Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 24, 1 (1994), 91-102

Review of Research
Faculty of Science
Mathematics Series

THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATION IN THE FIELD OF MIKUSINSKI OPERATORS

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Abstract

In this paper we construct the numerical solution for a class of differential equations in the field of Mikusinski operators, $\mathcal{F}$, which corresponds to a class of partial differential equations with nonconstant coefficients. This solution is obtained from the discrete analoques, for special classes of the considered problem, using the good algebraic properties (see [1]), of the field $\mathcal{F}$.

AMS Mathematics Subject Classification (1991): 65J10, 47A60.
Key words and phrases: Mikusiński operator calculus, difference equations.

1. Introduction

In this paper we consider the following linear partial differential equation

$$(1)\quad -\frac{\partial^{2+p}u(\lambda,t)}{\partial t^p\partial\lambda^2}+\sum_{i=0}^{m}A_i(\lambda)\frac{\partial^{1+i}u(\lambda,t)}{\partial t^i\partial\lambda}+\sum_{i=0}^{r}B_i(\lambda)\frac{\partial^i u(\lambda,t)}{\partial t^i}=f_1(\lambda,t),$$

with the appropriate conditions

$$(2)\qquad \frac{\partial^{\mu+\nu}u(\lambda,0)}{\partial\lambda^{\mu}\partial t^{\nu}}=0;\qquad \begin{array}{l}\mu=0;\nu=0,1,...,r-1,\\ \mu=1;\nu=0,1,...,m-1,\\ \mu=2;\nu=0,1,...,p-1,\end{array}$$

$$(3)\qquad x(0,t)=C(t),\qquad x(1,t)=D(t),$$

where $t\geq 0$, $\lambda\in[0,1]$, $p,m,r\in N$, $p\geq m, p\geq r$, $A_i(\lambda), i=0,...,m$, $B_i(\lambda), i=0,...,n$, are real valued continuous functions of one variable, while $f_1(\lambda,t)$ is a continuous real valued function of two variables. $C(t)$ and $D(t)$ are continuous functions for $t\geq 0$.

The ring of continuous functions $\mathcal{C}$ (or locally integrable functions $\mathcal{L}$) with usual addition and multiplication given by the convolution

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau,$$

where f and g are from $\mathcal{C}$ (or from $\mathcal{L}$), has no divisors of zero, hence its quotient field can be defined. The elements of this field, called the field of Mikusinski operators $\mathcal{F}$, are of the form

$$\frac{f}{g},$$

where this division is observed in the sense of convolution. The most important operators are the integral operator l and its inverse operator, the differential operator s, while I is the identical operator. It holds

$$ls=I,\qquad l^{\alpha}=\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\},\alpha>0$$

$$\{x^{(n)}(t)\}=s^n x-s^{n-1}x'(0)-...-x^{(n-1)}(0)I.$$

In the field of Mikusinski operators $\mathcal{F}$ equation (1) with conditions (2) and (3) corresponds to the problem

$$(4)\qquad -s^p u''(\lambda)+\sum_{i=0}^{m}A_i(\lambda)s^i u'(\lambda)+\sum_{i=0}^{r}B_i(\lambda)s^i u(\lambda)=f_2(\lambda)$$

$$(5)\qquad u(0)=C,\qquad u(1)=D,$$

where s is the differential operator, $A_i(\lambda), B_i(\lambda), f_2(\lambda) = \{f_1(\lambda,t)\}$ are continuous operator functions, while $C = \{C(t)\}$ and $D = \{D(t)\}$ are operators.

Multiplying equation (4) with l^p we obtain the following equation

$$(6) \qquad -u''(\lambda) + P(\lambda)u'(\lambda) + Q(\lambda)u(\lambda) = f(\lambda)$$

with

$$(7) \qquad P(\lambda) = \sum_{i=0}^{m} A_i(\lambda)l^{p-i}, \qquad Q(\lambda) = \sum_{i=0}^{r} B_i(\lambda)l^{p-i}.$$

and

$$f(\lambda) = l^p f_2(\lambda)$$

It is obvious that if $p > m$ and $p > r$, then $P(\lambda)$ and $Q(\lambda)$ represent continuous functions of two variables.

If $p = m = r$, then one can write

$$(8) \qquad P(\lambda) = A_m(\lambda)I + \sum_{i=0}^{m-1} A_i(\lambda)l^{p-i} = A_m(\lambda)I + P_1(\lambda)$$

and

$$(9) \qquad Q(\lambda) = B_r(\lambda)I + \sum_{i=0}^{r-1} B_i(\lambda)l^{p-i} = B_r(\lambda)I + Q_1(\lambda),$$

where $P_1(\lambda)$, and $Q_1(\lambda)$ are operator functions representing continuous functions of two variables.

Let us suppose that $P(\lambda), Q(\lambda)$, (or $P_1(\lambda), Q_1(\lambda)$) and $f(\lambda)$ are operator functions which have continuous its second derivatives and the solution $u(\lambda)$ has a continuous fourth derivative (see[2]). Let us use the following notations:

$$h = \frac{1}{n+1}, \quad n \in N, \quad \lambda_j = jh, \quad j = 1, 2, \dots n+1,$$

$$Q_j = Q(\lambda_j), \quad P_j = P(\lambda_j), \quad u_j = u(\lambda_j), \quad f_j = f(\lambda_j).$$

As is usual in numerical analysis, we shall use the following difference quotients

$$(10) \qquad -u''(\lambda) = \frac{-u(\lambda-h) + 2u(\lambda) - u(\lambda+h)}{h^2} + \frac{h^2 u^{(4)}(\xi_1)}{12},$$

$$u'(\lambda) = \frac{u(\lambda+h) - u(\lambda-h)}{2h} - \frac{h^2 u"(\xi_2)}{6}, \tag{11}$$

$$u'(\lambda) = \frac{u(\lambda) - u(\lambda-h)}{h} - \frac{h u"(\xi_3)}{2}, \tag{12}$$

$$u'(\lambda) = \frac{u(\lambda+h) - u(\lambda)}{h} - \frac{h u"(\xi_4)}{2}, \tag{13}$$

where $\xi_1 \in (\lambda - h, \lambda + h), \xi_2 \in (\lambda - h, \lambda + h), \xi_3 \in (\lambda - h, \lambda + h)$ and $\xi_4 \in (\lambda - h, \lambda + h)$, in order to obtain an appropriate discrete analogues.

2. Discrete anlogoue I

Taking the expressions given by relations (10), (11) instead of the $u"$ and u' (without reminder) we obtain the following discrete analogue

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + P_j \frac{v_{j+1} - v_{j-1}}{2h} + Q_j v_j = f_j, \qquad j = 1, ..., n, \tag{14}$$

$$v_0 = C, \qquad v_{n+1} = D. \tag{15}$$

Remark. In the system (14) and (15) we take the approximations of the solution u denoted by v (as is usual in numerical analysis).

The system (14) and (15) can be expressed as

$$a_j v_{j-1} + b_j v_j + c_j v_{j+1} = f_j \quad v_0 = C, \quad v_{n+1} = D, \tag{16}$$

where $a_j, b_j, c_j, j = 1, .., n$ are operators having the form

$$a_j = -\frac{1}{h^2}(I + \frac{hP_j}{2}) \tag{17}$$

$$b_j = \frac{1}{h^2}(2I + h^2 Q_j) \tag{18}$$

$$c_j = -\frac{1}{h^2}(I - \frac{hP_j}{2}.) \tag{19}$$

The field of Mikusinski operators has very good algebraic properties (see[2]), so for obtaining operators v_j, $j = 1, 2, ...n$, one can use the same methods from numerical analysis which are applicable in similar situations (see [3]), when a_j, b_j, and c_j are numerical constants.

The system (16) can be written in the form

$$Av = d \tag{20}$$

where A, v, d are corresponding matrices in the field $\mathcal{F}$

$$A = \begin{bmatrix} b_1 & c_1 & . & . & 0 & 0 \\ a_2 & b_2 & c_2 & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & a_{n-1} & b_{n-1} & c_{n-1} \\ . & . & . & . & a_n & b_n \end{bmatrix}, \tag{21}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_{n-1} \\ v_n \end{bmatrix}, \qquad d = \begin{bmatrix} f_1 - a_1 C \\ f_2 \\ . \\ . \\ . \\ f_{n-1} \\ f_n - c_n D \end{bmatrix}. \tag{22}$$

I. First, we shall consider such classes of equation (1) with conditions (2) and (3) where $p > m$ and $p > r$.

Theorem 1. *If matrix A and vector d have the forms given by relations (21), and (22), respectively, then the system $Av = d$ (given by relation(16)) has unique solution given by*

$$v_n = \frac{r_n}{\alpha_n}, \qquad v_j = \frac{r_j - c_j v_{j+1}}{\alpha_j}, \qquad j = n-1, n-2, ..., 1, \tag{23}$$

where

$$\alpha_1 = b_1, \quad \beta_j = \frac{a_j}{\alpha_{j-1}}, \qquad \alpha_j = b_j - \beta_j c_{j-1}, \quad j = 2, 3, ...n, \tag{24}$$

and

$$r_1 = d_1, \qquad r_j = d_j - \beta_j r_{j-1}, \quad j = 2, ...n. \tag{25}$$

Proof. It is obvious that there exist matrices L and U such that $A = LU$,where

$$L = \begin{bmatrix} 1 & . & . & . & . & . & 0 \\ \beta_2 & 1 & . & . & . & . & 0 \\ 0 & \beta_3 & 1. & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & \beta_n & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} \alpha_1 & c_1 & . & . & . & . & 0 \\ 0 & \alpha_2 & c_2 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & \alpha_{n-1} & c_{n-1} \\ 0 & . & . & . & . & . & \alpha_n \end{bmatrix}$$

where $\alpha_j, j = 1,...,$ and $\beta_j, j = 2,...,n$, has the forms which are given in relation (17).

From relation (20) in the field $\mathcal{F}$ it follows

$$\alpha_1 = b_1 = h^{-2}(2I + h^2Q_1) = \gamma_1 I + \alpha_{1,c}$$

$$\beta_2 = \frac{a_2}{\alpha_1} = -\frac{I + \frac{hP_2}{2}}{2I + h^2Q_1} = -\frac{1}{2}(I + \frac{hP_2}{2})\sum_{i=0}^{\infty}(-1)^i(\frac{h^2Q_1}{2})^i.$$

Since, for $p > m, p > r$ operators $P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n$ represent continuous functions, then the infinite series in the previous relation is an operationally convergent series (see [1], pp 180), so we can write

$$\beta_2 = \delta_2 I + \beta_{2,c},$$

where δ_2 is a numerical constant and $\beta_{2,c}$ is an operator representing continuous function. Also, it holds

$$\alpha_2 = b_2 - \beta_2 c_1 = h^{-2}((2I + h^2Q_2) + (\delta_2 I + \beta_{2,c})(I - \frac{hP_1}{2}))$$

$$\alpha_2 = h^{-2}(\gamma_2 I + \alpha_{2,c})$$

$$\beta_3 = \frac{a_3}{\alpha_2} = -\frac{I + \frac{hP_3}{2}}{\gamma_2(I + \frac{\alpha_{2,c}}{\gamma_2})} = \delta_3 I + \beta_{3,c}$$

and finally

$$(26) \qquad \alpha_j = (\gamma_j I + \alpha_{j,c}), \qquad \beta_j = \delta_j I + \beta j, c \qquad j = 3, ..n.$$

where $\gamma_j, j = 1, ...n$ and $\delta_j, j = 2, ...n$, are numerical constants and $\alpha_{j,c}, j = 1, ...n$, $\beta_{j,c}, j = 2, ...n$, are operators which represent continuous functions.

It is clear that in this case $\gamma_1 = 2h^{-2} \neq 0$ so $\delta_2 = -1/2 \neq 0$, and using the mathematical induction it can be proved easily, that

$$\gamma_k = \frac{k+1}{k} h^{-2}, \qquad \delta_k = -\frac{k-1}{k}, \qquad k = 2, ...n$$

This means that $\gamma_j \neq 0, j = 1, ...n$ and consequently from relation (26) it follows $\alpha_j \neq 0, j = 1, ...n$.

As in linear algebra, in this case we can say that the matrix A is a regular matrix. Using the trivial algebraic operations one can easily prove that the unique solution of this system has the form given in relations (23),(24) and (25).

Now, we can prove the following

Corollary 1. *The numerical solution of the differential equation (4) with conditions (5), with $p > m, p > r$ denoted by $v_k, k = 1, ...n$, obtained as the solutions of the algebraic system given by relation (16) are operators from $\mathcal{F}$ representing continuous functions.*

Proof. Since the operators $C, D, P_1, P_2, ..., P_n, Q_1, Q_2, ..., Q_n$ and $l^p f_j, j = 1, ...n$, are operators representing continuous functions, then $r_1 = d_1$ represents a continuous function, too. If we suppose that each operator $r_j, j = 2, .., k-1$, represents a continuous function, then from the relation

$$r_k = d_k - (\delta_k I + \beta_{k,c}) r_{k-1}$$

it follows that r_k is also an operator which represents a continuous function.

So, we can write

$$v_n = \frac{1}{\gamma_n} r_n \sum_{i=o}^{\infty} (\frac{\alpha_{1,n}}{\gamma_n})^i$$

and also

$$v_j = \frac{1}{\gamma_j}(r_j - c_j v_{j+1}) \sum_{i=0}^{\infty} (\frac{\alpha_{1,j}}{\gamma_j})^i, \quad j = n - 1, ..., 1.$$

Therefore we can conclude that v_j, $j = 1, 2, ...n$ are operators from $\mathcal{F}$ are representing continuous functions.

The functions $v_j(t)$ such that $v_j = \{v_j(t)\}, j = 1, ...n$, we call the numerical solutions of the partial differential equation (1) with conditions (2) and (3) (in the plains $\lambda_k = \frac{1}{n+1}k$).

Remark. In the case when $p > m$, $p > r$ we do not need any more supposition about the functions $A_j, j = 1, 2, ..., m$ and $B_j, j = 1, 2, ..., r$ except those that make operator functions $A(\lambda)$ and $B(\lambda)$ continuous operator functions.

II. Now, let us consider differential equations of the form (4) with conditions (5) where in $P(\lambda)$ and $Q(\lambda)$ it holds that $p = m, p = r$. In that case the coefficients in the system (16) a_j, b_j and c_j can be written as

$$a_j = a_j^1 I + P_j^1, \qquad a_j^1 = -\frac{1}{h^2}(1 + \frac{hA_m(\lambda_j)}{2}), \tag{27}$$

$$b_j = b_j^1 I + Q_j^1, \qquad b_j^1 = \frac{1}{h^2}(2 + h^2 B_r(\lambda_j)) \tag{28}$$

$$c_j = c_j^1 I - P_j^1, \qquad c_j^1 = -\frac{1}{h^2}(1 - \frac{hA_m(\lambda_j)}{2}), \tag{29}$$

where a_j^1, b_j^1 and c_j^1 are numerical factors and P_j^1 and Q_j^1 are operators representing continuous functions.

Previous three relations are giving the form of the elements of matrix A given by (21). In order to obtain the solution of the system (14) and (15) in this case we need some suppositions about $P(\lambda)$ and $Q(\lambda)$.

Theorem 2. *If in relations (8) and (9) ($p = m$. $p = r$) the numerical factors A_m and B_r satisfy the following inequalities*

$$B_r(\lambda) > 0, \quad \frac{hA_m(\lambda)}{2} < 1, \quad \lambda \in [0, 1],$$

Then the system $Av = \mathbf{d}$ (given by (20), where the elements of matrix A satisfy the conditions (27), (28), and (29), and vector $\mathbf{d}$ has the form given in relation (22), has a unique solution given by

$$v_n = \frac{r_n}{\alpha_n}, \qquad v_j = \frac{r_j - c_j v_{j+1}}{\alpha_j}, \qquad j = n-1, n-2, ..., 1, \tag{30}$$

where

$$(31) \qquad \alpha_1 = b_1, \quad \beta_j = \frac{a_j}{\alpha_{j-1}}, \qquad \alpha_j = b_j - \beta_j c_{j-1}, \quad j = 2, 3, ...n,$$

and

$$(32) \qquad r_1 = d_1, \qquad r_j = d_j - \beta_j r_{j-1}, \quad j = 2, ...n.$$

Proof. Let us remark that in this case the solution of the system given by relation (20) has the same form as in Theorem 1. So ,again we have to prove that $\alpha_j \neq 0$. Using that

$$B_r(\lambda) > 0, \quad \text{and} \quad \frac{A_m(\lambda}{2} < 1$$

we have

$$(33) \qquad \begin{aligned} | b_1^1 | &> | c_1^1 | > 0 \\ | b_n^1 | &> | a_n^1 | > 0 \\ | b_j^1 | &> | a_j^1 | + | c_j^1 |, \quad j = 2, ...n-1. \end{aligned}$$

Since α_j and β_j for $j = 1, ..., n$, are operators from $\mathcal{F}$, let us decompose them as

$$\alpha_1 = b_1^1 I + Q_1^1 = \alpha_1^1 I + \alpha_{1,c}^1,$$

$$\beta_2 = \frac{a_2}{\alpha_1} = \frac{a_2^1 I - \frac{P_j^1}{21h}}{b_1^1 I + Q_1^1} = \frac{a_2^1}{\alpha_1^1} I + \beta_{2,c}^1 = \beta_2^1 I + \beta_{2,c}^1.$$

and

$$\begin{aligned} \alpha_2 &= b_2^1 I + Q_2^1 - (\beta_2^1 + \beta_{2,c}^1)(c_1^1 I - P_1^1) = \\ &= (b_2^1 - \beta_2^1 c_1^1) I + \alpha_{2,c}^1 = \alpha_2^1 I + \alpha_{2,c}^1 \end{aligned}$$

and finally

$$(34) \qquad \alpha_j = \alpha_j^1 I + \alpha_{j,c}^1 \quad \beta_j = \beta_j^1 I + \beta_{j,c}^1$$

where

$$(35) \qquad \alpha_j^1 = b_j^1 - \beta_j^1 c_{j-1}^1; \qquad \beta_j^1 = \frac{a_j^1}{\alpha_{j-1}^1}.$$

where $\alpha_1^1, \alpha_2^2, ..., \alpha_n^n$, and $\beta_1^1, \beta_2^2, ..., \beta_2^2$, are numerical constants and $\alpha_{1,c}^1, \alpha_{2,c}^2, ..., \alpha_{n,c}^n$, and $\beta_{1,c}^1, \beta_{2,c}^2, ..., \beta_{2,c}^2$, are operators representing continuous functions.

For α_j^1, β_j^1 we can prove that it holds

$$\alpha_j^1 \neq 0; \qquad \left| \frac{c_j^1}{\alpha_j^1} \right| < 1.$$

In the proof we use the inequalities from (33) and it is analogous as the same one, for the matrix where a_j, b_j, and c_j in matrix A are numerical constants (see[3]).

If $f(\lambda)$ on the right hand side of the equation (4) and the operators C and D which are conditions given by relation(5) represent continuous functions then as in Corollary 1. we can prove that the solutions of the considered system are operators $v_j, j = 1, 2, .., n$ representing continuous functions.

3. Discrete analogue II

Let us consider the case when u" and u' can be approximated by relations (10) and (12) (without remainder). Then we obtain the following discrete analogue

$$a_j v_{j-1} + b_j v_j + c_j v_{j+1} = f_j \quad v_0 = C, \quad v_{n+1} = D, \tag{36}$$

where $a_j, b_j, c_j, j = 1, .., n$ are operators having the form

$$a_j = -\frac{1}{h^2}(I + hP_j), \tag{37}$$

$$b_j = \frac{1}{h^2}(2I + hP_j) + Q_j \tag{38}$$

$$c_j = -\frac{1}{h^2}I. \tag{39}$$

The system $Av = d$ corresponds to the system given by relation (36), with matrix A, which has form (21) with elements $a_j, b_j, c_j, j = 1, ..., n$ satisfying the relations (37), (38) and (39) and vector d has the form given in relation (22).

In this case, also we can have

Theorem 3. *If in relations (8) and (9) we have $A_m(\lambda) \geq 0$ and $B_r(\lambda) \geq 0$, for $\lambda \in [0, 1]$ then the system $Av = d$ corresponding to the system given by (36) has unique solution and can be expressed as in relation (30), (31), and (32).*

Proof. Since in this case also, we can express a_j as in relation (27) and

$$b_j = b_j^1 I + \frac{P_j^1}{h} + Q_j^1$$

where

$$b_j^1 = \frac{1}{h^2}(2 + hA_m(\lambda_j) + h^2 B_r(\lambda_j))$$

we have the same inequalities as in (33) , so we can similarly obtain that $\alpha_j \neq 0$, for $j = 1, ..., n$.

4. Discrete anologue III

If we consider the case when u" and u' can be approximated by relations (10) and (13) (without remainder). Then we obtain the following discrete analogue

$$a_j v_{j-1} + b_j v_j + c_j v_{j+1} = f_j \quad v_0 = C, \quad v_{n+1} = D, \tag{40}$$

where $a_j, b_j, c_j, j = 1, .., n$ are operators having the form

$$a_j = -\frac{1}{h^2} I, \tag{41}$$

$$b_j = \frac{1}{h^2}(2I - hP_j) + Q_j, \tag{42}$$

$$c_j = -\frac{1}{h^2}(I - hP_j). \tag{43}$$

Also,we have

Theorem 4. *If in relations (8) and (9) we have $A_m(\lambda) \geq 0$ and $B_r(\lambda) \geq 0$, for $\lambda \in [0, 1]$ then the system $Av = d$ corresponding to the system given by (40) has unique solution and can be expressed as in relations (30), (31), and (32).*

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REZIME

PRIBLIŽNO REŠENJE DIFERENCIJALNE JEDNAČINE U POLJU OPERATORA MIKUSINSKOG

U radu se konstruiše približno rešenje klase diferencijalnih jednačina u polju operatora Mikusinskog, $\mathcal{F}$ koje odgovaraju klasi parcijalnih diferencijalnih jednačina. Numeričko rešenje se dobija kao rešenje diferentnih jednačina (diskretnog analogona) u polju $\mathcal{F}$.

Received by the editors December 13, 1991