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$f(2 \cdot 2^k + 1, -1)$-STRUCTURE IN $(k+1)$-LAGRANGIAN SPACE

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Abstract

The theory of $(k+1)$-Lagrangian spaces has been investigated in several papers as [1], [4]. In this paper an adapted frame in a $(k+1)$-Lagrangian space is chosen for an $f(2t+1,-1)$-structure, and matricies of the tensors g and f with respect to this adapted frame are obtained for $t = 2^k$.

Given is the necessary and sufficient condition for the $(k+1)$-Lagrangian space E to admit a tensor field f of type (1,1) and rank $f = r = (k+1) \cdot n$, such that $f^{2 \cdot 2^k + 1} - f = 0, f^{2i+1} - f \neq 0$ for $1 \leq i < 2^k, k \in N$.

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1. Introduction

In the $n + n(k+1)$ dimensional differentiable manifold E, special coordinate transformations are allowed. Using the nonlinear connection N in $T(E)$ a canonical basis is introduced.

Let E be an $n + n \cdot (k+1)$ dimensional differentiable manifold. If u is one point of E, then in some local chart u has the coordinates

$$u = ((x^i), (y^i_1), \dots, (y^i_{(k+1)})),$$

where

$$(x^i) = (x^1, x^2, \dots, x^n) = (x)$$
$$(y^i_\alpha) = (y^1_\alpha, y^2_\alpha, \dots, y^n_\alpha) = (y_\alpha)$$

and the allowed coordinate transformations are given by

$$\text{(1.1)} \qquad \begin{array}{ll} \text{(a)}\ x^{i'} = x^{i'}(x^1, \dots, x^n) & \text{rank}[\frac{\partial x^{i'}}{\partial x^i}] = n \\ \text{(b)}\ y^{i'}_\alpha = y^i_\alpha \partial x^{i'} / \partial x^i & \end{array}$$

In the sequel Latin and Greek indices assume the values $1, \dots, n$ and $1, \dots, k+1$, respectively $k+1 < n$, the summation convention being used in both.

The natural basis in $T_u E$ is formed by n vectors of the type

$$\partial_i = \frac{\partial}{\partial x^i}$$

and $n \cdot k$ vectors

$$\partial^\alpha_i = \partial / \partial y^i_\alpha.$$

Let us introduce the notation $B = \{(\partial_i), (\partial^1_i), \dots, (\partial^{k+1}_i)\}$ for this basis of $T_u(E)$. Any vector $X \in T_u E$ may be written in the form

$$\text{(1.2)} \qquad X = X^i \partial_i + \overline{X}^i_\alpha \partial^\alpha_i.$$

With respect to the coordinate transformation (2.1), the basic vectors have the following law of transformation:

$$\text{(1.3)} \qquad \begin{array}{c} \text{(a)}\ \partial_i = (\partial_i x^{i'}) \partial_{i'} + (\partial_j \partial_i x^{i'}) y^j_\alpha \partial^\alpha_{i'}, \\ \text{(b)}\ \partial^\alpha_i = (\partial_i x^{i'}_{\cdot}) \partial^\alpha_{i'}. \end{array}$$

It is obvious from (1.3a) that $\partial_i \quad i = 1, 2, \dots, n$ are not transformed as tensors. We shall introduce a family of functions $N^{io}_{\alpha j}(x, y_1, \dots, y_{k+1})$, the so-called nonlinear connection coefficients, which, with respect to (1.1) are transformed in the following way

$$N^{i'o}_{\alpha j'} = N^{io}_{\alpha j}(\partial_i x^{i'})(\partial_{j'} x^j) - (\partial_i\partial_j x^{i'})(\partial_{j'} x^j) y^i_\alpha. \tag{1.4}$$

Using these functions, we introduce in $T_u(E)$ the so-called adapted basis

$$B' = \{(\partial^o_i), (\partial^1_i), \ldots, (\partial^{k+1}_i)\},$$

where

$$\partial^o_i = \partial_i - N^{jo}_{\alpha i}\partial^\alpha_j. \tag{1.5}$$

Substituting ∂^o_i form (1.5) into (1.2), we obtain

$$X = X^i_o \partial^o_i + X^i_\alpha \partial^\alpha_i, \tag{1.6}$$

where

$$X^i_o = X^i, \quad X^i_\alpha = \overline{X}^i_\alpha + X^j_o N^{io}_{\alpha j}. \tag{1.7}$$

From (1.4) and (1.5) it follows that under transformation (1.1) ∂^o_i is transformed as a covariant vector, i.e.

$$\partial^o_{i'} = (\partial_{i'} x^i)\partial^o_i. \tag{1.8}$$

Introducing the capital Latin indices which take the values $0, 1, \ldots, k+1$, we can write (1.6) in the form

$$X = X^i_A \partial^A_i \tag{1.9}$$

further, (1.3b) and (1.8) in the form

$$\partial^A_{i'} = (\partial_{i'} x^i)\partial^A_i (A = \overline{0, k+1}). \tag{1.10}$$

Let us consider two coordinate systems (x^i, y^i_α) and $(x^{i'}, y^{i'}_\alpha)$ in some neighbourhood of point $u \in E$, for which the corresponding adapted bases are

$$\{\partial_i, \partial^\alpha_i\} = \{(\partial^o_i), (\partial^1_i), \ldots, (\partial^{k+1}_i)\}, \;\; \{\partial_{i'}, \partial^\alpha_{i'}\} = \{(\partial^o_{i'}), (\partial^1_{i'}), \ldots, (\partial^{k+1}_{i'})\}.$$

For any vector field $X \in T_u(E)$ we have

$$X = X^i_A \partial^A_i = X^{i'}_A \partial^A_{i'}.$$

From the above equation and (1.8) we get

$$(1.11) \qquad \text{(a)}\ X_A^i = X_A^{i'}\partial_{i'}x^i \ , \ \text{(b)}\ X_A^{i'} = X_A^i\partial_i X^{i'}.$$

The vectors (∂_i^o) span the n-dimensional space $T_H(E)$, and the vectors $(\partial_i^\alpha) = (\partial_1^\alpha, \partial_2^\alpha, \ldots, \partial_n^\alpha)$ the n-dimensional ${}_\alpha T_V(E)$, and,

$$T(E) = T_H(E) + T_V(E), \qquad T_V(E) = \oplus_1^{k+1}\ {}_\alpha T_V(E).$$

On the space $T(E)\otimes T(E)$, a metric tensor G is defined such that $T(E)$ can be decomposed into two orthogonal parts $T_H(E)$ and $T_V(E)$.

2. Tensorial structures $f(2.2^k+1,-1)$

Now, let us first observe the structure f satisfying the condition $f^{2t+1} - f = 0$.

Definition 1. *Let $\mathcal{M}^m$ be a differentiable manifold of class C^∞, and let there be a tensor field $f \neq 0$ of the type (1,1) and of class C^∞ such that*

$$(2.1) \qquad f^{2t+1} - f = 0, \ \ f^{2i+1} - f \neq 0 \ \ \textit{for} \ \ 1 \leq i < t,$$

where t is a fixed integer greater than 1. Let rank $f = r$ be constant. We call such a structure an $f(2t+1,-1)$-structure or an f-structure of the rank r and of degree $2t+1$.

Theorem 1. *For a tensor field $f, f \neq 0$ satisfying (2.1), the operators*

$$(2.2) \qquad m = 1 - f^{2t}, \ \ l = f^{2t}$$

are the complementary projection operators where 1 denotes the identity operator applied to the tangent space at a point of the manifold.

Proof. We have

$$l + m = 1, \ l^2 = l, \ m^2 = m, \ ml = lm = 0$$

by virtue of (2.1), which proves the theorem.

Let L and M be the complementary distributions corresponding to the operators l and m, respectively. If rank $f = r$ is constant and dim $L = r$, then dim $M = m - r$.

Theorem 2. *For f satisfying (2.1) and l, m, defined by (2.2), we have*

$$(2.3)\qquad \begin{aligned} &(a)\ lf = fl = f,\\ &(b)\ mf = fm = 0,\\ &(c)\ f^{2t}m = 0,\\ &(d)\ (m+f^t)^2 = 1. \end{aligned}$$

Proof. Trivial.

Theorem 3. *Suppose that there is a projection operator m on $\mathcal{M}^m$ and that there exists a tensor field f such that (2.3b) and (2.3d) are satisfied, then f satisfies (2.1).*

Proposition 1. *Let an f-structure of the rank r and degree $2t+1$ be given on $\mathcal{M}^m$, then $f^{2t}l = l$ and $f^{2t}m = 0$, i.e. f^t acts on L as an almost product structure operator and on M as a null operator.*

We shall assume that $\mathcal{M}^m$ is a $(k+1)$-Lagrange space of dimension $m = n + (k+1)n$, and that rank $f = r = (k+1)n$. Then $\dim L = n(k+1), \dim M = n$ and $M = T_H(E), L = T_V(E)$.

If we denote by h the projection morphism of $T(E)$ to $T_H(E)$ we have

$$hX = hX_A^i\partial_i^A = X_0^i\partial_i^0$$

The mapping α which is defined in [5] by

$$\alpha(X,Y) = \frac{1}{2}[\overline{h}(lX,lY)] + \overline{h}(mX,mY)],\ \ \forall X,Y\in T(E),$$

where $\overline{h} = Gh$, is a pseudo-Riemannian structure on $T(E)$, such that $\alpha(X,Y)=0$,
$\forall X\in M, Y\in L$.

If we put $g(X,Y) = \frac{1}{2t}[\alpha(X,Y)+\alpha(fX,fY)+\cdots+\alpha(f^{2t-1}X,f^{2t-1}Y)]$, it is easy to see that $g(X,Y)=0, \forall X\in M, Y\in L$.

Also, using (2.2) and Theorem 2. we get $g(fX,fY) = \frac{1}{2t}[\alpha(fX,fY)+\alpha(f^2X,f^2Y)+\cdots+\alpha(X,Y)] = g(X,Y)$. Thus f is an isometry with respect to g.

We assume that f_L^i (the restriction from f^i on L, $(i<2t)$) is not identity operator of L. Then f_L is a linear transformation of L with the minimal polynomial $x^{2t} - 1 = 0$. (We know that $f^{2t} = 1$ on L.) The polynomial $(x^t - 1)(x^t + 1) = 0$ has simple roots:

$$e^{2\frac{\pi i}{t}}, e^{4\frac{\pi i}{t}}, \ldots, e^{2t\frac{\pi i}{t}}, e^{\frac{\pi i}{t}}, e^{3\frac{\pi i}{t}}, \ldots, e^{(2t-1)\frac{\pi i}{t}}.$$

The eigenvectors which correspond to these eigenvalues are $e_2, e_4, \ldots, e_{2t}, e_1, e_3, \ldots, e_{2t-1}$, respectively. Let us denote by L_1 the vector space generated by the vectors $e_2, e_4, \ldots, e_{2t}$ and by L_2 the vector space generated by the vectors $e_1, e_3, \ldots, e_{2t-1}$. Then

$$f^t = 1 \text{ on } L_1, \; f^t = -1 \text{ on } L_2.$$

For $X \in L_1$ and $Y \in L_2$, we have

$$g(X,Y) = g(fX, fY) = g(f^tX, f^tY) = g(X, -Y) = -g(X,Y).$$

Hence, L_1 and L_2 are orthogonal with respect to the metric g.

We assume that $f^j - 1 \neq 0$ on $L_1, j < t$ and $f^j + 1 \neq 0$ on $L_2, j < t$. Then, f is a linear transformation of L_2 with the minimal polynomial $x^t + 1 = 0$, with the eigenvalue $\sqrt[t]{-1}$, to which correspond the eigenvectors $e'_1, e'_2, \ldots, e'_t$ and $L_2 = L_2^1 \oplus L_2^2 \oplus \ldots \oplus L_2^t$ where L_2^s is the subspace of L_2 generated by the vector $e'_s, s = 1, \ldots, t$.

It is also an f linear transformation on L_1 with the minimal polynomial $x^t - 1 = 0$, with the eigenvalue $\sqrt[t]{1}$, to which the eigenvectors $e'_{t+1}, e'_{t+2}, \ldots, e'_{2t}$ correspond. Now $L_1 = L_1^{t+1} \oplus L_1^{t+2} \oplus \ldots \oplus L_1^{2t}$, where L_1^{t+s} is the subspace of L_1 generated by the vector $e'_{t+s}, s = 1, \ldots, t$.

L_1^{t+p} and $L_1^{t+r}, (p, r < t)$, are orthogonal with respect to g if $t = 2^k$, $k \in N$, which is then shown by induction, see [6]. In the sequel $t = 2^k, k \in N$.

In [3] the following theorem is proved: If $f^t = \begin{bmatrix} 0 & E_p \\ -E_p & 0 \end{bmatrix}$, then $t \leq p$ and p is divisible by $t, (p = s \cdot t)$.

An analogous situation is on the space $L_2(\dim L_2 = 2p, p = s \cdot 2^{k-1})$.

If we assume that $2p = n, n + (k+1)n$ is the dimension of the $(k+1)$-Lagrangian space E, we can identify L_2 by ${}_1T_V(E), L_1$ by $\oplus_2^{k+1} T_V(E)$.

Let $u_1, \ldots, u_{2p}$ be an orthogonal basis of L_2 and $u_{2p+1}, u_{2p+2}, \ldots, u_{r-2p}$ be an orthogonal basis of L_1, both with respect to g, then $u_1, \ldots, u_{2p}, u_{2p+1}, \ldots, u_{r-2p}$ is an orthogonal basis of L such that

$$L_2:\begin{cases} f(u_i)=u_{i+\frac{2p}{2^k}}, & f(u_{j+2p-\frac{2p}{2^k}})=-u_j,\\ i=1,2,\dots,2p-2p/2^k, & j=1,2,\dots,2p/2^k. \end{cases}$$

$$L_1:\begin{cases} f(u_{2p+i})=u_{2p+i+\frac{2p}{2^{k-1}}}, & f(u_{4p+j-\frac{2p}{2^{k-1}}})=-u_{2p+j}\\ i=1,2,\dots,2p-2p/2^{k-1}, & j=1,2,\dots,2p/2^{k-1}\\ f(u_{4p+i})=u_{4p+i+\frac{2p}{2^{k-2}}}, & f(u_{6p+j-\frac{2p}{2^{k-2}}})=-u_{4p+j}\\ i=1,2,\dots,2p-2p/2^{k-2}, & j=1,2,\dots,2p/2^{k-2}\\ \vdots & \\ f(u_{2(p-1)k+i})=u_{2(p-1)k+i+p}, & f(u_{2pk-j})=-u_{2(p-1)k+j}\\ i=1,2,\dots,p, & j=1,2,\dots,p\\ f(u_{2kp+i})=u_{2pk+i}, & f(u_{2(k+1)p-i})=-u_{2p(k+1)-i}\\ i=1,2,\dots,p. & \end{cases}$$

Next, we choose in $M=T_H(E)$ an orthogonal basis $u_{r+1},\dots,u_{n+(k+1)n}$ with respect to $g, m=n+r=n+(k+1)n$. Then, with respect to the orthogonal frame $u_1,\dots,u_m$, the tensors g and f have components as in [6]:

$$f=\begin{bmatrix} 0 & E_{2p-\frac{2p}{2^k}} & & & & & & & \\ -E_{\frac{2p}{2^k}} & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & E_{2p-\frac{p}{2}} & & & & \\ & & & -E_{\frac{p}{2}} & 0 & & & & \\ & & & & & 0 & E_p & & \\ & & & & & -E_p & 0 & & \\ & & & & & & & E_p & 0 & \\ & & & & & & & 0 & -E_p & \\ & & & & & & & & & 0_{m-r} \end{bmatrix}$$

(2.4)

$$g=\begin{bmatrix} E_{2p} & & & \\ & \ddots & & \\ & & E_{2p} & \\ & & & E_{m-r} \end{bmatrix}$$

We call such a frame an adapted frame of $f(2 \cdot 2^k + 1, -1)$ structure.

Let $\overline{u}_1, \ldots, \overline{u}_m$ be another adapted frame with respect to which the metric tensor g and the tensor f have the same components as (2.4). We put $\overline{u}_i = \gamma_i^j u_j$, then we find that γ has the form

$$\tag{2.5} \gamma = \begin{bmatrix} S_{(\frac{2p}{2^k})} & & & & & & & \\ & S_{(\frac{2p}{2^{k-1}})} & & & & & & \\ & & \ddots & & & & & \\ & & & S_{(\frac{p}{2})} & & & & \\ & & & & A_p & B_p & & \\ & & & & -B_p & A_p & & \\ & & & & & & 0_{2p} & \\ & & & & & & & 0_{m-r} \end{bmatrix}$$

where $S_{(\frac{2p}{i})}, i = 2, 4, \ldots, 2^k$ is a matrix of format $2p = n$ and has the form

$$\tag{2.6} S_{(\frac{2p}{i})} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & \ldots & A_i \\ -A_i & A_1 & A_2 & A_3 & \ldots & A_{i-1} \\ -A_{i-1} & -A_i & A_1 & A_2 & \ldots & A_{i-2} \\ \vdots & & & & & \\ -A_3 & -A_4 & -A_5 & & & A_2 \\ -A_2 & -A_3 & -A_4 & & \ldots & A_1 \end{bmatrix}$$

where each matrix $A_l, l = 1, \ldots, i$ has a format $(\frac{2p}{i}) \times (\frac{2p}{i})$, i.e. $s \times s$.

Let $\overline{S}_{(\frac{2p}{i})}$ be the tangent group defined by $S_{(\frac{2p}{i})}$. Then we can say that the group of the tangent bundle of the manifold can be reduced to

$$\overline{S}_{(\frac{2p}{2^k})} \times \overline{S}_{(\frac{2p}{2^{k-1}})} \times \ldots \times \overline{S}_{(\frac{2p}{4})} \times U_p \times 0_{2p} \times 0_{m-r}.$$

Theorem 4. *A necessary and sufficient condition for a $(k+1)$-Lagrangian space E of dimension $n + (k+1)n$ to admit a tensor field $f \neq 0$ of type (1,1) and of rank $r = (k+1)n$, such that $f^{2 \cdot 2^k + 1} - f = 0$, is that*

i) $r = (k+1)2p$,

ii) $2p = s \cdot 2^k = n$ *and*

iii) *the group of the tangent bundle of the manifold be reduced to the group*

$$\overline{S}_{(\frac{2p}{2^k})} \times \overline{S}_{\frac{2p}{2^{k-1}}} \times \ldots \times \overline{S}_{(\frac{2p}{4})} \times U_p \times O_{2p} \times O_{m-r}.$$

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