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ON SOME FIXED POINT THEOREMS IN METRIC SPACES

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Abstract

We proved a common fixed point theorem involving two pairs of weakly commuting mappings on a complete metric space and two fixed point theorems in non-complete metric space, generalizing some results of [4], [6].

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1.

Two mappings S and I of metric space (X, d) into itself are said to be weakly commuting [9] if

$$d(SIx, ISx) \leq d(Ix, Sx), \qquad (\forall)x \in X.$$

Two commuting mappings commute weakly but two weakly commuting mappings do not necessarily commute (see [9]).

We consider the set $\mathcal{L}$ of all real continuous functions $g : [0,\infty)^5 \to [0,\infty)$ satisfying the following properties:

(i) g is nondecreasing in the 4th and 5th variable,

(ii) there is an $h_1 > 0$ an $h_2 > 0$ such that $h = h_1 h_2 < 1$ and if $u, v \in [0, \infty)$ satisfy $u \leq g(v, v, u, u+v, 0)$ or $u \leq g(v, u, v, u+v, 0)$ then $u \leq h_1 v$ and if $u, v \in [0, \infty)$ satisfy $u \leq g(v, v, u, 0, u+v)$, or $u \leq g(v, u, v, 0, u+v)$, then $u \leq h_2 v$,

(iii) if $u \in [0, \infty)$ is such that $u \leq g(u, 0, 0, u, u)$ or $u \leq g(0, u, 0, u, u)$ or $u \leq g(0, 0, u, u, u)$, then $u = 0$.

Theorem 1. *Let S and I be weakly commuting mappings and let T and J be weakly commuting mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$\begin{aligned} (1) \qquad & d(Sx, Ty) \\ \leq \; & g(d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Sx, Jy)), \\ & (\forall) x, y \in X \end{aligned}$$

where q is in $\mathcal{L}$. If the range of I contains the range of T and the range of J contains the range of S and if one of S, T, I and J is continuous, then S, T, I and J have a unique common fixed point z, and z is the unique common fixed point of S and I and of T and J.

Proof. Let $x_0 \in X$. Since the range of J contains the range of S and the range of I contains the range of T we can choose x_{2n}, x_{2n+1} and x_{2n+2} such that

$$Sx_{2n} = Jx_{2n+1}, \; Tx_{2n+1} = Ix_{2n+2}, \; n = 0, 1, 2, \ldots.$$

Using (1) we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq g(d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}),$$

$$d(Ix_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, Sx_{2n})) = g(d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}),$$

$$d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Tx_{2n+1}), 0) \leq g(d(Tx_{2n-1}, Sx_{2n}),$$

$$d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n+1}), 0)$$

since $d(Tx_{2n-1}, Tx_{2n+1}) \leq d(Tx_{2n}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n+1})$, and by property (ii), we deduce that

$$d(Sx_{2n}, Tx_{2n+1}) \leq h_1 d(Sx_{2n}, Tx_{2n-1}).$$

Similarly,

$$d(Tx_{2n-1}, Sx_{2n}) \leq h_2 d(Sx_{2n-2}, Tx_{2n-1})$$

and so

$$d(Sx_{2n}, Tx_{2n+1}) \leq hd(Sx_{2n-2}, Tx_{2n-1})$$

from where we deduce that

$$d(Sx_{2n}, Tx_{2n+1}) \leq h^n d(Sx_0, Tx_1)$$

$$d(Tx_{2n+1}, Sx_{2n+2}) \leq h_2 h^n d(Sx_0, Tx_1)$$

for $n = 1, 2, \ldots$. Since $h < 1$, we have that the sequence

$$\{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\}$$

is a Cauchy sequence. Since (X, d) is a complete metric space we deduce that this sequence has a limit z in X and the subsequence $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n+1}\} = \{Ix_{2n+2}\}$ converge to the point z.

We suppose that the mapping I is continuous, so that the sequences $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the point Iz. Since S and I weakly commute, we have

$$d(ISx_{2n}, SIx_{2n}) \leq d(Ix_{2n}, Sx_{2n})$$

so that the sequence $\{SIx_{2n}\}$ converges to the point Iz.

From

$$d(SIx_{2n}, Tx_{2n+1}) \leq g(d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}),$$

$$d(Jx_{2n+1}, Tx_{2n+1}), d(I^2x_{2n}, Tx_{2n+1}), d(SIx_{2n}, Jx_{2n+1}))$$

letting $n \to \infty$, we obtain

$$d(Iz, z) \leq g(d(Iz, z), 0, 0, d(Iz, z), d(Iz, z)).$$

By property (iii) this implies $Iz = z$.

Since

$$d(Sz, Tx_{2n+1}) \leq g(d(Iz, Jx_{2n+1}), d(Iz, Sz),$$

$$d(Jx_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Sz, Jx_{2n+1}))$$

letting $n \to \infty$, we have

$$d(Sz, z) \leq g(0, d(z, Sz), 0, 0, d(Sz, z))$$

and so, by (iii) we deduce that $Sz = z$.

Since the range of J contains the range of S, there is a $z' \in X$ such that $Jz' = z$. We deduce that

$$d(z, Tz') = d(Sz, Tz') \leq g(d(Iz, Jz'), d(Iz, Sz), d(Jz', Tz'), d(Iz, Tz'),$$

$$d(Sz, Jz')) = g(0, 0, d(z, Tz'), d(z, Tz'), 0)$$

and by (iii) we deduce that $Tz' = z$.

Since T and J weakly commute, we have

$$d(Tz, Jz) = d(TJz', JTz') \leq d(Jz', Tz') = 0$$

and so $Tz = Jz$. We deduce that

$$d(z, Tz) = d(Sz, Tz) \leq g(d(Iz, Jz), d(Iz, Sz),$$

$$d(Jz, Tz), d(Iz, Tz), d(Sz, Jz)) = g(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz))$$

and by (iii) we have $z = Tz = Jz$. Since $Iz = Sz = z$ we deduce that

$$z = Tz = Jz = Iz = Sz.$$

If the mapping J is continuous the proof is similar.

We suppose that the mapping S or T is continuous. In a similar way for the above and to the proof of Theorem 3 [4] we deduce that z is again a common fixed point of S, T, I and J.

We suppose that there is a second common fixed point y of S and I. We have that

$$d(Sy, Tz) = d(y, z) \leq g(d(Iy, Jz), d(Iy, Sz), (Jz, Tz),$$

$$d(Iy, Tz), d(Sy, Jz)) = g(d(y, z), 0, 0, d(y, z), d(y, z))$$

and for property (iii) it follows that $y = z$. Similarly, it is proved that z is the unique common fixed point of T and J.

Remark 1. In [3] Delbosco considered the set $\mathcal{G}$ of all continuous functions $g : [0, \infty)^3 \to [0, \infty)$ satisfying the following properties:

(i)' $g(1, 1, 1) = h < 1$.

(ii)' let $u, v \geq 0$ be such that either $u \leq g(u, v, v)$ or $u \leq g(v, v, u)$ or $u \leq g(v, u, v)$. Then $u \leq hv$.

We observe that $\mathcal{G} \subset \mathcal{L}$. Remark 3 shows that $\mathcal{G} \neq \mathcal{L}$.

Corollary 1. *([3], Th. 3) let S and I be weakly commuting mappings and let T and J be weakly commuting mappings of a complete metric space (X, d) into itself satisfying*

$$(2) \qquad d(Sx, Ty) \leq g(d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)), \qquad (\forall)x, y \in X$$

where g is in $\mathcal{G}$. If the range of I contains the range of T and the range of J contains the range of S, and if one of S, T, I and J is continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and T and J.

Proof. We apply Theorem 1 since $\mathcal{G} \subset \mathcal{L}$.

Corollary 2. *([4], Th. 4) Let S and I be weakly commuting mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$(3) \qquad d(Sx, Ty)$$

$$\leq \max\{cd(Ix, Jy), cd(Ix, Sx), cd(Jy, Ty), ad(Ix, Ty) + bd(Jy, Sx)\},$$

for all $x, y \in X$ where a, b and c are real numbers such that $0 \leq c < 1$, $0 \leq a + b < 1$ and $c \max\{a/(1-a), b/(1-b)\} < 1$. If the range of I contains the range of T and the range of J contains the range of S, and if one of S, T, I and J is continuous, then S, T, I and J have a unique fixed point z. Further, z is the unique common fixed point of S and I and T and J.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$

$$g(x_1, x_2, x_3, x_4, x_5) = \max\{cx_1, cx_2, cx_3, ax_4 + bx_5\}$$

and observe that $g \in \mathcal{L}$. The result follows from Theorem 1.

Remark 2. We observe that the hypotheses about a, b, c are to restrictive in Corollary 2; it is sufficient to suppose that $0 \leq c < 1$, $0 \leq a + b < 1$.

Remark 3. We define a metric on $X = \{1,2,3,4\}$ by

$$d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1,\ d(1,2) = d(3,4) = 2.$$

Let I be identity on X and define S, T, J by

$$S1 = 2,\ S2 = S3 = 1,\ S4 = 3,$$

$$T1 = T2 = T3 = T4 = 4,$$

$$J1 = 2,\ J2 = 1,\ J3 = 3,\ J4 = 4.$$

As has been shown in Example 3 [4], inequality (2) is not satisfied but inequality (3) holds. We deduce that $\mathcal{G} \neq \mathcal{L}$.

Remark 4. We recall [11] that a generalized ϕ-contraction is an application $T : X \to X$ which satisfies the inequality

$$d(Tx,Ty) \leq \phi(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)),\ (\forall)x,y \in X$$

where $\phi : \mathbf{R}_+^5 \to \mathbf{R}_+$ is increasing and has the property

$$\lim_{x\to t+0} \sup \phi(z,\dots,z) < t, \qquad (\forall)t \in (0,\infty).$$

For fixed point theorems concerning generalized ϕ-contractions see C. B. Cirić [2], O. Hadžić [5] and M. Tasković [10].

If we consider the family $\mathcal{F}$ of all real functions $f : [0,\infty) \to [0,\infty)$ such that f is increasing, continuous from the right and $f(t) < t$ for any $t > 0$, we have that if S and T are two mappings such that

$$d(Sx,Ty) \leq f(\max\{d(x,y), d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx)\}),$$

$$(\forall)x,y \in X$$

then S and T hava a unique common fixed point ([7])

D. Delbosko [3] proved that $\mathcal{G} \not\subset \mathcal{F}$ and so we have also that $\mathcal{L} \not\subset \mathcal{F}$ (for counterexample see Example 1 of [4]).

2.

We recall the following

Theorem A. ([6], Th. 1) *Let (X,d) be a metric space. Let T be a map of X into itself such that*

(i) $d(Tx,Ty) \leq \alpha(d(x,Tx)+d(y,Ty))$, $0<\alpha<1/2$, $(\forall)x,y \in X$,

(ii) T is continuous at a point $u \in X$,

(iii) there exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ converging to u.

Then u is the unique fixed point of T.

We generalize Theorem A with the following result

Theorem 2. *Let (X,d) be a metric space and T be a map of X into itself such that*

(i) $d(Tx,Ty) \leq g(d(x,y),d(x,Tx),d(y,Ty))$ *for all $x,y \in X$ where $g \in \mathcal{G}$,*

(ii) T is continuous at a point $u \in X$,

(iii) there exists a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a subsequence $\{T^{n_i}(x)\}$ onverging to u.

Then u is the unique fixed point of T.

We will give an example of a mapping T that satisfies the conditions given in Theorem 2 and to which Theorem A could not be applied.

Consider the mapping

$$T:[0,1) \to [0,\frac{1}{3}),\ T(x)=\frac{x}{3}$$

and let suppose that there is a $\alpha \in (0,1/2)$ so that (i) is satisfied for the euclidean distance

$$d(Tx,Ty) \leq \alpha(d(x,Tx)+d(y,Ty)),\ (\forall)x,y \in [0,1).$$

We would have that

$$\frac{|x-y|}{3} \leq \alpha(\frac{2x}{3}+\frac{2y}{3}) < \frac{x+y}{3}, \; x>0, \; y \geq 0,$$

which is not satisfied for $x > y = 0$.

On the other hand, the function

$$g : [0,\infty)^3 \to [0,\infty), \; g(x,y,z) = \alpha(x+y+z), \; \frac{1}{5} < \alpha < \frac{1}{3}$$

is in the set $\mathcal{G}$ and

$$g(1,1,1) = 3\alpha < 1,$$

$$u \leq g(u,v,v) = g(v,u,v) = g(v,v,u) \;\; \Rightarrow \;\; u \leq \alpha(2v+u)$$
$$\Rightarrow \;\; u(1-\alpha) \leq 2\alpha v \Rightarrow u \leq \frac{2\alpha}{1-\alpha}v$$

and we have

$$\frac{2\alpha}{1-\alpha} \leq 3\alpha \Leftrightarrow \frac{2}{1-\alpha} \leq 3 \Leftrightarrow 2 \leq 3-3\alpha \Leftrightarrow \alpha \leq \frac{1}{3}.$$

Hence

$$u \leq g(u,v,v) = g(v,u,v) = g(v,v,u) \Rightarrow u \leq (3\alpha)v,$$

and for $x \geq y$ we have that (i) from Theorem 2 holds if and only if

$$\frac{x-y}{3} \leq \alpha(x-y+\frac{2x}{3}+\frac{2y}{3}) \Leftrightarrow (5\alpha-1)x+(1-\alpha)y \geq 0$$

and so we can apply Theorem 2 to this function, but not Theorem A.

Proof of Theorem 2. The continuity at u implies that $\{T^{n_i+1}x\}$ converges to Tu. Suppose $u \neq Tu$ and consider two open balls $B(u,r)$ and $B'(Tu,r)$ centered at u (resp. at Tu) and of radius $r>0$ where $r < d(u,Tu)/3$.

We have that there is an integer $N > 0$ such that

$$T^{n_i}x \in B(u,r) \text{ and } T^{n_i+1}x \in B'(Tu,r) \text{ for } i \geq N$$

and hence we obtain that

$$(*) \qquad d(T^{n_i}x, T^{n_i+1}x) > r \text{ for } i > N.$$

From the condition given in the theorem we obtain that

$$\begin{aligned} & d(T^{n_i+1}x, T^{n_i+2}x) \\ \leq \;& g(d(T^{n_i}x, T^{n_i+1}x), d(T^{n_i}x, T^{n_i+1}x), d(T^{n_i+1}x, T^{n_i+2}x)) \end{aligned}$$

and so we deduce that

$$d(T^{n_i+1}x, T^{n_i+2}x) \leq hd(T^{n_i}x, T^{n_i+1}x).$$

For $k > j \geq N$ we obtain

$$\begin{aligned} & d(T^{n_k}x, T^{n_k+1}x) \leq hd(T^{n_k-1}x, T^{n_k}x) \\ \leq \;& h^2 d(T^{n_k-2}x, T^{n_k-1}x) \leq \ldots \leq h^{n_k-n_j} d(T^{n_j}x, T^{n_j+1}x). \end{aligned}$$

As $k \to \infty$, the last expression approaches 0 and we obtain a contradiction with relation (*). Hence $Tu = u$.

If u' is another fixed point of T we would obtain that

$$\begin{aligned} & d(Tu, Tu') \leq g(d(u, u'), d(u, Tu), d(u', Tu')) \\ \Leftrightarrow \;& d(u, u') \leq g(d(u, u'), 0, 0) \Rightarrow d(u, u') = 0 \end{aligned}$$

and hence $u = u'$, i. e., u is the unique fixed point of T.

Now we recall that D. Delbosko [3] proved that every pair of T and S of maps on a complete metric space (X, d) onto itself satisfying the condition

$$d(Sx, Ty) \leq g(d(x, y), d(x, Sx), d(y, Ty))$$

for all $x, y \in X$, where g is in $\mathcal{G}$, have a unique common fixed point.

We will prove the following

Theorem 3. *Let S and T be two maps of a metric space (X, d) into itself, satisfying*

(i) $d(Sx, Ty) \leq g(d(x, y), d(x, Sx), d(y, Ty))$ for all $x, y \in X$ and $g \in \mathcal{G}$,

(ii) there exists a point $u \in X$ so that S is continuous at u and T is continuous at the point Su,

(iii) there exists a point $x \in X$ such that the sequence $\{(T \circ S)^n(x)\} = \{(TS)^n(x)\}$ has a subsequence $\{(TS)^{n_i}(x)\}$ onverging to u.

Then $u' = Su$ is the unique common fixed point of T and S.

Proof. We have that

$$d(Sx, TSx) \leq g(d(x, Sx), d(x, Sx), d(Sx, TSx))$$

and so we obtain

$$d(Sx, TSx) \leq hd(x, Sx) \tag{4}$$

A similar inequality can be likely proved

$$d(STx, Tx) \leq hd(x, Tx). \tag{5}$$

in a similar argument.

We observe that

$$\begin{aligned} & d(STSx, TSTSx) \\ \leq \; & g(d(TSx, STSx), d(TSx, STSx), d(STSx, TSTSx)) \end{aligned}$$

leads to

$$d(STSx, TSTSx) \leq hd(TSx, STSx).$$

Using inequality (5), we have

$$d(TSx, STSx) \leq hd(Sx, TSx)$$

and

$$d(STSx, TSTSx) \leq h^2 d(Sx, TSx).$$

We deduce that

$$d(S(TS)^{n+1}x, (TS)^{n+2}x) \leq h^2 d(S(TS)^n x, (TS)^{n+1}x). \tag{6}$$

Considering the sequence $\{(TS)^{n_i}x\}$ and taking account of condition (ii) we obtain

$$\begin{aligned} & S(TS)^{n_i}x \to Su \\ & (TS)^{n_i+1}x = TS(TS)^{n_i}s \to TSu. \end{aligned}$$

From (6) we obtain that

$$\lim_{i\to\infty} d(S(TS)^{n_i}x, (TS)^{n_i+1}) = 0$$

and so we have

$$d(Su, TSu) = 0.$$

We have obtained that $Su = u'$ is a fixed point for T. Introducing in inequality (i) we deduce that

$$\begin{aligned} d(Su', Tu') &\le g(d(u', u'), d(u', Su'), d(u', Tu')) \\ &\Leftrightarrow d(Su', u') \le g(0, d(u', Su'), 0) \end{aligned}$$

and so $u' = Su'$.

Supposing that u'' is another common fixed point of T and S we obtain

$$\begin{aligned} d(Su', Tu'') &\le g(d(u', u''), d(u', Su'), d(u'', Tu'')) \\ &\Leftrightarrow d(u', u'') \le g(d(u', u''), 0, 0) \Rightarrow u' = u'' \end{aligned}$$

and the theorem is proved.

To compare this theorem with Delbosco we see that we have omitted the completness of the space and, instead, we have assumed conditions (ii) and (iii). These two conditions together do not guarantee the completness of the space.

Example. Let $X = [0,1] \cap Q$, $g(x,y,z) = \alpha(y+z)$ with $\frac{1}{3} < \alpha < \frac{1}{2}$ and $T, S : X \to X$, $Tx = \frac{x}{4}$, $Sx = \frac{x}{5}$ and d the euclidean metric.

i) We have that

$$d(Sx, Ty) = \left|\frac{x}{5} - \frac{y}{4}\right| = \frac{|4x - 5y|}{20}$$

$$g(d(x,y), d(x,Sx), d(y,Ty)) = \alpha(d(x,Sx) + d(y,Ty)) = \alpha\frac{16x + 15y}{20}.$$

Because $\alpha > \frac{1}{3}$ we have that

$$\alpha\frac{16x + 15y}{20} \ge \alpha\frac{5x + 4y}{20} \ge \alpha\frac{|5y - 4x|}{20}, \quad x, y \in [0,1) \cap Q,$$

ii) S is continuous at 0 and T is continuous at $S0 = 0$.

iii) $TSx = \frac{x}{20}$ and if we take $x = 0$, then the existence of a convergent subsequence is evident.

3.

As an application of foregoing, we prove a theorem on nonlinear functional equations.

Let us consider the nonlinear functional equation

$$Ax = Px$$

in a complete metric space X, where A is a nonlinear operator of X onto itself and P is also a nonlinear operator mapping. The condition of solvability of this kind of functional equation was also investigated by R. Sen [8].

Theorem 4. *Assume that there exists $\alpha > 0$ such that $d(Ax, Ay) \geq \alpha d(x, y)$ for all $x, y \in X$ and let P be a nonlinear operator on X. If for some $m \in \mathbf{N}$, P^{m-1} commutes with A and*

$$\begin{aligned} d(P^m x, P^m y) \quad \leq \quad & \beta g(d(P^{m-1}x, y), d(P^{m-1}x, A^{-1}P^m x) \\ & d(y, A^{-1}P^m x), d(P^{m-1}x, A^{-1}P^m x) d(A^{-1}P^m x, y)) \end{aligned}$$

for all $x, y \in X$ where $\beta \in (0, \alpha]$ and $g \in \mathcal{L}$, then the equation $Ax = Px$ has one and only one solution.

Proof. Note that if $Ax = Ay$ then $x = y$ so that A is bijective and therefore A^{-1} exists. Moreover, A^{-1} commutes with P^{m-1}.

We have

$$\begin{aligned} & d(A^{-1}P^m x, A^{-1}P^m y) \leq \frac{1}{\alpha} d(P^m x, P^m y) \\ \leq \quad & \frac{\beta}{\alpha} g(d(P^{m-1}x, y), d(P^{m-1}x, A^{-1}P^m x), d(y, A^{-1}P^m y), \\ & d(P^{m-1}x, A^{-1}P^m y), d(A^{-1}P^m x, y)) \end{aligned}$$

and since P^{m-1} commutes with $A^{-1}P^m$ we deduce from Theorem 1 (take $S = T = A^{-1}P^m$, $I = P^{m-1}$ and J the identity) that $A^{-1}P^m$ and P^{m-1} have a unique common fixed point x_0.

We have that

$$A^{-1}P^m x_0 = P^{m-1}x_0 = x_0$$

and we deduce $A^{-1}Px_0 = x_0$. It follows that $Px_0 = Ax_0$ and the unicity is immediate.

Remark 5. Taking $m = 1$ and $g : \mathbf{R}^5 \to \mathbf{R}_+$, $g(x_1, x_2, x_3, x_4, x_5) = hx_1$ with $0 < h < 1$ we obtain a result of R. Sen [8]. Our generalization is different from that of S. K. Chatterjea [1] because we do not assume the continuity of $A^{-1}P$, and the condition on P^m is also different.

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