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A LOCALLY CONVEX VERSION OF ADJOINT THEOREM

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Abstract

In this paper the adjoint theorem (the boundedness of the adjoint operator for a linear operator) for locally convex topological vector spaces is generalized. The obtained results on the closed graph theorem are applied.

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In [7], E. Pap established a very interesting theorem concerning the boundedness of the adjoint operator for a linear (not necessarily continuous) operator on inner product spaces. The result was extended to normed spaces in [2], [8], and [9] and was employed in [9] to give a proof of the closed graph theorem for normed spaces which does not rely on the Bair Category Theorem. In this note we establish a general form of Pap's adjoint theorem for locally convex spaces. We then use this result to establish a closed graph theorem for closed linear operators between locally convex spaces.

Throughout this note let X and Y be Hausdorff locally convex topological vector spaces and $T : X \to Y$ a linear mapping. The domain of the adjoint operator, T', is defined to be $D(T') = \{y' \in Y' : y'T \in X'\}$ and $T' : D(T') \to X'$ is defined by $T'y' = y'T$. Pap's adjoint theorem asserts that the adjoint operator is always a bounded linear operator when X and Y are normed spaces and X is a K-space ([2] 3.11, [8], [9]). We show below in Theorem 3 that T' is always a bounded linear operator with respect to the relative weak * topology on $D(T')$ and a particular locally convex topology on X'.

If (E, τ) is a topological vector space, a sequence $\{x_k\}$ in E is said to be a $\tau - K$-sequence if every subsequence of $\{x_k\}$ has a further subsequence $\{x_{n_k}\}$ such that the subseries $\sum x_{n_k}$ is τ-convergent to an element of E ([2] Ch. 3). A subset A of E is said to be $\tau - K$-bounded if for every sequence $\{x_k\} \subset A$ and every scalar sequence $\{t_k\}$ which converges to 0, the sequence $\{t_n x_n\}$ is a $\tau - K$-sequence ([1], [2] Ch. 3). A K-sequence obviously converges to 0 and a K-bounded set is bounded but the reverse implications do not hold in general although they do hold in a complete metric linear space ([2] Ch. 3). The notions of K-convergence and K-boundedness have proven to be very useful as substitute for completeness in many of the classical results of functional analysis (see [2], [11]).

We begin by establishing a useful property of sets which are K-bounded in the weak topology. The weak topology of X (weak * topology of X') will be denoted by $\sigma(X, X')$ $(\sigma(X', X))$.

Proposition 1. *If $A \subset X'$ is $\sigma(X', X)$ bounded and $B \subset X$ is $\sigma(X, X') - K$ -bounded, then*

$$\sup\{|< x', x >| : x' \in A, x \in B\} < \infty,$$

i.e., A is uniformly bounded on B.

Proof. It suffices to show that $t_k < x'_k, x_k > \to 0$ for every $\{x_k\} \subset B$, $\{x'_k\} \subset A$ and a positive sequence of scalars $t_k \to 0$. To show this we use the Basic Matrix Theorem of [2], 2.1 or [6], 2.1. For this consider the matrix $M = [< \sqrt{t_i} x'_i, \sqrt{t_j} x_j >]$. The columns of M converge to 0 since A is weak * bounded. If $\{m_j\}$ is any subsequence, then there is a further subsequence $\{n_j\}$ such that the subseries $\sum_{j=1}^{\infty} \sqrt{t_{n_j}} x_{n_j}$ is $\sigma(X, X')$ convergent to some

$x \in X$. Then

$$\sum_{j=1}^{\infty} < \sqrt{t_i}x_i', \sqrt{t_{n_j}}x_i > = < \sqrt{t_i}x_i, x > \rightarrow 0.$$

Hence, M is a K-matrix and its diagonal converges to 0 as desired ([2], 2.1).

We have the following elementary property of the transpose operator.

Proposition 2. *T' carries $\sigma(Y', Y)$ bounded subsets of $D(T')$ to $\sigma(X', X)$ bounded subsets of X'.*

Proof. If $A \subset D(T')$ is weak * bounded and $x \in X$, then $< T'A, x >$ $= < A, Tx >$ is bounded.

If we let $K(X', X)$ be the locally convex topology on X' of uniform convergence on the $\sigma(X, X') - K$-bounded subsets of X ([10]), then we obtain the following result from Proposition 1 and 2.

Theorem 1. *T' carries $\sigma(Y', Y)$ bounded subsets of $D(T')$ to $K(X', X)$ bounded sets.*

Since the topology $K(X', X)$ is stronger than the Mackey topology $\tau(X', X)$ ([10]), it follows, in particular, from Theorem 1 that the transpose map T' carries weak * bounded subsets of $D(T')$ to $\tau(X', X)$ bounded sets.

A topological vector space (E, τ) is said to be K-space if every sequence which converges to 0 is a $\tau - K$-sequence ([2] Ch.3). For K-spaces we obtain from Theorem 1 the following extension of Pap's Adjoint Theorem.

Theorem 2. *If X is a K-space for some topology τ which is compatible with the duality between X and X', i.e., $\sigma(X, X') \subset \tau \subset \tau(X, X')$, then T' carries $\sigma(Y', Y)$ bounded subset of $D(T')$ to strongly bounded subsets of X'.*

Proof. If a subset of X is $\sigma(X, X')$ bounded, it is τ-bounded and, hence, $\tau - K$ bounded since (X, τ) is a K-space, and therefore, $\sigma(X, X') - K$ bounded. By Theorem 1 map T' carries $\sigma(Y', Y)$ bounded subsets of $D(T')$ onto sets which are uniformly bounded on $\sigma(X, X') - K$ bounded sets and, therefore, uniformly bounded on $\sigma(X, X')$ bounded sets.

If X is a normed space, then the strong topology of X' is just the dual norm topology so if X is a normed K-space, T' carries weak * bounded subsets of $D(T')$ to norm bounded subsets of X', and if Y is a normed space too, then T' carries norm bounded subsets of $D(T')$ to norm bounded subsets of X'. This is just the norm version of Pap's adjoint theorem ([2], 3.11, [8] Theorem 2, [9]).

As an application of Theorem 1 we obtain form of the closed graph theorem for locally convex spaces. To accomplish this give sufficient conditions which insure that $D(T') = Y'$, and then we use Theorem 1 to obtain the continuity of the operator T. First we derive continuity properties for T assuming that $D(T') = Y'$.

Let $\beta(Y, Y')$ be the strong topology on Y, i.e., topology on Y of uniform convergence on $\sigma(Y', Y)$ bounded sets.

Let $\beta^*(Y, Y')$ be the topology on Y of uniform convergence on $\beta(Y', Y)$ bounded subsets of Y' ([3] p.220). We have $\beta^*(Y, Y') \subset \beta(Y, Y')$ and the equality holds iff Y is infrabarrelled ([3] p.220).

Theorem 3. *Let (X, τ) be a K-space for some compatible topology τ, i.e., $\sigma(X, X') \subset \tau \subset \beta(X, X')$. Assume that $D(T') = Y'$. Then T is continuous with respect to $\beta^*(X, X')$ and $\beta(Y, Y')$.*

Proof. Let $B \subset Y'$ be $\sigma(Y', Y)$ bounded. By Theorem 2, $T'B$ is $\beta(X', X)$ bounded in X'. Therefore, $(T'B)^\circ$, the polar of $T'B$ in X, is a basic $\beta^*(X, X')$ neighborhood of 0 in X. Since $(T'B)^\circ = T^{-1}(B^\circ)$ this shows that T is $\beta^*(X, X') - \beta(Y, Y')$ continuous.

If X in Theorem 3 is also infrabarrelled, then T is continuous with respect to the original topology of X and the strong topology of Y.

Next, we consider conditions which will insure that $D(T') = Y'$. First recall that if T is closed, then $D(T')$ is $\sigma(Y', Y)$ dense in Y' ([5] 34.5.3). Thus, to show that $D(T') = Y'$ for a closed operator T is suffices to give conditions which insure that $D(T')$ is $\sigma(Y', Y)$ closed.

A locally convex space Y is an infra-Pták space if every $\sigma(Y', Y)$ dense subspace $D \subset Y'$ which is such that $D \cap U^\circ$, where U° is polar of U in Y', is $\sigma(Y', Y)$ closed for every neighborhood U of 0 in Y ([4] 34.2).

Theorem 4. *Let X be infrabarrelled and Y be an infra-Pták space. If T is closed and T' carries equicontinuous subsets of $D(T')$ to strongly bounded subsets of X', then $D(T') = Y'$.*

Proof. Let U be a neighborhood of 0 in Y. By the assumption that Y is an infra-Pták space, it suffices to show that $D(T') \cap U^\circ$ is $\sigma(Y', Y)$ closed. Let $\{y'_\delta\}$ be a net in $D(T') \cap U^\circ$ which is $\sigma(Y', Y)$ convergent to some $y' \in U^\circ$. We must show that $y'T$ is continuous. For each $x \in X$, we have $< T'y'_\delta, x > = < y'_\delta, Tx > \to < y', Tx >$, and since $\{T'y'_\delta\} \subset T'(D(T') \cap U^\circ)$, the net $\{y'_\delta T\}$ is contained in a strongly bounded subset of Y' by hypothesis. Therefore, by the version of the Banach-Steinhaus Theorem for infrabarrelled spaces ([4] 39.5 Remark 2), $y'T$ is continuous.

Note that the condition on T' assumed in Theorem 4 is satisfied if the hypothesis of Theorem 2 is satisfied.

We can now combine the results above to obtain a version of the Closed Graph Theorem (CGT).

Theorem 5. *Let X be infrabarrelled and such that (X, τ) is a K-space for some compatible topology τ. Let Y be an infra-Pták space. If T is closed, then T is continuous with respect to the original topology of X $(= \beta^*(X, X'))$ and $\beta(Y, Y')$.*

Proof. The result follows immediately from Theorems 2, 3, 4.

Remark. Note that it follows from Theorem 5 that T is continuous with respect to the original topologies of X and Y; this is the conclusion in the 'usual' form of CGT ([5] 34.6 (9)). The standard locally convex version of the CGT assumes that the domain space is barrelled so the operator T is continuous with respect to the topology $\beta(X, X')$ on the domain space. The topology $\beta^*(X, X')$ is weaker than $\beta(X, X')$ so Theorem 5 yields a stronger continuity result that the usual CGT; of course, this is obtained at the expense of more complicated assumptions on the domain space. Also, the standard locally convex version of the CGT only assures that the operator T is continuous with respect to the original topology of Y; it requires an additional argument to obtain the continuity with respect to $\beta(Y, Y')$ where as the proof of Theorem 3 gives the continuity with respect to $\beta(Y, Y')$ directly.

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