Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 24, 2 (1994), 31-38

Review of Research
Faculty of Science
Mathematics Series

GENERALIZED CONTRACTIONS IN σ-COMPLETE VECTOR LATTICES

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Abstract

A fixed point theorem for nonlinear generalized contractions in σ-complete vector lattices is given.

AMS Mathematics Subject Classification (1991): 47H10
Key words and phrases: σ-complete vector lattices, comparison operator, φ-contraction.

1. Introduction

Any metrical fixed point theorem is stated in terms directly related to the metric structure of the ambient space, i.e. metric space, K-metric space, locally convex or uniform space, etc. [11].

Many vector lattices which have importance in analysis do not possess such a structure. However, in order to obtain a metrical fixed point result, we can take $d(x,y) = \mid x - y \mid$, where $\mid x \mid = \sup\{x, -x\}$, instead of the usual distance of two elements x, y.

Several papers have been devoted to this subject: [8], [9], [13].

In this paper we shall generalize a result from the last above quoted papers, using the notion of φ-contraction (see [2], [3], [12]).

Referring to vector lattices and generalized contractions we shall follow, both in the terminology and notation, the monographs by Cristescu, R. [6], [7], and Rus, A. I. [12].

2. σ-complete vector lattices

Let $(X, \leq)$ be an ordered set and $A \subset X$ a majorized (minorized) nonempty subset.

We denote by $\sup A$ ($\inf A$) the supremum (infimum) of A.

If $A = \{x_j \mid j \in J\}$, then we denote $\sup A$ ($\inf A$) by $\bigvee_{j\in J} x_j$ ($\bigwedge_{j\in J} x_j$).

A sequence $\{x_n\}$ in X is said to be *increasing (decreasing)* and we denote $x_x \uparrow$ ($x_n \downarrow$) if $x_n \leq x_{n+1}$, for each $n \in N$ ($x_n \geq x_{n+1}$, respectively).

If $x_n \uparrow$ ($x_n \downarrow$) and $x = \bigvee_{n\in N} x_n$ ($x = \bigwedge_{n\in N} x_n$) we denote $x_n \uparrow x$ ($x_n \downarrow x$).

Definition 1. *A sequence $\{x_n\}$ of elements from X* (0)-converges to an element $x \in X$ *if there exist two sequences $\{a_n\}, \{b_n\}$ in X such that*

$$a_n \leq x_n \leq b_n, \quad \text{for each } n \in N,$$

and, in addition, $a_n \uparrow x$, $b_n \downarrow x$. We denote

$$x = (0) - \lim_n x_n \quad \text{or} \quad x_n \to^\circ x$$

Remark 1. If a sequence of X is (0)-convergent, then its (0)-limit is unique.

Definition 2. *An ordered set X is called* lattice *if there exist $x \vee y$ and $x \wedge y$ for each $x, y \in X$.*

A lattice X is called *σ-complete* if there exist $\sup A$ and $\inf A$ for each numerable subset A of X.

Let X be a linear space and $K \subset X$ a *cone* in X, i.e. a closed subset of X satisfying

$$K \cap (-K) = \{\emptyset\},\ K + K \subset K \ \text{ and } \ t \cdot K \subset K \ \text{ for all } \ t > 0,$$

where $\emptyset$ denotes the zero element of E.

The condition

$$x \leq y \ \text{ iff } \ y - x \in K$$

defines a partial linear order relation on X.

The linear space X endowed with this order relation is called *linear ordered space*, while K is termed its *positive cone.*

A *vector (linear) lattice* is a linear ordered space which is a lattice with respect to the considered order.

A vector lattice X is called *σ-complete vector lattice* if, for any bounded numerable subset A of X, there exist $\sup A$ and $\inf A$.

Let X be a vector lattice and $x \in X$. Then we denote

$$\mid x \mid = \sup\{x, -x\},$$

the *modulus* of x.

The following properties are immediate consequences of the above definitions (see [7]).

If X is a vector lattice, then

$$(1) \qquad \mid \alpha x \mid = \mid \alpha \mid \cdot \mid x \mid, \ \alpha \in R;$$

$$(2) \qquad \mid x + y \mid \leq \mid x \mid + \mid y \mid;$$

$$(3) \qquad \|x \mid - \mid y\| \leq \mid x - y \mid,$$

for each $x, y \in X$.

In any linear ordered space we have

(4) If $x = (0) - \lim_n x_n$ and $x_n \geq 0$, $n \in N$, then $x \geq 0$;

(5) If $0 \leq x_n \leq y_n$, for each $n \in N$ and $(0) - \lim_n y_n = 0$, then

$$(0) - \lim_n x_n = 0$$

Definition 3. *Let X, Y be two linear ordered spaces. A mapping $U : X \to Y$ is called* (0)-continuous *in $a \in X$ if, for any sequence $\{x_n\}$ in X, such that $x_n \to^\circ a$, we have $U(x_n) \to^\circ U(a)$.*

Definition 4. *Let X be a linear ordered space and $\{x_n\}$ a sequence in X. We define*

$$(0) - \sum_{n=1}^{\infty} x_n = (0) - \lim_n \sum_{j=1}^{n} x_j,$$

if the right-hand side limit exists, and we say that the series $\sum_{n=1}^{\infty} x_n$ is (0)-convergent.

If $\sum_{n=1}^{\infty} \mid x_n \mid$ *is* (0)*-convergent we say that the series* $\sum_{n=1}^{\infty} x_n$ *is* absolute (0)-convergent.

Lemma 1. *(Cristescu, R. [7]). In a* σ*-complete vector lattice any absolute* (0)*-convergent series is* (0)*-convergent.*

Definition 5. *Let* X *be a vector lattice and let* K *be its positive cone. A mapping* $\varphi : K \to K$ *which satisfies:*

$$\varphi \text{ is monotone increasing (isotone);} \tag{6}$$

$$(0) - \lim_n \varphi^n(t) = \emptyset, \quad \text{for each } t \in K, \tag{7}$$

is called comparison operator *(*φ^n *stands for the nth iterate of* φ*).*

Remark 2. It is easy to see that a comparison operator possesses all the properties of comparison functions ([2], [3]). We need the following generalized ratio test in σ-complete vector lattices, proved in [4], [5] for series of real positive terms.

Theorem 1. *Let* X *be a* σ*-complete vector lattice and let* K *be its positive cone.*

If $\sum_{n=1}^{\infty} u_n$ *is a series of positive terms in* X *(i.e.* $u_n \in K\setminus\{\emptyset\}$*) satisfying the following condition:*

there exist an (0)*-convergent series* $\sum_{n=1}^{\infty} v_n$, $v_n \in K$ *and a real number* $a, 0 \le a < 1$, *such that*

$u_{n+1} \le a u_n + v_n$, *for each* $n \in N$ *(fixed), then the series* $\sum_{n=1}^{\infty} u_n$ *is* (0)*-convergent.*

Proof. It follows by analogous arguments to these in [4] or [5]. □

Definition 6. *Let* X *be a* σ*-complete vector lattice and let* K *be its positive cone. An isotone mapping* $\varphi : K \to K$ *which satisfies the following convergence condition*
(c) there exist an (0)*-convergent series* $\sum_{n=1}^{\infty} v_n$ *in* K *and a real number* a, $0 \le a < 1$, *such that*
$\varphi^{k+1}(t) \le a\varphi^k(t) + v_k$, *for each* $t \in K$ *and* $n \in N$ *(fixed), is called (c)-*comparison operator.

Example. If $X = R$, the real axis, when $K = R^+$, a tipical comparison operator is $\varphi : R^+ \to R^+$,

$$\varphi(t) = at, \ \ 0 \le a < 1.$$

Lemma 2. *Any (c)-comparison operator is also a comparison operator.*

Proof. We apply Theorem 1. □

Lemma 3. *Let X be a σ-complete vector lattice, K its positive cone and $\varphi : K \to K$ a (c)-comparison operator. Let $s : K \to K$, given by*

$$s(t) = \sum_{n=0}^{\infty} \varphi^k(t), \ \ t \in K.$$

Then φ is continuous in $\emptyset$.

Proof. See [2], [3] for the scalar comparison operators (comparison functions). □

3. Generalized contractions

Let X be a vector lattice and K its positive cone.

Definition 7. *A mapping $f : X \to X$ is called φ -* contraction *if there exists a comparison operator $\varphi : K \to K$ such that*

(8) $$| f(x) - f(y) | \le \varphi(| x - y |), \ \ \text{for each} \ \ x, y \in X.$$

Remark 3. Any φ-contraction is (0) - continuous, as, for each comparison operator we have

$$\varphi(t) \le t, \ \ t \in K.$$

The main result of this paper is given by

Theorem 2. *Let X be a σ-complete vector lattice and $f : K \to K$ a φ-contraction, with φ (c)-comparison operator. Then*

(9) $$F_f = \{x^*\}; \ \ \text{where} \ \ F_f = \{x \in X \mid f(x) = x\};$$

(10) $$f^n(x_0) \to^\circ x^* \quad \textit{for each} \quad x_0 \in X.$$

(11) $$| f^n(x_0) - x^* | \le s\left(| f^{n+1}(x_0) - f^n(x_0) |\right), \; n \in N,$$

where $s(t)$ denotes the sum of the series

$$\sum_{k=0}^{\infty} \varphi^k(t)$$

Proof. Let $\{x_n\}$, $x_n = f(x_{n-1})$, $n \in N$, $x \in X$, be the sequence of successive approximations.

From (8) and (2) we obtain

$$| x_{n+p} - x_n | \le | x_{n+p} - x_{n+p-1} | + \cdots + | x_{n+1} - x_n | \le$$
$$\le \sum_{k=0}^{p-1} \varphi^k(| x_{n+1} - x_n |) \le \sum_{k=n}^{n+p-1} \varphi^k(| x_0 - x_1 |), \; n, p \in N.$$

Since φ is a (c)-comparison operator, it results that $\{x_n\}$ is (0)-Cauchy sequence. But X is σ-complete, hence $\{x_n\}$ is (0)-convergent.

Let $x^* = (0)-\lim_n x_n$. From the continuity property of each φ-contraction we deduce

$$x^* = f(x^*),$$

that is $x^* \in F_f$.

The unicity of fixed point follows in a standard way. Assume $x^*, y^* \in F_f$, $x^* \ne y^*$. Then

$0 < | x^* - y^* | = | f^n(x^*) - f^n(y^*) | \le \varphi^n(| x^* - y^* |)$, and letting $n \to \infty$, we obtain

$$0 < | x^* - y^* | \le 0,$$

contradiction. Hence $F_f = \{x^*\}$.

To obtain (11), we take $p \to \infty$ in the inequality

$$| x_n - x_{n+p} | \le \sum_{k=0}^{p-1} \varphi^k(| x_n - x_{n+1} |)$$

The proof is now complete. □

Corollary 1. *Let X be a σ-complete vector lattice and $f : X \to X$ a mapping such that, for certain $n \in N^*$, f^n is a φ-contraction, with $\varphi(c)$-comparison operator.*

Then f has a unique fixed point.

Proof. We apply Theorem 2. □

Remark 4.

a) For $\varphi(t) = \alpha t$, $\alpha \in [0, 1)$, $t \in K$, from Theorem 2 we obtain Theorem 2.1 from [13].

b) For other results based on the comparison method and various applications, see [8], [9].

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Received by the editors October 13, 1994.