude that

$$b_\square = k^{-1}\cdot\varphi(k^{-1})\cdot\ldots\cdot\varphi^{n-1}(k^{-1})\cdot b =$$
$$= \mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})\cdot\varphi(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))\cdot\ldots\cdot\varphi^{n-1}(\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3}))\cdot b =$$
$$= A(\overset{n}{\overline{\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})}}),$$

i.e. that

$$b_\square = A(\overset{n}{\overline{\mathbf{e}(b_1^{i-1}, x_i, b_i^{n-3})}}),$$

and hence
$\hat{\mathrm{C}}_A \subseteq \mathrm{C}_A$
also holds. □

## 6. Remark

E. I. Sokolov in [7] for the proof of Hosszú-Glushkin Theorem uses the unary operation *skew element* (:[4-7]). In [4] (see [6], p. 53), the following theorem is proved: For $n \geq 3$, an $n$-semigroup $(Q, A)$ is an $n$-group iff there is a unary operation $^-$ in $Q$ such that the following laws are satisfied:

$$A(x, \overset{n-2}{a}, \bar{a}) = x, \quad A(\bar{a}, \overset{n-2}{a}, x) = x,$$

$$A(x, \overset{n-3}{a}, \bar{a}, a) = x, \quad A(a, \bar{a}, \overset{n-2}{a}, x) = x$$ [7] .

In an $n$-group $(Q, A)$ these laws are fulfilled by a unary operation $^-$ defined in the following way:

$$\bar{a} \overset{def}{=} \mathbf{e}(\overset{n-2}{a})$$

[7] $\bar{a} \in Q$ is said to be a *skew element* of the element $a \in Q$.

for every $a \in Q$, where $\mathbf{e}$ is a $\{1, n\}$-neutral operation of the $n$-group $(Q, A)$.

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