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FIXED POINT THEOREMS IN FUZZY METRIC AND PROBABILISTIC METRIC SPACES

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Abstract

Caristi's type fixed point theorem and it's equivalent: weak statement of Ekeland's variational principle are given in fuzzy metric and Menger's probabilistic metric spaces.

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1. Introduction

The following result is well known in nonlinear functional analysis.

Theorem 1. *(Caristi's fixed point theorem) Let (E, d) be a complete metric space and $\Phi : E \to (-\infty, +\infty]$ be a proper lower semicontinuous function, bounded from bellow. If $f : E \to E$ satisfies the inequality*

$$d(x, f(x)) \leq \Phi(x) - \Phi(f(x)), \text{ for every } x \in E,$$

then there exists $x_0 \in E$ such that $x_0 \in f(x_0)$.

In this paper we give a fixed point theorem in fuzzy metric spaces, which is of Caristi's type. The weak statement of Ekeland's variational principle is equivalent to Caristi's theorem, and we shall give it's statement in fuzzy metric spaces. These results are applied in Menger's spaces. The obtained results are generalizations of the results from [4] where the same is done, but only for $R = \max$.

2. Preliminaries

Throughout the paper, let $\mathbf{R}$ denote the set of real numbers, $\mathbf{R}^+$ the set of non-negative real numbers, $\mathbf{N}$ the set of natural numbers.

A **fuzzy set on the real axis** is a mapping $x : \mathbf{R} \to [0,1]$. A real fuzzy set is said to be **normal** if there exists t_0 in $\mathbf{R}$ such that $x(t_0) = 1$. A real fuzzy set is **convex** if

$$\forall t_1, t_2 \in \mathbf{R}\ ,\ \forall \mu \in [0,1]\,,\ x(\mu t_1 + (1-\mu)t_2) \geq \min(x(t_1), x(t_2))\,.$$

We denote the set of all upper semicontinuous, normal, convex real fuzzy sets by $\mathcal{E}$ and we call it's elements **fuzzy numbers**. We denote by $\mathcal{E}^+$ the set of **non-negative** fuzzy numbers, that are fuzzy numbers x from $\mathcal{E}$ which satisfy $\forall t < 0\ ,\ x(t) = 0$. **$\alpha$-level cuts** of fuzzy sets are defined by $[x]_\alpha := \{t \in \mathbf{R} : x(t) \geq \alpha\}$. It can be showed that α-level cuts of a fuzzy number x are nonempty closed intervals $[a^\alpha, b^\alpha]$, where values $-\infty$ and $+\infty$ are admissible.

We denote by $I_{\{a\}}(t) := \begin{cases} 0, & t \neq a \\ 1, & t = a \end{cases}$ the **indicator function** of a.

We say that a sequence of fuzzy numbers $\{x_n\}_{n\in\mathbf{N}}$ is **α-level converging** to x if

$$\forall \alpha \in (0,1]\,,\ \lim_{n\to\infty} a_n^\alpha = a^\alpha \text{ and } \lim_{n\to\infty} b_n^\alpha = b^\alpha\ ,$$

where $[x_n]_\alpha = [a_n^\alpha, b_n^\alpha]$ and $[x]_\alpha = [a^\alpha, b^\alpha]$, and we write

$$\lim_{n\to\infty} x_n = x\ .$$

We define a **partial ordering** $\leq$ in $\mathcal{E}$

$$x \leq y \Leftrightarrow a_1^\alpha \leq a_2^\alpha \text{ and } b_1^\alpha \leq b_2^\alpha\ ,$$

where $[x]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[y]_\alpha = [a_2^\alpha, b_2^\alpha]$.

Throughout the paper $\mathcal{E}$ will be endowed with the partial ordering $\leq$ and α-level convergence.

Let L and R be mappings from $[0,1] \times [0,1]$ to $[0,1]$, symmetric, non-decreasing in both arguments, which satisfy $L(0,0) = 0$ and $R(1,1) = 1$. Let X be a non-empty set and $d : X \times X \to \mathcal{E}^+$. We denote by $\lambda_\alpha(x,y)$ and $\rho_\alpha(x,y)$ left and right end points of α-level intervals of $d(x,y)$

$$[\lambda_\alpha(x,y), \rho_\alpha(x,y)] := [d(x,y)]_\alpha \, , \ \alpha \in (0,1] \, .$$

The quadruple (X,d,L,R) is called a **fuzzy metric space** and d a **fuzzy metric** if

i) $\forall x, y \in X \, , \ d(x,y) = I_{\{0\}} \Leftrightarrow x = y$

ii) $\forall x, y \in X \, , \ d(x,y) = d(y,x)$

iii) $\forall x, y, z \in X$

a) $d(x,z)(s+t) \geq L(d(x,y)(s), d(y,z)(t))$ whenever $s \leq \lambda_1(x,y)$, $t \leq \lambda_1(y,z)$ and $s + t \leq \lambda_1(x,z)$

b) $d(x,z)(s+t) \leq R(d(x,y)(s), d(y,z)(t))$ whenever $s \geq \lambda_1(x,y)$, $t \geq \lambda_1(y,z)$ and $s + t \geq \lambda_1(x,z)$.

In this paper we shall consider fuzzy metric spaces which satisfy

$$\lim_{a \to 0_+} R(a,a) = 0 \, . \tag{1}$$

The following theorems are proved in [2].

Theorem 2. *Let (X,d,L,R) be a fuzzy metric space and R satisfy (1). Then the family*

$$\mathcal{B} := \{U(\varepsilon,\alpha) : \varepsilon > 0, \alpha \in (0,1]\}$$

of sets

$$U(\varepsilon,\alpha) := \{(x,y) \in X \times X : \rho_\alpha(x,y) < \varepsilon\}$$

forms a basis for a Hausdorf uniformity $\mathcal{U}$ on X. Moreover, the sets

$$\mathcal{N}_x(\varepsilon,\alpha) := \{y \in X : \rho_\alpha(x,y) < \varepsilon\}$$

form a basis for a family of neighborhoods of a point x, and the topology $\mathcal{T}_d$ induced by the uniformity $\mathcal{U}$ is metrizable.

Theorem 3. *In fuzzy metric space* $(X, d, L, \max)$ *relation* iii) b) *is equivalent to the triangle inequality for* $\rho_\alpha(x, y)$

$$\forall x, y, z \in X \ \ \forall \alpha \in (0,1] \ \rho_\alpha(x, z) \le \rho_\alpha(x, y) + \rho_\alpha(y, z) .$$

Hence, if $R = \max$ it is easy to generalize fixed point theorems from metric to fuzzy metric spaces.

It can be showed that a sequence $\{x_n\}_{n \in \mathbf{N}}$ of elements from X is converging to a point $x \in X$ if and only if

$$\lim_{n \to \infty} d(x_n, x) = I_{\{0\}} ,$$

which is equivalent to

$$\forall \alpha \in (0,1] \ \lim_{n \to \infty} \rho_\alpha(x_n, x) = 0 .$$

A sequence $\{x_n\}_{n \in \mathbf{N}}$ of elements from a fuzzy metric space (X, d, L, R) is a **Cauchy sequence** if

$$\forall \alpha \in (0,1] \ \lim_{m,n \to \infty} \rho_\alpha(x_m, x_n) = 0 .$$

A fuzzy metric space (X, d, L, R) is said to be **complete** if every Cauchy sequence from X converges in X.

Two argument function $t : [0,1] \times [0,1] \to [0,1]$ is called **t-norm** if it is symmetric, non-decreasing in both arguments, associative and $\forall a \in [0,1]$, $t(a,1) = t(1,a) = a$.

We denote by Δ the set of **distribution functions** $F : \mathbf{R} \to [0,1]$ (F is non-decreasing, lower semicontinuous, $\inf_{s \in \mathbf{R}} F(s) = 0$ and $\sup_{s \in \mathbf{R}} F(s) = 1$). We denote by Δ^+ the set of elements from Δ which satisfy the condition $F(0) = 0$.

The triple $(X, \mathcal{F}, t)$ is a Menger space if X is a non-empty set, $\mathcal{F} : (x, y) \mapsto \{F_{x,y}(s)\} \in \Delta^+$, t a t-norm and the following conditions are satisfied.

1. $\forall x, y \in X$, $F_{x,y}(s) = H(s)$, $\forall s > 0 \ \Leftrightarrow x = y$,
2. $\forall x, y \in X$, $F_{x,y} = F_{y,x}$,

3. $\forall x, y, z \in X$, $\forall s, u > 0$, $F_{x,z}(s+u) \geq t(F_{x,y}(s), F_{y,z}(u))$,

where $H(s) := \begin{cases} 0, & s \leq 0 \\ 1, & s > 0 \end{cases}$.

Example 1. Let $(\Omega, \mathcal{A}, P)$ be a probability space, (M, d') a separable metric space. We shall denote by X the set of all the equivalence classes of random variables $x : \Omega \to M$. We define for $x, y \in X$, $s \in \mathbf{R}$:

$$F_{x,y}(s) := P(\{\omega \in \Omega : d(x(\omega), y(\omega)) < s\}) .$$

It is well known that $(X, \mathcal{F}, t_m)$ is a Menger space, where the t-norm t_m is defined by $t_m(a, b) := \max(a + b - 1, 0)$ for $a, b \in [0, 1]$.

The topology in Menger spaces is obtained by the two-parameter family $\mathcal{O}_x := \{O_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1)\}$, which is a base for the family of neighborhoods of a point x, where $O_x(\varepsilon, \lambda) := \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$. We call this topology the (ε, λ) topology. The convergence in the (ε, λ) topology is equivalent to the convergence in the probability.

We say that $\{x_n\}_{n \in \mathbf{N}}$, a sequence of elements from a Menger space $(X, \mathcal{F}, t)$, is a **Cauchy sequence** if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists an $n_0 = n_0(\varepsilon, \lambda)$ so that

$$\forall n \geq n_0 \; \forall p \in \mathbf{N} \, , \; F_{x_n, x_{n+p}}(\varepsilon) > 1 - \lambda \, .$$

A Menger space $(X, \mathcal{F}, t)$ is said to be **complete** if every Cauchy sequence in X converges in X.

A Menger space $(X, \mathcal{F}, t)$ is also a fuzzy metric space. Indeed, if we define

$$L(a, b) := 0 \text{ and } R(a, b) := 1 - t(1 - a, 1 - b) \, , \text{ for } a, b \in [0, 1] \, ,$$

$$d(x, y)(s) := \begin{cases} 0, & s < s_{x,y} := \sup\{s : F_{x,y}(s) = 0\} \\ 1 - F_{x,y}(s), & s \geq s_{x,y} \, , \end{cases}$$

we can see:

- $L(0, 0) = 0$, $R(1, 1) = 1 - t(0, 0) = 1$, $L \equiv 0$ is non-decreasing in both arguments and since t is non-decreasing in both arguments it follows that R is non-decreasing in both arguments.

- For all $x, y \in X$, $\{F_{x,y}(s)\}$ is non-decreasing and lower semicontinuous $\Rightarrow$ $\{d(x,y)(s)\}$ is non-increasing and upper semicontinuous on $[s_{x,y}, +\infty)$ $\Rightarrow$ $\{d(x,y)(s)\}$ is convex and $d(x,y)(s_{x,y}) = 1$, $s_{x,y} \geq 0$. So, $d(x,y)$ is a non-negative fuzzy number.

- 1. and 2. from the definition of Menger spaces are equivalent to i) and ii) in the definition of fuzzy metric spaces. iii) a) is trivially satisfied. 3. $\Rightarrow$ iii) b). Indeed, for all $s \geq \lambda_1(x,y) = s_{x,y}$ and all $u \geq \lambda_1(y,z) = s_{y,z}$ if $s + u \geq \lambda_1(x,z) = s_{x,z}$ then

$$1 - F_{x,z}(s+u) \leq 1 - t(F_{x,y}(s), F_{y,z}(u)) = R(1 - F_{x,y}(s), 1 - F_{y,z}(u)) .$$

So, it follows that (X, d, L, R) is a fuzzy metric space.

It is easily checked that the (ε, λ) topology and $\mathcal{T}_d$ topology are equivalent. Even more, the definition of a Cauchy sequence in a Menger space is equivalent to the definition of a Cauchy sequence in the fuzzy metric space.

3. Caristi's fixed point theorem and variational principle in fuzzy metric spaces

We begin this section with a metatheorem, which is similar to the one in [1].

Theorem 4. *If X is a non-empty set and $p(x,y)$ a sentence formula, then the next four statements are equivalent.*

1. *If a function $f : X \to X$ for all $x \in X$ satisfies $p(x, f(x))$ then there exists a point x^* in X so that $f(x^*) = x^*$.*

2. *If a set-valued mapping $F : X \to 2^X \backslash \{\emptyset\}$ satisfies that for all $x \in X$ there exists $y \in F(x)$ such that $p(x,y)$ is true then there exists an $x^* \in X$ so that $x^* \in F(x^*)$.*

3. *If a set-valued mapping $F : X \to 2^X \backslash \{\emptyset\}$ satisfies that for all $x \in X$ there exists $y \in F(x)$ such that $p(x,y)$ is true then there exists an $x^* \in X$ so that $\{x^*\} = F(x^*)$.*

4. *There exists an element x^* in X such that for all elements y from $X \backslash \{x^*\}$ it is not true $p(x^*, y)$.*

Proof. We shall make a full circle of implications

1. $\Rightarrow$ 2. We shall make a selection f of F that satisfies $\forall x \in X\ p(x, f(x))$. Following 1. it must have a fixed point $x^* = f(x^*) \in F(x^*)$.

2. $\Rightarrow$ 3. If we suppose the opposite of 3.: that

$$\forall x \in X\ F(x) \neq \{x\}\ (\text{ and } F(x) \neq \emptyset)\,,$$

then we get that $G(x) := F(x) \backslash \{x\}$ satisfies the conditions of 2., so it must have a fixed point $x^* \in G(x^*) = F(x^*) \backslash \{x^*\}$ which is a contradiction.

3. $\Rightarrow$ 1. Since a singlevalued function is a special case of set-valued mapping, 1. follows from 3. immediately.

1. $\Rightarrow$ 4. Suppose the opposite of 4. Then for all elements x in X there exists an element y from $X \backslash \{x\}$ such that $p(x, y)$. Then by choosing $f(x) = y$ we get a function which using 1. must have a fixed point. That is a contradiction.

4. $\Rightarrow$ 1. By 4. there exists an x^* satisfying $\neg p(x^*, y)$ for all y from $X \backslash \{x^*\}$. Then supposing $p(x^*, f(x^*))$ and $x^* \neq f(x^*)$ gives a contradiction. □

A subset A of a fuzzy metric space (X, d, L, R) is called **fuzzy bounded** if there exists a $u \in \mathcal{E}$, with $\lim_{t \to +\infty} u(t) = 0$, such that $d(x, y) \leq u$, for every $x, y \in A$. For a fuzzy bounded set A and $\alpha \in (0, 1]$ we define

$$\overline{d}_\alpha(A) = \sup_{x,y \in A} \rho_\alpha(x, y)\,.$$

Lemma 1. *Let (X, d, L, R) be a fuzzy metric space such that $\lim_{a \to 0_+} R(a, a) = 0$. Then for every $\alpha \in (0, 1]$ there exists an $\alpha' \in (0, 1]$ such that the following implication holds for all $u, v, w \in X$:*

$$\forall t_1, t_2 > 0\ (\rho_{\alpha'}(u, v) < t_1) \wedge (\rho_{\alpha'}(v, w) < t_2) \Rightarrow (\rho_\alpha(u, w) < t_1 + t_2)\,.$$

Proof. Let $\alpha' \in (0, 1]$ be such that $R(\alpha', \alpha') < \alpha$. Since $\lim_{a \to 0_+} R(a, a) = 0$, such an element exists.

Let $\rho_{\alpha'}(u,v) < t_1$ and $\rho_{\alpha'}(v,w) < t_2$. Then $d(u,v)(t_1) < \alpha'$ and $d(v,w)(t_2) < \alpha'$, which imply, since $t_1 > \lambda_1(u,v)$ and $t_2 > \lambda_1(v,w)$, that for every $s = s_1 + s_2 \geq \max(t_1 + t_2, \lambda_1(u,w))$, $s_1 \geq t_1$, $s_2 \geq t_2$:

$$d(u,w)(s) \leq R(d(u,v)(s_1), d(v,w)(s_2) \leq R(\alpha', \alpha') < \alpha .$$

If $t_1 + t_2 \leq \lambda_1(u,w)$ then we have that $d(u,w)(\lambda_1(u,w)) = 1 < \alpha$, which is a contradiction. Hence $t_1 + t_2 > \lambda_1(u,w)$ and $d(u,w)(t_1 + t_2) < \alpha$. This implies that $\rho_\alpha(u,w) < t_1 + t_2$. □.

Lemma 2. *Let (X, d, L, R) be a fuzzy metric space such that* $\lim_{a\to 0_+} R(a, a) = 0$ *and A be a fuzzy bounded subset of X. Then for every $\alpha \in (0,1]$ there exists $\tilde{\alpha} \in (0,1]$ (independent on A) such that*

$$\overline{d}_\alpha(\bar{A}) \leq \overline{d}_{\tilde{\alpha}}(A) . \tag{2}$$

Proof. Let $\bar{u}, \bar{v} \in \bar{A}$ and $\alpha \in (0,1]$. Let $\alpha' \in (0,1]$ be such that $R(\alpha', \alpha') < \alpha$ and $\alpha'' \in (0,1]$ such that $R(\alpha'', \alpha'') < \alpha'$. If $\varepsilon > 0$ we shall prove that $\rho_\alpha(\bar{u}, \bar{v}) < \overline{d}_{\alpha''}(A) + \varepsilon$. Since $\bar{u}, \bar{v} \in \bar{A}$ it follows that there exist $u, v \in A$ such that $\rho_{\alpha'}(u, \bar{u}) < \varepsilon/4$ and $\rho_{\alpha''}(v, \bar{v}) < \varepsilon/4$. Since $\rho_{\alpha''}(u,v) < \overline{d}_{\alpha''}(A) + \varepsilon/2$, we have from Lemma 1 that

$$\rho_{\alpha'}(u, \bar{v}) < \overline{d}_{\alpha''}(A) + \varepsilon/2 .$$

Applying Lemma 1 again we obtain that

$$\rho_\alpha(\bar{u}, \bar{v}) < \overline{d}_{\alpha''}(A) + \varepsilon .$$

Hence $\overline{d}_\alpha(\bar{A}) \leq \overline{d}_{\alpha''}(A) + \varepsilon$ and since ε is an arbitrary number (2) is proved. Now we shall prove a generalization of Theorem 4.1 from [4].

Theorem 5. *Let (X, d, L, R) be a complete fuzzy metric space such that* $\lim_{a\to 0_+} R(a,a) = 0$. *Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping and $f : X \to X$ such that for every $x \in X$*

$$\rho_\alpha(x, f(x)) \leq \Phi(x) - \Phi(f(x)) , \text{ for every } \alpha \in (0,1] . \tag{3}$$

Then f has at least one fixed point.

Proof. As in [4] we shall define a partial ordering in X in the following way

$$x \preceq y \Leftrightarrow \rho_\alpha(x,y) \leq \Phi(x) - \Phi(y)\,, \text{ for all } \alpha \in (0,1]\,.$$

The reflexivity and antisymmetry of $\preceq$ follow immediately. Hence, we shall only prove the transitivity which, in the case that $R = \max$, follows from the Theorem 3. Let $x \preceq y$ and $y \preceq z$. We shall prove that $x \preceq z$. From $x \preceq y$ and $y \preceq z$ we have that for every $\alpha \in (0,1]$:

$$\rho_\alpha(x,y) \leq \Phi(x) - \Phi(y)\,,\ \rho_\alpha(y,z) \leq \Phi(y) - \Phi(z)\,.$$

Then for every s and t such that $s > \Phi(x) - \Phi(y)$, $t > \Phi(y) - \Phi(z)$ we have that $d(x,y)(s) = 0$ and $d(y,z)(t) = 0$. If $u > \Phi(x) - \Phi(z) = \Phi(x) - \Phi(y) + \Phi(y) - \Phi(z)$ there exist s_u and t_u such that $u = s_u + t_u$, $s_u > \Phi(x) - \Phi(y)$ and $t_u > \Phi(y) - \Phi(z)$. Suppose that it is not true $x \preceq z$. Then there exists $\alpha \in (0,1]$ such that $u = \rho_\alpha(x,z) > \Phi(x) - \Phi(z)$ which implies that $d(x,z)(u) > 0$. It is obvious that $s_u > \lambda_1(x,y)$, $t_u > \lambda_1(y,z)$ and $s_u + t_u \geq \lambda_1(x,z)$ since from $u' = s' + t' \geq \lambda_1(x,z) > t_u + s_u$, $s' > s_u$, $t' > t_u$ we have:

$$d(x,z)(u') \leq R(d(x,y)(s'), d(y,z)(t')) = R(0,0) = 0$$

and so $d(x,z)(\lambda_1(x,z)) = 1 \leq 0$, which is a contradiction. Hence for $u = \rho_\alpha(x,z) = s_u + t_u$ we have that

$$0 < d(x,z)(u) \leq R(d(x,y)(s_u), d(y,z)(t_u)) = R(0,0) = 0$$

which is a contradiction. Hence, we proved the transitivity.

As in [4] let C be any chain in X and $\beta := \inf_{x\in C} \Phi(x)$. Then there exists a sequence $\{x_n\}_{n\in \mathbf{N}}$ in C such that $\Phi(x_n) \downarrow \beta$, $n \to \infty$. From $\Phi(x_1) \geq \Phi(x_2) \geq \cdots$ it follows that $x_1 \leq x_2 \leq \cdots$. Let

$$W_n := \{x : x \in C, \Phi(x) \leq \Phi(x_n)\}\,,\ \ n \in \mathbf{N}\,.$$

We shall prove that there exists $\xi \in C$ such that $\{\xi\} = \bigcap_{n\in\mathbf{N}} \bar{W}_n$. As in [4] $\lim_{n\to\infty} \overline{d}_\alpha(W_n) = 0$ for all $\alpha \in (0,1]$, and from Lemma 2 it follows that $\lim_{n\to\infty} \overline{d}_\alpha(\bar{W}_n) = 0$ for all $\alpha \in (0,1]$. Since for $m > n$:

$$\rho_\alpha(x_n, x_m) \leq \Phi(x_n) - \Phi(x_m)\,,$$

it follows that $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence and let $\xi = \lim_{n\to\infty} x_n$. From $x_m \in W_n$ $(m \geq n)$ it follows that $\xi \in \bigcap_{n\in\mathbf{N}} \bar{W}_n$ and $\lim_{n\to\infty} \overline{d}_\alpha(\bar{W}_n) = 0$ implies that

$$\{\xi\} = \bigcap_{n\in\mathbf{N}} \bar{W}_n\,.$$

We shall prove that $x \preceq \xi$, for every $x \in C$. If $x \in C$, $x \neq \xi$ it follows that $x \preceq x_n$, for $n \geq n_0(x)$. Indeed, in the opposite case there exists a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ such that

$$x_{n_k} \preceq x \ , \ k \in \mathbf{N}$$

and so $\Phi(x) \leq \Phi(x_{n_k})$, $k \in \mathbf{N}$. This means that

$$x \in \bigcap_{k\in\mathbf{N}} W_{n_k} = \bigcap_{n\in\mathbf{N}} W_n \subseteq \bigcap_{n\in\mathbf{N}} \bar{W}_n = \{\xi\}$$

and so $x = \xi$ which is a contradiction. Hence

$$\rho_\alpha(x, x_n) \leq \Phi(x) - \Phi(x_n) \ , \ n \geq n_0(x) \, .$$

We have to prove that $\rho_\alpha(x,\xi) \leq \Phi(x) - \Phi(\xi)$, for every $\alpha \in (0,1]$. Let $\alpha \in (0,1]$ and $\alpha' \in (0,1]$ be such that $R(\alpha',\alpha') < \alpha$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = \xi$, there exists $n_1 \in \mathbf{N}$ such that

$$\rho_{\alpha'}(x_n, \xi) < \frac{\varepsilon}{2} \ , \ n \geq n_1 \, .$$

Since $\rho_{\alpha'}(x, x_n) < \Phi(x) - \Phi(x_n) + \varepsilon/2$, for all $n \geq \bar{n}_0(x)$, we have that

$$\rho_\alpha(x,\xi) < \varepsilon + \Phi(x) - \Phi(x_n) \ , \ n \geq n_2 = \max(n_1, \bar{n}_0(x)) \, ,$$

and so

$$\rho_\alpha(x,\xi) \leq \varepsilon + \Phi(x) - \lim_{n\to\infty} \inf \Phi(x_n) \leq \varepsilon + \Phi(x) - \Phi(\xi) \, .$$

Since ε is an arbitrary positive number we conclude that $\rho_\alpha(x,\xi) \leq \Phi(x) - \Phi(\xi)$. Hence $x \preceq \xi$. Let η be a maximal element of X. Suppose that $\eta \neq f(\eta)$. Then (3) implies that $\eta < f(\eta)$ which is a contradiction with the maximality of η. □

Using Theorem 4 we have immidiately the following three theorems.

Theorem 6. *Let (X, d, L, R) be a complete fuzzy metric space such that $\lim_{a\to 0_+} R(a,a) = 0$. Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping and $F : X \to 2^X \setminus \{\emptyset\}$ a set-valued mapping such that*

$$\forall x \in X \ \exists y \in F(x) \ , \ \rho_\alpha(x,y) \leq \Phi(x) - \Phi(y) \ , \ \textit{for every}\ \alpha \in (0,1] \, .$$

Then F has at least one fixed point $x^* \in F(x^*)$.

Theorem 7. *Let (X,d,L,R) be a complete fuzzy metric space such that $\lim_{a\to 0+} R(a,a)=0$. Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping and $F : X \to 2^X\backslash\{\emptyset\}$ a set-valued mapping such that*

$$\forall x \in X\ \forall y \in F(x)\,,\ \rho_\alpha(x,y) \le \Phi(x)-\Phi(y)\,,\ \textit{for every}\ \alpha \in (0,1]\,.$$

Then F has at least one invariant point $\{x^*\}=F(x^*)$.

Theorem 8. *Let (X,d,L,R) be a complete fuzzy metric space such that $\lim_{a\to 0+} R(a,a)=0$. Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping. Then for any real number $\varepsilon > 0$ there exists an $x_\varepsilon \in X$ such that for each given $y \ne x_\varepsilon$*

$$\exists \alpha \in (0,1]\,,\ \Phi(y) > \Phi(x_\varepsilon) - \varepsilon\rho_\alpha(x_\varepsilon,y)\,.$$

Proof. Suppose that the statement of the Theorem is not true. Then there exists an $\varepsilon > 0$ such that for all $x \in X$ there exists $y = f(x) \in X\backslash\{x\}$ satisfying for all $\alpha \in (0,1]$

$$\rho_\alpha(x,f(x)) \le \frac{\Phi(x)}{\varepsilon} - \frac{\Phi(f(x))}{\varepsilon}\,.$$

Thus we obtain a mapping $f : X \to X$ which has no fixed point. This is a contradiction to the statement of Theorem 5. □

The last Theorem is analogous to the weak statement of Ekelands variational principle.

4. Caristi's fixed point theorem and a variational principle in probabilistic metric spaces

Using results from section 2. we can immediately apply theorems from section 3. in Menger spaces.

Theorem 9. *Let $(X,\mathcal{F},t)$ be a complete Menger space. Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping and $f : X \to X$ such that for every $x \in X$*

$$\text{(4)}\qquad \forall\alpha \in (0,1]\,,\ \sup\{s : F_{x,f(x)}(s) \le 1-\alpha\} \le \Phi(x) - \Phi(f(x))\,.$$

Then f has at least one fixed point.

Proof. We have to see only that for each $\alpha \in (0,1]$ and $x, y \in X$

$$\begin{aligned} \rho_\alpha(x,y) &= \sup\{s : d(x,y)(s) \geq \alpha\} \\ &= \sup\{s : F_{x,y}(s) \leq 1 - \alpha\}, \end{aligned}$$

and the Theorem is proved using Theorem 5. □

It easily verified that the statement (4) is equivalent to the following statement

$$\forall s > 0, \; F_{x,f(x)}(s) \geq H(s - (\Phi(x) - \Phi(f(x)))). \tag{5}$$

In the same manner as in the Theorem 9 we can prove the following theorems.

Theorem 10. *Let $(X, \mathcal{F}, t)$ be a complete Menger space. Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping and $F : X \to 2^X \backslash \{\emptyset\}$ a set valued mapping such that for every $x \in X$ there exists an y in $F(x)$ and*

$$\forall \alpha \in (0,1], \; \sup\{s : F_{x,y}(s) \leq 1 - \alpha\} \leq \Phi(x) - \Phi(y).$$

Then F has at least one fixed point $x^* \in F(x^*)$.

Theorem 11. *Let $(X, \mathcal{F}, t)$ be a complete Menger space. Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping and $F : X \to 2^X \backslash \{\emptyset\}$ a set valued mapping such that for every $x \in X$ and every y from $F(x)$*

$$\forall \alpha \in (0,1], \; \sup\{s : F_{x,y}(s) \leq 1 - \alpha\} \leq \Phi(x) - \Phi(y).$$

Then F has at least one invariant point $\{x^*\} = F(x^*)$.

Theorem 12. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and (M, d') a complete separable metric space. We denote by X the set of equivalence classes of random variables $x : \Omega \to M$, and let $(X, \mathcal{F}, t_m)$ be as in example 1. If $\Phi : X \to \mathbf{R}^+$ is a lower semicontinuous function and $f : X \to X$ satisfies for all x from X*

$$\forall s, \; s > \Phi(x) - \Phi(f(x)) \Rightarrow P(\{\omega \in \Omega : d(x(\omega), (fx)(\omega)) \leq s) = 1,$$

then f has at least one fixed point.

We can also prove the probabilistic version of the Theorem 8.

Theorem 13 *Let $(X, \mathcal{F}, t)$ be a complete Menger space. Let $\Phi : X \to \mathbf{R}^+$ be a lower semicontinuous mapping. Then for any real number $\varepsilon > 0$ there exists an $x_\varepsilon \in X$ such that for each given $y \neq x_\varepsilon$*

$$\exists \alpha \in (0,1]\,,\ \Phi(y) > \Phi(x_\varepsilon) - \varepsilon \sup\{s : F_{y,x_\varepsilon}(s) \leq 1 - \alpha\}\,.$$

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