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ON THE NUMBER OF HAMILTONIAN CYCLES OF $P_m \times P_n$

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Abstract

A new characterization of Hamiltonian cycles in $P_m \times P_n$, given in this paper, makes it possible to determine a special digraph for each number m. In this way, the enumeration of Hamiltonian cycles in $P_m \times P_n$ amounts to the enumeration of all oriented walks of the length $(n-2)$ in the digraph with the initial and final vertices in given sets. A recurrence relation for the number of Hamiltonian cycles in $P_6 \times P_n$ is derived as well.

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1. Introduction

Let $P_m \times P_n$ denote the cartesian product of two paths with m and n vertices, respectively. In this graph, there are $(m-1) \times (n-1)$ squares (cycles of order 4) which we call the *windows* of the graph. We can associate with the graph $P_m \times P_n$ its *window lattice graph* $W_{m,n}$ whose vertices are the windows of $P_m \times P_n$, two vertices being adjacent in $W_{m,n}$ iff the two windows of $P_m \times P_n$ which correspond to those vertices have a common edge.

We denote by $H_m(n)$ the number of Hamiltonian cycles in $P_m \times P_n$. The value of $H_4(n)$ and $H_5(n)$ was studied in [1] and [2] .

It is easy to prove the following statement:

$P_m \times P_n$ $(m, n > 1)$ has a Hamiltonian cycle iff the number of vertices is even.

Consider a Hamiltonian graph $P_m \times P_n$ $(H_m(n) > 0)$.

Now, as in [1] , for each Hamiltonian cycle, we associate with the graph $W_{m,n}$ a binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ defining its elements in the following way (see the fig. 1):

$$a_{i,j} = \begin{cases} 1 & \text{if } \ w_{i,j} \ \text{ belongs to the interior of the Hamiltonian cycle} \\ 0 & \text{otherwise.} \end{cases}$$

This matrix satisfies the following necessary conditions which are easy to verify (these conditions correspond to conditions (BC), (IC), (CC) and (EC) from [2]):

- *Adjacency of Column Conditions:*
  $(\forall j)(1 \le j \le (n-2))$

  (1) $$\neg(a_{1,j} = a_{1,j+1} = 0 \ \vee \ a_{m-1,j} = a_{m-1,j+1} = 0)$$

  $(\forall i)(1 \le i \le m-2)(\forall j)(1 \le j \le (n-2))$

  $(a_{i,j}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}) \notin \{(0,0,0,0),(1,1,1,1),(1,0,0,1),(0,1,1,0)\}$
  (2)

- *First and Last Column Conditions:*

  (3) $$a_{1,1} = a_{m-1,1} = a_{1,n-1} = a_{m-1,n-1} = 1$$

  $(\forall i)(1 \le i \le (m-2))$

  (4) $$\neg(a_{i,1} = a_{i+1,1} = 0 \ \vee \ a_{i,n-1} = a_{i+1,n-1} = 0)$$

- *Root Condition:*
  The subgraph of $W_{m,n}$ induced by windows belonging to the exterior of the Hamiltonian cycle forms a forest such that each component (we

call it *exterior tree (ET)*) contains exactly one window $w_{i,j}$ (we call it *root of the exterior tree*) that satisfies the next condition:

$$(i \in \{1,(m-1)\} \wedge j \notin \{1,(n-1)\}) \vee (j \in \{1,(n-1)\} \wedge i \notin \{1,(m-1)\}) \quad (5)$$

The converse is also satisfied: every binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ which satisfies adjacency of column conditions , first and last column conditions and root condition determines exactly one Hamiltonian cycle of the graph $P_m \times P_n$.

Using these conditions some new values of $H_m(n)$ were obtained in [3]. However, that algorithm is very slow because it generates binary matrices which fulfill (1) – (4) and root condition, one by one.

Definition 1. *Two windows* $w_{i,l}$ *and* $w_{j,s}$ *which satisfy:* $a_{i,l} = 0$, $a_{j,s} = 0$ *and* $l, s \leq k$ *are said to be surly in the same exterior tree at the k-th level (i.e. in relation k-SISET) iff they belong to the same component in the subgraph of* $W_{m,n}$ *which is induced by the set of all windows* $w_{p,t}$ *which satisfy* $a_{p,t} = 0$ *and* $t \leq k$.

Note that the relation k- SISET represents a RST - relation in the set of all windows $w_{i,k}$ which satisfy $a_{i,k} = 0$ $(1 \leq i \leq m-1)$ and k-fixed. There are at most $\lceil \frac{m-1}{2} \rceil$ classes of the RST-relation. (It is possible that two different classes belong to the same ET, but we cannot conclude that if we know only the first k column of the matrix A.) Further, every class belongs to exactly one ET so it can be in relation k-SISET with at most one root.

Let C denote the set $\{2, 3, \ldots, \lceil \frac{m-1}{2} \rceil\}$. Now, for each Hamiltonian cycle, we associate with binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ which satysfies adjacency of column conditions, first and last column conditions and root condition the matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$, $b_{i,j} \in C \cup \{0, 1\}$ in the following way (see the fig. 2):

(a) $b_{i,j} = 1$ iff $a_{i,j} = 1$ $(1 \leq i \leq (m-1))$ $(1 \leq j \leq (n-1))$;

(b) if the window $w_{i,j}$ is the root of an ET and $(i = 1$ or $i = (m-1)$ or $j = 1)$ then $b_{i,j} = 0$;

(c) if the window $w_{i,j}$ isn't root of an ET but it is in relation j-SISET with a root then $b_{i,j} = 0$;

(d) we associate with the left windows some elements of C considering the ordinal numbers of left classes in the fixed j -th column. (Till now, we associated with some classes in the fixed column 0-elements (see (b) and (c)). Thus, the first of the left classes in the fixed column (looking at from above) is associated with number 2, the second one with number 3, etc..

1	1	1	1	1	1	1	1	0	1	1	0	1	1
0	0	1	0	0	0	0	0	0	1	0	0	1	0
1	1	1	1	1	0	1	1	1	1	1	0	1	1
1	0	0	0	0	0	0	0	1	0	0	0	1	0
1	0	1	0	1	1	1	1	1	0	1	1	1	1
1	1	1	0	1	0	0	0	0	0	0	0	1	0
1	0	0	0	1	1	1	1	1	1	1	0	1	1
1	1	1	0	0	0	0	0	0	0	1	0	1	0
0	0	1	1	1	1	1	1	1	1	1	0	1	1
1	0	1	0	0	0	0	0	0	0	0	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Fig. 1

1	1	1	1	1	1	1	1	0	1	1	0	1	1
0	0	1	2	2	2	2	2	0	1	2	0	1	2
1	1	1	1	1	2	1	1	1	1	1	0	1	1
1	2	2	3	3	2	2	2	1	2	3	0	1	3
1	2	1	3	1	1	1	1	1	2	1	1	1	1
1	1	1	3	1	3	3	3	2	2	3	0	1	4
1	3	3	3	1	1	1	1	1	1	1	0	1	1
1	1	1	3	3	2	2	2	0	0	1	0	1	5
0	0	1	1	1	1	1	1	1	1	1	0	1	1
1	0	1	4	4	4	4	4	3	3	4	0	1	6
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Fig. 2

We need still two definitions:

Definition 2. *The base of the integer word* $d_1 d_2 \ldots d_{m-1}$ *is the binary* *word* $\bar{d}_1\bar{d}_2\ldots\bar{d}_{m-1}$ *where*

$$\bar{d}_i = \begin{cases} 1 & \textit{if} \quad d_i = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 3. *The base of the integer matrix* $[d_{i,j}]$ *is the binary matrix* $[\bar{d}_{i,j}]$ *of the same format where*

$$\bar{d}_{i,j} = \begin{cases} 1 & \textit{if} \quad d_{i,j} = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

2. Characteristics of New Characterisation

From definition of the matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$, we can easily obtain the following characteristics of that matrix:

1. The base of the matrix B i.e. matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ satisfies adjacency of column conditions ((1) and (2)) and first and last column conditions ((3) and (4)).

2. The first column is equal to its base
   i.e.
   $$(\forall i)(1 \le i \le (m-1))(b_{i,1} = a_{i,1})$$

3. The last , $(n-1)$-th column does not contain any 0-element and if the number p of all 1-elements is not equal to $(m-1)$ then the subword of the $(n-1)$-th column which is obtained removing all 1-elements is equal to the word $23\cdots(m-p)$.

4. For every k-th column $(2 \le k \le (n-1))$ of the matrix B it is satisfied:

   (a)
   $$b_{1,k} = a_{1,k}; \quad b_{m-1,k} = a_{m-1,k}$$

(b) If $b_{i,k} \neq 1$ $(2 \leq i \leq (m-2))$ then $b_{i-1,k} \in \{b_{i,k}, 1\}$ and $b_{i+1,k} \in \{b_{i,k}, 1\}$
(Two windows in the same class must be associated with the same number.)

(c) If $b_{i,k-1} = 0$ $(2 \leq i \leq m-2)$ then $b_{i,k} \in \{0, 1\}$
(If the window $w_{i,k-1}$ is in relation (k-1)-SISET with a root (i.e. $b_{i,k-1} = 0$) then it is in relation k-SISET with the same root , as well; and if it is in relation k-SISET with $w_{i,k}$ (i.e. $a_{i,k} = 0$) then $w_{i,k}$ must be in relation k-SISET with the mentioned root.)

(d) For each number $b \in C$ which appears in $(k-1)$-th column there must be the window $w_{i,k-1}$ with $b_{i,k-1} = b$ and $b_{i,k} \neq 1$.
(There are no ET without root.)

(e) If there are p and l $(p \neq l$, $2 \leq p, l \leq (m-1))$ where $b_{p,k-1} = b_{l,k-1} \neq 1$ and $a_{p,k} = a_{l,k} = 0$ then $b_{p,k} = b_{l,k}$
(If $w_{p,k-1}$ and $w_{l,k-1}$ are in relation $(k-1)$ -SISET and $a_{p,k} = a_{l,k} = 0$ then the windows $w_{p,k}$ and $w_{l,k}$ must be in relation k - SISET.)

(f) If $b_{i,k-1} = b_{j,k-1} \in C$ and $b_{i,k} = b_{j,k} = b \neq 1$ $(i \neq j,$ $2 \leq i, j \leq (m-2))$ then there is no maximal sequence of consecutive appearance of number $b \in C \cup \{0\}$ in the k - th column which contains both $w_{i,k}$ and $w_{j,k}$. (In the opposite, we would get a cycle in a ET.)

(g) For every maximal sequence of consecutive appearance of number 0 in the k - th column there is exactly one sequence $v_1, v_2, \ldots, v_p$ $(p \geq 1)$ of different maximal sequence of consecutive appearance of number 0 in the same column which satisfies:

- if $p = 1$ then
  sequence v_1 is either adjacent with exactly one 0 -windows from the $(k-1)$ - th column or contains exactly one of the elements $w_{1,k}$ and $w_{m-1,k}$;
- if $p > 1$ then

  – for every i $(1 \leq i \leq (p-1))$, there is exactly one $w_{j_i,k-1}$ with $b_{j_i,k-1} \in C$ for which $w_{j_i,k} \in v_i$ and there is exactly one $w_{s_{i+1},k-1}$ with $b_{s_{i+1},k-1} \in C$ for which $w_{s_{i+1},k} \in v_{i+1}$ and $b_{j_i,k-1} = b_{s_{i+1},k-1}$;

- the p - th sequence v_p is either adjacent with exactly one 0 -th window from the $(k-1)$ - th column or contains exactly one of the windows $w_{1,k}$ and $w_{m-1,k}$.

(h) If v and u are two different maximal sequence of consecutive appearance of number $b \in C$ in the k -th column (i.e. if we are 'sure ' that v and u are in the same ET knowing the first k columns) then there is exactly one sequence $v = v_1, v_2, \ldots, v_p = u$ of p different maximal sequence of consecutive appearance of number b in the k -th column which satisfies: for every i $(1 \le i \le p-1)$ there is exactly one $w_{j_i,k-1}$ with $b_{j_i,k-1} \in C$ for which $w_{j_i,k} \in v_i$ and there is exactly one $w_{s_{i+1},k-1}$ with $b_{s_{i+1},k-1} \in C$ for which $w_{s_{i+1},k} \in v_{i+1}$ and $b_{j_i,k-1} = b_{s_{i+1},k-1}$

(i) Consider the first appearances of elements from the set C in the k-th column from above (from $w_{1,k}$ to $w_{m-1,k}$). Let them be $w_{p_1,k}, w_{p_2,k}, \ldots, w_{p_l,k}$ $(l < \lceil \frac{m-1}{2} \rceil)$. Then, that is satisfied $b_{p_i,k} = i + 1$. (It appears from the definition of matrix B.)

Vice versa, every integer matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$ with elements from the set $C \cup \{0,1\}$ which satisfies 1. - 4. determines exactly one Hamiltonian cycle in the graph $P_m \times P_n$ i.e. the base of the matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$ fulfills the root condition besides adjacency of column conditions and the first and last column conditions, which is not so difficult to prove.

Now, we can create for each number m $(m \ge 3)$ a digraph D_m in the following way: the set of vertices $V(D_m)$ consists of all possible columns in the matrix B (integer words $d_1 d_2 \ldots d_{m-1}$ of the alphabet $C \cup \{0,1\}$); a (directed) line joins the vertex v to the vertex u $(v, u \in V(D_m))$ i.e. $v \to u$ iff vertex v (as a integer word $b_{1,k-1} b_{2,k-1} \ldots b_{m-1,k-1}$) might be previous column for the vertex u (as a word $b_{1,k} b_{2,k} \ldots b_{m-1,k}$) i.e. these words satisfy characteristics 1. and 4..

The subset of $V(D_m)$ which consists of all possible first columns in the matrix B (see the characteristics 1. and 2.) will be called the set of the *emphasized* vertices.

The subset of $V(D_m)$ which consists of all possible last columns in the matrix B (see the characteristics 1. and 3.) will be called the set of the *last* vertices.

Note that these two subsets of $V(D_m)$ has exactly one common element (word $11\ldots1$).

So, in this way, our problem of enumeration of all Hamiltonian cycles in $P_m \times P_n$ amounts to enumeration of all oriented walks of the length $(n-2)$ in the digraph D_m with the *emphasized* initial vertices and the *last* final verices.

For every m $(m \geq 3)$ we can create a digraph D_m using the characteristics of the matrix B.

3. The Adjacency Matrices of D_m

For $m = 3, 4, 5, 6$ the adjacency matrices of their digraphs are given below.

m=5		1	1	1	1	1	1	1	1	0	0	1	1	1	0	0	0	0	0	1
		0	2	1	1	1	0	0	1	0	0	0	2	2	1	1	1	1	0	1
		1	1	0	2	1	0	1	1	0	1	0	2	1	0	1	0	2	1	0
		1	1	1	1	1	1	0	0	1	0	0	1	0	1	1	0	1	1	0
1.	* 1011	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
2.	+ 1211	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0
3.	* 1101	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0
4.	+ 1121	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	1	1	0	0
5.	* + 1111	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	1	1	0	0
6.	1001	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7.	1010	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8.	1110	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
9.	0001	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10.	0010	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11.	1000	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12.	1221	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
13.	1210	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
14.	0101	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15.	0111	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
16.	0100	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17.	0121	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
18.	0011	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
19.	1100	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0

$*$ - emphasized vertex
$+$ - last vertex

m=3

		1 0 1
		1 1 0
1.	* + 11	0 1 1
2.	01	1 0 0
3.	10	1 0 0

m=4

		1 1 1 0 1 0
		0 2 1 0 0 1
		1 1 1 1 0 0
1.	* 101	1 0 1 0 0 0
2.	+ 121	0 1 0 1 1 0
3.	* + 111	0 1 0 1 1 1
4.	001	1 0 1 0 0 0
5.	100	1 0 1 0 0 0
6.	010	0 0 1 0 0 0

m=6

		1 1 1 1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 1 0 1 0 0 0 0 0 0 0 0 1 1
		0 2 0 2 1 1 1 1 1 0 0 0 0 2 2 0 0 0 0 1 0 2 1 1 1 1 1 1 0 0 2 2
		1 1 1 1 0 2 1 1 1 0 1 1 0 2 1 0 0 1 1 1 0 2 0 0 1 0 2 2 1 1 1 1
		0 3 1 1 1 1 0 2 1 0 0 2 0 2 0 0 1 0 2 0 1 1 0 1 1 0 2 1 0 1 0 2
		1 1 1 1 1 1 1 1 1 1 0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 1 1
1.	* 10101	1 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
2.	+ 12131	0 1 0 0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3.	* 10111	0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
4.	+ 12111	0 1 0 0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0
5.	* 11011	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0
6.	+ 11211	0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 1 1 1 0 0 0 0
7.	* 11101	0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 1 1 1 0
8.	+ 11121	0 1 0 0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0
9.	* + 11111	0 1 0 0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 1 1 0 0 0 1 1 1 0 0 0 0
10.	00001	1 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
11.	00100	1 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
12.	00121	0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
13.	10000	1 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
14.	12221	0 0 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 1 1 0 1
15.	12100	0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0
16.	10001	1 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
17.	10010	0 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
18.	10100	1 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
19.	10121	0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
20.	11100	0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 1 1 1 0
21.	00010	0 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
22.	12210	0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
23.	01001	0 0 0 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
24.	01010	0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
25.	01110	0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
26.	01000	0 0 0 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
27.	01221	0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
28.	01210	0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
29.	00101	1 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
30.	00111	0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
31.	12101	0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0
32.	12121	0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 1 1 0 1

4. Value of $H_6(n)$

Let f_i^k denote the number of all walks of the length k having the initial vertex which corresponds to the i-th row (column) in the adjacency matrix $D = [d_{i,j}]$ of D_m $(m \geq 3)$. If we denote by E the set of all emphasized vertices in the digraph D_m, then the value of $H_m(n)$ is obtained from the following system of recurrence relations:

$$\left.\begin{array}{r} H_m(k+2) \stackrel{def}{=} f_0^k = \sum_{i\in E} f_i^k \\ f_i^k = \sum_{j=1}^{|V(D_m)|} d_{i,j} f_j^{k-1}, \quad i = 1, \ldots, |V(D_m)| \end{array}\right\} \tag{6}$$

In [4] it is proved that the above system of recurrence relations is solvable by the unknown function f_0^k $(H_m(n))$ i.e. there is a recurrence relation of $H_m(k)$ with order at most $2^{|V(D_m)|-1}$.

Solving the corresponding systems of recurrence relations for $m = 4$ and $m = 5$ we can obtain the recurrence relations for $H_4(n)$ and $H_5(n)$ (given in [1] and [2]) again:

Theorem 1. $H_4(n)$ *satisfies the recurrence relation*

$$H_4(n) = 2H_4(n-1) + 2H_4(n-2) - 2H_4(n-3) + H_4(n-4)$$

for $n \geq 4$, *with initial values* $H_4(0) = H_4(1) = 0$, $H_4(2) = 1$ *and* $H_4(3) = 2$.

Theorem 2. $H_5(n)$ *satisfies the recurrence relation*

$$H_5(n) = 11H_5(n-2) + 2H_5(n-6)$$

for $n \geq 6$, *with initial values* $H_5(0) = H_5(1) = H_5(3) = H_5(5) = 0$, $H_5(2) = 1$ *and* $H_5(4) = 14$.

In this paper, we derive a recurrence relation for $H_6(n)$:

Theorem 3. $H_6(n)$ *satisfies the recurrence relation*

$$H_6(n) = 6H_6(n-1)+13H_6(n-2)-86H_6(n-3)-45H_6(n-4)+187H_6(n-5)+$$

$$+775H_6(n-6)-378H_6(n-7)-2185H_6(n-8)-1699H_6(n-9)+3690H_6(n-10)+$$

$$+5807H_6(n-11)-1420H_6(n-12)-8130H_6(n-13)-1450H_6(n-14)+$$

$$+5673H_6(n-15)-165H_6(n-16)-6128H_6(n-17)+4264H_6(n-18)+$$

$$+9962H_6(n-19)-5278H_6(n-20)-10904H_6(n-21)+4056H_6(n-22)+$$

$$+5910H_6(n-23)-1266H_6(n-24)-900H_6(n-25)+180H_6(n-26)-72H_6(n-27) \quad (7)$$

The initial values are given in Section 5.

Lemma 1. *Let*

$$\sum_{i=0}^{p} a_i f^{k-i} = \sum_{i=0}^{q} b_i g^{k-i} + \varepsilon^k \tag{8}$$

$$\sum_{i=0}^{r} c_i g^{k-i} = \sum_{i=0}^{s} d_i f^{k-i} + \xi^k \tag{9}$$

$a_i, b_i, c_i, d_i \in Z;\; p, q, r, s \in N;\; f, g, \varepsilon, \xi : N \cup 0 \mapsto Z$
be a system of recurrence relations. Then, it is solvable by function f *"removing " function* g *i.e.*
it satisfies

$$\sum_{i=0}^{r}\sum_{j=0}^{p} c_i a_j f^{k-i-j} - \sum_{j=0}^{q}\sum_{i=0}^{s} d_i b_j f^{k-i-j} = \sum_{i=0}^{q} b_i \xi^{k-i} + \sum_{i=0}^{r} c_i \varepsilon^{k-i}$$

Proof. From (8) we obtain

$$\varphi^k \overset{def}{=} \sum_{i=0}^{p} a_i f^{k-i} - \sum_{i=0}^{q} b_i g^{k-i} - \varepsilon^k = 0$$

and further

$$\sum_{i=0}^{r} c_i \varphi^{k-i} = 0 \quad \text{i.e.}$$

$$\sum_{i=0}^{r}\sum_{j=0}^{p} c_i a_j f^{k-i-j} - \sum_{i=0}^{r}\sum_{j=0}^{q} c_i b_j g^{k-i-j} - \sum_{i=0}^{r} c_i \varepsilon^{k-i} = 0$$

If we take

$$\psi^k \stackrel{def}{=} \sum_{i=0}^{r} c_i g^{k-i}$$

then the above expression is equivalent to the following one

$$\sum_{i=0}^{r}\sum_{j=0}^{p} c_i a_j f^{k-i-j} - \sum_{j=0}^{q} b_j \psi^{k-j} - \sum_{i=0}^{r} c_i \varepsilon^{k-i} = 0 \tag{10}$$

Substituing (9) into (10) we obtain

$$\sum_{i=0}^{r}\sum_{j=0}^{p} c_i a_j f^{k-i-j} - \sum_{j=0}^{q} b_j \sum_{i=0}^{s} d_i f^{k-i-j} - \sum_{j=0}^{q} b_j \xi^{k-j} - \sum_{i=0}^{r} c_i \varepsilon^{k-i} = 0$$

$\square$

Proof of Theorem 3. Let us create system (6) for D_6. Because of symmetry of some couples of words from set $V(D_6)$, we can see that

$$f_7 = f_3;\quad f_8 = f_4;\quad f_{13} = f_{10};\quad f_{15} = f_{12};\quad f_{23} = f_{17}$$
$$f_{26} = f_{21};\quad f_{27} = f_{22};\quad f_{29} = f_{18};\quad f_{30} = f_{20};\quad f_{31} = f_{19}$$

Using this we obtain a reduced system where

$$f_{11} = f_1;\quad f_{16} = f_{10};\quad f_{18} = f_1;\quad f_{19} = f_{12}$$
$$f_{20} = f_3;\quad f_{21} = f_{17};\quad f_{28} = f_{25}$$

Further, removing the spare functions the system is transformed into

$$H_6(k+2) \stackrel{def}{=} f_0^k = f_1^k + 2f_3^k + f_5^k + f_9^k$$

$$f_1^k = f_1^{k-1} + 2f_3^{k-1} + f_9^{k-1}$$

$$f_2^k = f_1^{k-1} + f_2^{k-1} + 2f_{10}^{k-1} + 2f_{12}^{k-1} + f_{14}^{k-1}$$

$$f_3^k = f_1^{k-1} + f_3^{k-1} + f_4^{k-1} + f_{10}^{k-1} + f_{12}^{k-1} + f_{17}^{k-1}$$

$$f_4^k = f_1^{k-1} + f_2^{k-1} + 2f_{10}^{k-1} + 2f_{12}^{k-1} + f_{14}^{k-1} + f_{17}^{k-1} + f_{22}^{k-1}$$
$$f_5^k = f_{10}^{k-1} + 2f_{17}^{k-1} + f_{24}^{k-1} + f_{25}^{k-1}$$
$$f_6^k = 2f_{10}^{k-1} + f_{14}^{k-1} + 2f_{17}^{k-1} + 2f_{22}^{k-1} + f_{25}^{k-1}$$
$$f_9^k = f_1^{k-1} + f_2^{k-1} + 2f_{10}^{k-1} + 2f_{12}^{k-1} + f_{14}^{k-1} + 2f_{17}^{k-1} + 2f_{22}^{k-1} + f_{25}^{k-1}$$
$$f_{10}^k = f_1^{k-1} + 2f_3^{k-1} + f_5^{k-1} + f_9^{k-1}$$
$$f_{12}^k = f_1^{k-1} + f_3^{k-1} + f_4^{k-1} + f_{10}^{k-1} + f_{12}^{k-1}$$
$$f_{14}^k = 2f_1^{k-1} + 2f_3^{k-1} + 2f_4^{k-1} + f_6^{k-1} + f_{32}^{k-1}$$
$$f_{17}^k = f_3^{k-1} + f_5^{k-1} + f_9^{k-1}$$
$$f_{22}^k = f_3^{k-1} + f_4^{k-1} + f_6^{k-1}$$
$$f_{24}^k = f_5^{k-1} + f_9^{k-1}$$
$$f_{25}^k = f_6^{k-1}$$
$$f_{32}^k = 2f_1^{k-1} + 2f_3^{k-1} + 2f_4^{k-1} + f_{32}^{k-1}$$

i.e.

$$f_0^k = f_1^k + 2f_3^k + f_5^k + f_9^k$$
$$f_1^k = f_0^{k-1} - f_5^{k-1}$$
$$f_2^k = f_1^{k-1} + f_2^{k-1} + 2f_{10}^{k-1} + 2f_{12}^{k-1} + f_{14}^{k-1}$$
$$f_3^k = f_1^{k-1} + f_3^{k-1} + f_4^{k-1} + f_{10}^{k-1} + f_{12}^{k-1} + f_{17}^{k-1}$$
$$f_4^k = f_2^k + f_{17}^{k-1} + f_{22}^{k-1}$$
$$f_5^k = f_{10}^{k-1} + 2f_{17}^{k-1} + f_{24}^{k-1} + f_{25}^{k-1}$$
$$f_6^k = 2f_{10}^{k-1} + f_{14}^{k-1} + 2f_{17}^{k-1} + 2f_{22}^{k-1} + f_{25}^{k-1}$$
$$f_9^k = f_2^k + f_6^k - 2f_{10}^{k-1} - f_{14}^{k-1}$$
$$f_{10}^k = f_0^{k-1}$$
$$f_{12}^k = f_3^k - f_{17}^{k-1}$$
$$f_{14}^k = 2f_1^{k-1} + 2f_3^{k-1} + 2f_4^{k-1} + f_6^{k-1} + f_{32}^{k-1}$$
$$f_{17}^k = f_3^{k-1} + f_5^{k-1} + f_9^{k-1}$$
$$f_{22}^k = f_3^{k-1} + f_4^{k-1} + f_6^{k-1}$$

$$f_{24}^k = f_{17}^k - f_3^{k-1}$$
$$f_{25}^k = f_6^{k-1}$$
$$f_{32}^k = f_{14}^k - f_6^{k-1}$$

Removing f_1, f_{10} and f_{25} we obtain

$$f_0^k - f_0^{k-1} = 2f_3^k + f_5^k - f_5^{k-1} + f_9^k$$
$$f_2^k - f_2^{k-1} = 3f_0^{k-2} - f_5^{k-2} + 2f_{12}^{k-1} + f_{14}^{k-1}$$
$$f_3^k - f_3^{k-1} = 2f_0^{k-2} + f_4^{k-1} - f_5^{k-2} + f_{12}^{k-1} + f_{17}^{k-1}$$
$$f_4^k = f_2^k + f_{17}^{k-1} + f_{22}^{k-1}$$
$$f_5^k = f_0^{k-2} + f_6^{k-2} + 2f_{17}^{k-1} + f_{24}^{k-1}$$
$$f_6^k - f_6^{k-2} = 2f_0^{k-2} + f_{14}^{k-1} + 2f_{17}^{k-1} + 2f_{22}^{k-1}$$
$$f_9^k = -2f_0^{k-2} + f_2^k + f_6^k - f_{14}^{k-1}$$
$$f_{12}^k = f_3^k - f_{17}^{k-1}$$
$$f_{14}^k = 2f_0^{k-2} + 2f_3^{k-1} + 2f_4^{k-1} - 2f_5^{k-2} + f_6^{k-1} + f_{32}^{k-1}$$
$$f_{17}^k = f_3^{k-1} + f_5^{k-1} + f_9^{k-1}$$
$$f_{22}^k = f_3^{k-1} + f_4^{k-1} + f_6^{k-1}$$
$$f_{24}^k = -f_3^{k-1} + f_{17}^k$$
$$f_{32}^k = -f_6^{k-1} + f_{14}^k$$

Removing f_4, f_9, f_{12}, f_{24} and f_{32} we obtain

$$f_0^k - f_0^{k-1} + 2f_0^{k-2} = f_2^k + 2f_3^k + f_5^k - f_5^{k-1} + f_6^k - f_{14}^{k-1} \tag{11}$$
$$f_2^k - f_2^{k-1} = 3f_0^{k-2} + 2f_3^{k-1} - f_5^{k-2} + f_{14}^{k-1} - 2f_{17}^{k-2} \tag{12}$$
$$f_3^k - 2f_3^{k-1} = 2f_0^{k-2} + f_2^{k-1} - f_5^{k-2} + f_{17}^{k-1} + f_{22}^{k-2} \tag{13}$$
$$f_5^k = f_0^{k-2} - f_3^{k-2} + f_6^{k-2} + 3f_{17}^{k-1}$$
$$f_6^k - f_6^{k-2} = 2f_0^{k-2} + f_{14}^{k-1} + 2f_{17}^{k-1} + 2f_{22}^{k-1}$$
$$f_{14}^k - f_{14}^{k-1} = 2f_0^{k-2} + 2f_2^{k-1} + 2f_3^{k-1} - 2f_5^{k-2} + f_6^{k-1} - f_6^{k-2} + 2f_{17}^{k-2} + 2f_{22}^{k-2} \tag{14}$$
$$f_{17}^k = -2f_0^{k-3} + f_2^{k-1} + f_3^{k-1} + f_5^{k-1} + f_6^{k-1} - f_{14}^{k-2} \tag{15}$$
$$f_{22}^k - f_{22}^{k-2} = f_2^{k-1} + f_3^{k-1} + f_6^{k-1} + f_{17}^{k-2} \tag{16}$$

We are going to remove f_2. From (11) and (13) we obtain

$$(17) \qquad f_2^k = f_0^k - f_0^{k-1} + 2f_0^{k-2} - 2f_3^k - f_5^k + f_5^{k-1} - f_6^k + f_{14}^{k-1}$$

$$(18) \qquad f_2^{k-1} = f_3^k - 2f_3^{k-1} - 2f_0^{k-2} + f_5^{k-2} - f_{17}^{k-1} - f_{22}^{k-2}$$

and using (17) and (18) in (12)

$$(19)\ 3f_3^k = f_0^k - f_0^{k-1} + f_0^{k-2} - f_5^k + f_5^{k-1} - f_6^k + f_{17}^{k-1} + 2f_{17}^{k-2} + f_{22}^{k-2}$$

Further, using (17) in (18) we obtain (20); using (20) in (19) we obtain (21); using (18) in (14), (15), (16) we obtain (24), (25), (26).

So, our system is transformed into

$$(20) \qquad f_3^k = f_0^{k-1} + f_0^{k-2} + 2f_0^{k-3} - f_5^{k-1} - f_6^{k-1} + f_{14}^{k-2} + f_{17}^{k-1} + f_{22}^{k-2}$$

$$(21) \qquad - f_0^k + 4f_0^{k-1} + 2f_0^{k-2} + 6f_0^{k-3} =$$

$$= -f_5^k + 4f_5^{k-1} - f_6^k + 3f_6^{k-1} - 3f_{14}^{k-2} - 2f_{17}^{k-1} + 2f_{17}^{k-2} - 2f_{22}^{k-2}$$

$$(22) \qquad f_5^k = f_0^{k-2} - f_3^{k-2} + f_6^{k-2} + 3f_{17}^{k-1}$$

$$(23) \qquad f_6^k - f_6^{k-2} = 2f_0^{k-2} + f_{14}^{k-1} + 2f_{17}^{k-1} + 2f_{22}^{k-1}$$

$$(24) \qquad f_{14}^k - f_{14}^{k-1} = -2f_0^{k-2} + 2f_3^k - 2f_3^{k-1} + f_6^{k-1} - f_6^{k-2} - 2f_{17}^{k-1} + 2f_{17}^{k-2}$$

$$(25) \quad f_{17}^k + f_{17}^{k-1} = -2f_0^{k-2} - 2f_0^{k-3} + f_3^k - f_3^{k-1} + f_5^{k-1} + f_5^{k-2} + f_6^{k-1} - f_{14}^{k-2} - f_{22}^{k-2}$$

$$(26) \qquad f_{22}^k = -2f_0^{k-2} + f_3^k - f_3^{k-1} + f_5^{k-2} + f_6^{k-1} - f_{17}^{k-1} + f_{17}^{k-2}$$

We are going to remove f_5.

From (22), (20) and (26) we obtain respectively

$$(27) \qquad f_5^k = f_0^{k-2} - f_3^{k-2} + f_6^{k-2} + 3f_{17}^{k-1}$$

$$(28) \qquad f_5^{k-1} = f_0^{k-1} + f_0^{k-2} + 2f_0^{k-3} - f_3^k - f_6^{k-1} + f_{14}^{k-2} + f_{17}^{k-1} + f_{22}^{k-2}$$

$$(29) \qquad f_5^{k-2} = 2f_0^{k-2} - f_3^k + f_3^{k-1} - f_6^{k-1} + f_{17}^{k-1} - f_{17}^{k-2} + f_{22}^k$$

Using (27) and (28) we obtain (30); using (28) and (29) we obtain (31); using (27) and (28) in (21) we obtain (32); using (28) and (29) in (25) we obtain (35) so, the above system is transformed into:

$$(30) \quad f_{22}^{k-2} = -f_0^{k-1} - f_0^{k-2} - f_0^{k-3} + f_3^k - f_3^{k-3} + f_6^{k-1} + f_6^{k-3} - f_{14}^{k-2} - f_{17}^{k-1} + 3f_{17}^{k-2}$$

$$(31) \qquad f_0^{k-2} - f_0^{k-3} - 2f_0^{k-4} =$$

$$f_3^k - 2f_3^{k-1} + f_6^{k-1} - f_6^{k-2} + f_{14}^{k-3} - f_{17}^{k-1} + 2f_{17}^{k-2} - f_{22}^k + f_{22}^{k-3}$$

$$(32)\quad f_0^k + f_0^{k-2} + 2f_0^{k-3} = 4f_3^k - f_3^{k-2} + f_6^k + f_6^{k-1} + f_6^{k-2} - f_{14}^{k-2} + f_{17}^{k-1} - 2f_{17}^{k-2} - 2f_{22}^{k-2}$$

$$(33)\quad f_6^k - f_6^{k-2} = 2f_0^{k-2} + f_{14}^{k-1} + 2f_{17}^{k-1} + 2f_{22}^{k-1}$$

$$(34)\quad f_{14}^k - f_{14}^{k-1} = -2f_0^{k-2} + 2f_3^k - 2f_3^{k-1} + f_6^{k-1} - f_6^{k-2} - 2f_{17}^{k-1} + 2f_{17}^{k-2}$$

$$(35)\quad f_{17}^k - f_{17}^{k-1} + f_{17}^{k-2} = f_0^{k-1} + f_0^{k-2} - f_3^k - f_6^{k-1} + f_{22}^k$$

Now, we are going to remove f_{14}.

From (33) and (30) we obtain

$$(36)\quad f_{14}^{k-1} = -2f_0^{k-2} + f_6^k - f_6^{k-2} - 2f_{17}^{k-1} - 2f_{22}^{k-1}$$

and

$$(37)\quad f_{14}^{k-2} = -f_0^{k-1} - f_0^{k-2} - f_0^{k-3} + f_3^k - f_3^{k-3} + f_6^{k-1} + f_6^{k-3} - f_{17}^{k-1} + 3f_{17}^{k-2} - f_{22}^{k-2}$$

Using (36) in (34) we obtain

$$(38)\quad f_{14}^k = -4f_0^{k-2} + 2f_3^k - 2f_3^{k-1} + f_6^k + f_6^{k-1} - 2f_6^{k-2} + 2f_{17}^{k-2} - 4f_{17}^{k-1} - 2f_{22}^{k-1}$$

From (36) and (37) we obtain (39); from (36) and (38) we obtain (40); from (35) we obtain (41). Substituing (37) (taking $(k-1)$ instead of k) into (31) we obtain (42); and using (37) into (32) we obtain (43).

$$(39)\quad f_{22}^{k-2} = f_0^{k-1} + f_0^{k-2} - f_0^{k-3} - f_3^k + f_3^{k-3} - 2f_6^{k-3} + f_{17}^{k-1} - 5f_{17}^{k-2}$$

$$(40)\quad 2f_{22}^{k-1} - 2f_{22}^{k-2} = +$$

$$-2f_0^{k-2} + 4f_0^{k-3} - 2f_3^{k-1} + 2f_3^{k-2} + f_6^k - f_6^{k-1} - 2f_6^{k-2} + 2f_6^{k-3} - 2f_{17}^{k-1} + 4f_{17}^{k-2} - 2f_{17}^{k-3}$$

$$(41)\quad f_{22}^k = -f_0^{k-1} - f_0^{k-2} + f_3^k + f_6^{k-1} + f_{17}^k - f_{17}^{k-1} + f_{17}^{k-2}$$

$$(42)\quad f_{22}^k = -2f_0^{k-2} + f_0^{k-4} + f_3^k - f_3^{k-1} - f_3^{k-4} + f_6^{k-1} + f_6^{k-4} - f_{17}^{k-1} + f_{17}^{k-2} + 3f_{17}^{k-3}$$

$$(43)\quad f_{22}^{k-2} = -f_0^k + f_0^{k-1} - f_0^{k-3} + 3f_3^k - f_3^{k-2} + f_3^{k-3} + f_6^k + f_6^{k-2} - f_6^{k-3} + 2f_{17}^{k-1} - 5f_{17}^{k-2}$$

We are going to remove f_{22}. Using (39) in (40) we obtain (44)

$$(44)\quad 2f_{22}^{k-1} = 2f_0^{k-1} + 2f_0^{k-3} -$$

$$2f_3^k - 2f_3^{k-1} + 2f_3^{k-2} + 2f_3^{k-3} + f_6^k - f_6^{k-1} - 2f_6^{k-2} - 2f_6^{k-3} - 6f_{17}^{k-2} - 2f_{17}^{k-3}$$

Using (41) (taking $(k-1)$ instead of k) in (44) we obtain (47); using (39) and (44) (taking $(k-1)$ instead of k) we obtain (46).

From (41) and (42) we obtain (48); from (39) and (43) we obtain (45).

(45) $$f_0^k + f_0^{k-2} = 4f_3^k - f_3^{k-2} + f_6^k + f_6^{k-2} + f_6^{k-3} + f_{17}^{k-1}$$

(46) $$2f_3^k - 2f_3^{k-1} - 2f_3^{k-2} + 2f_3^{k-4} =$$

$$2f_0^{k-1} - 2f_0^{k-3} - 2f_0^{k-4} - f_6^{k-1} + f_6^{k-2} - 2f_6^{k-3} + 2f_6^{k-4} + 2f_{17}^{k-1} - 10f_{17}^{k-2} + 6f_{17}^{k-3} + 2f_{17}^{k-4}$$

(47) $$f_6^k - f_6^{k-1} - 4f_6^{k-2} - 2f_6^{k-3} =$$

$$-2f_0^{k-1} - 2f_0^{k-2} - 4f_0^{k-3} + 2f_3^k + 4f_3^{k-1} - 2f_3^{k-2} - 2f_3^{k-3} + 2f_{17}^{k-1} + 4f_{17}^{k-2} + 4f_{17}^{k-3}$$

(48) $$f_{17}^k - 3f_{17}^{k-3} = f_0^{k-1} - f_0^{k-2} + f_0^{k-4} - f_3^{k-1} - f_3^{k-4} + f_6^{k-4}$$

Using the given Lemma we can remove function f_{17} from (45), (46) and (47) using (48)

(49) $$f_0^k - 2f_0^{k-3} - 4f_0^{k-5} =$$

$$4f_3^k - 2f_3^{k-2} - 12f_3^{k-3} + 2f_3^{k-5} + f_6^k + f_6^{k-2} - 2f_6^{k-3} - 2f_6^{k-5} - 3f_6^{k-6}$$

(50) $$2f_3^k - 2f_3^{k-1} - 16f_3^{k-3} + 14f_3^{k-4} + 10f_3^{k-5} - 10f_3^{k-6} + 2f_3^{k-8} =$$

$$2f_0^{k-1} + 2f_0^{k-2} - 14f_0^{k-3} + 8f_0^{k-4} - 2f_0^{k-5} - 6f_0^{k-6} + 12f_0^{k-7} + 2f_0^{k-8}$$
$$-f_6^{k-1} + f_6^{k-2} - 2f_6^{k-3} + 5f_6^{k-4} - f_6^{k-5} - 4f_6^{k-6} + 2f_6^{k-8}$$

(51) $$f_6^k - f_6^{k-1} - 4f_6^{k-2} - 5f_6^{k-3} + 3f_6^{k-4} + 10f_6^{k-5} + 2f_6^{k-6} - 4f_6^{k-7} =$$

$$-2f_0^{k-1} - 2f_0^{k-3} + 6f_0^{k-4} + 4f_0^{k-5} + 16f_0^{k-6} + 4f_0^{k-7} +$$
$$+2f_3^k + 4f_3^{k-1} - 4f_3^{k-2} - 12f_3^{k-3} - 16f_3^{k-4} + 4f_3^{k-5} + 2f_3^{k-6} - 4f_3^{k-7}$$

Using Lemma again, but removing f_6 in (49) and (50) using (51) the above system is transformed into

$$f_0^k + f_0^{k-1} - 4f_0^{k-2} - 3f_0^{k-3} - 5f_0^{k-4} + 12f_0^{k-5} - 14f_0^{k-6} + 4f_0^{k-7}$$

$$-12f_0^{k-8} + 18f_0^{k-9} + 2f_0^{k-10} + 36f_0^{k-11} + 72f_0^{k-12} + 12f_0^{k-13} =$$

$$6f_3^k - 20f_3^{k-2} - 42f_3^{k-3} + 4f_3^{k-4} + 96f_3^{k-5} + 56f_3^{k-6} - 52f_3^{k-7}$$

$$-104f_3^{k-8} + 50f_3^{k-9} + 116f_3^{k-10} - 12f_3^{k-11} - 6f_3^{k-12} + 12f_3^{k-13}$$

$$2f_3^k - 2f_3^{k-1} - 4f_3^{k-2} - 22f_3^{k-3} + 36f_3^{k-4} + 44f_3^{k-5} + 16f_3^{k-6}$$

$$-122f_3^{k-7} - 68f_3^{k-8} + 102f_3^{k-9} + 84f_3^{k-10} - 84f_3^{k-11} - 18f_3^{k-12} + 36f_3^{k-13} =$$

$$2f_0^{k-1} + 2f_0^{k-2} - 26f_0^{k-3} + 10f_0^{k-4} + 24f_0^{k-5} + 68f_0^{k-6} - 58f_0^{k-7}$$

$$-60f_0^{k-8}+8f_0^{k-9}+10f_0^{k-10}-86f_0^{k-11}+66f_0^{k-12}+60f_0^{k-13}-12f_0^{k-14}$$

Using again our Lemma we obtain the recurrence relation (7) for the numbers f_0^k i.e. $H_6(n)$.

5. New Results

Some new results and numbers of vertices in difraph D_m for some values of m are given below.

$\mid V(D_m)\mid$ - **the number of vertices in** D_m

$H_m(n)$ - **the number of Hamiltonian cycles in** $P_m \times P_n$

n	$H_7(n)$
2	* 1
4	* 92
6	* 5320
8	* 301384
10	17066492
12	966656134
14	54756073582
16	3101696069920
18	175698206778318
20	9952578156814524
22	563772503196695338
24	31935387285412942410
26	1809007988782552388490
28	102472842263117124008066
30	5804663918990466729365476

m	$\mid V(D_m)\mid$
3	3
4	6
5	19
6	32
7	113
8	182
9	706

n	$H_6(n)$	$H_8(n)$
2	* 1	* 1
3	* 4	* 8
4	* 37	* 236
5	* 154	* 1696
6	* 1072	* 32675
7	* 5320	* 301384
8	* 32675	* 4638576
9	* 175294	49483138
10	1024028	681728204
11	5668692	7837276902
12	32463802	102283239429
13	181971848	1220732524976
14	1033917350	15513067188008
15	5824476298	188620289493918
16	32989068162	2365714170297014
17	186210666468	29030309635705054
18	1053349394128	361749878496079778
19	5950467515104	4459396682866920534
20	33643541208290	55391169255983979555
21	190115484271760	684363209103066303906
22	1074685815276400	8487168277379774266411
23	6073680777522430	104976660007043902770814
24	34330607094625734	1300854247070195164448395
25	194032156833259734	16098959403506801921858124
26	1096704136430950646	199418506963731877069653608
27	6198554011846307000	2468612432237087475265791106
28	35034883701169366742	30572953033472980838613625389
29	198018172380783203690	378515201134457658578140498814
30	1119214052513009716324	4687342384540802154353083423651

* - the old results given in [3]

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