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SOME GENERALIZATIONS OF CONTRACTION IN PROBABILISTIC SPACES

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Abstract

The problem of the existence and uniqueness of a common fixed point for a family of selfmappings in Menger spaces is investigated. That family is supposed to satisfy a generalization of the contraction condition.

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1. Introduction

The theory of probabilistic spaces started to develop rapidly after the publication of the paper of B. Schweizer and A. Sklar [7]. A.T. Bharucha–Raid and V.M. Sehgal [1] initiated the investigation of the fixed point problem in probabilistic metric spaces T. Hicks [5] introduced a very convinient definition of contraction which has properties quite similar to the properties of the classical contraction in metric spaces. Different generalizations of this type of contraction were given in [3], [6], V. Radu in [6] investigated a family

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of deterministic metrics in Menger spaces and some aspects related to the fixed point theory.

In § 2 we give some definition and concepts which are used in this article. For a detailed discussion of probabilistic spaces and their properties we refer to [8]. In § 3, which is the main section of this paper, we present a new generalization of Hicks–type contraction. Finally, in § 4, we discuss briefly a connection of this theorems with metric spaces.

2. Preliminaries

A mapping $F : R \to R^+ (R^+ = \{x \in R, x \geq 0\})$ is a distribution function if it is nondecreasing, leftcontinuous and $\inf_{t\in R} F(t) = 0$, $\sup_{t\in R} F(t) = 1$. In the sequel, we always denote by H the distribution function defined by

$$H(\varepsilon) = \begin{cases} 0 & \varepsilon \leq 0 \\ 1 & \varepsilon > 0. \end{cases}$$

A commutative, associative and nondecreasing mapping $t : [0,1] \times [0,1] \to [0,1]$ is a T–norm if $t(a,1) = a$ for all $a \in [0,1]$ and $t(0,0) = 0$.

A Menger space is a triplet $(X, \mathcal{F}, t)$, where X is an abstract set of elements. $\mathcal{F}$ is a mapping from $X \times X$ into the set of all distribution functions and t is a T–norm. We shall denote the distribution function $\mathcal{F}(x,y)$ by $F_{x,y}$ and $F_{x,y}(\varepsilon)$ will represent the value of $F_{x,y}$ at $\varepsilon \in R$. The functions $F_{x,y}, x, y \in X$ are assumed to satisfy the following conditions:

1. $F_{x,y}(\varepsilon) = H(\varepsilon)$ iff $x = y$,
2. $F_{x,y}(0) = 0$, for all $x, y \in X$,
3. $F_{x,y} = F_{y,x}$, for all $x, y \in X$,
4. $F_{x,y}(\varepsilon+\delta) = t(F_{x,z}(\varepsilon), F_{z,y}(\delta))$, for all $x, y, z \in X$ and all $\varepsilon, \delta \in R^+$.

The concept of neighbourhoods in Menger space was introduced by Schweizer and Sklar [7]. If $x \in X, \varepsilon > 0$ and $\lambda \in (0,1)$, then (ε, λ)–neighbourhood of x denoted by $U_x(\varepsilon, \lambda)$ is defined by $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

If $\sup_{a<1} t(a,a) = 1$, then $(X, \mathcal{F}, t)$ is a Hausdorff space in the topology induced by the family $\{U_x(\varepsilon, \lambda) : x \in X, \varepsilon > 0, \lambda \in (0,1)\}$ of neighbourhoods and that (ε, λ)–topology is uniformly metrisable.

Let M be the family of continuous mappings $m : R^+ \to R^+$ such that

1. $m(\varepsilon + \delta) \geq m(\varepsilon) + m(\delta)$, for all $\varepsilon, \delta \in R^+$,
2. $m(\varepsilon) = 0 \;\Leftrightarrow\; \varepsilon = 0$.

Let t be an Archimedean T–norm with additive generator g [8], that is,

$$g \cdot F_{x,y}(\varepsilon + \delta) \leq g \cdot F_{x,z}(\varepsilon) + g \cdot F_{z,y}(\delta)$$

for all $x, y, z \in X$ and all $\varepsilon, \delta \in R^+$.

According results from [6], we know that if $m_1, m_2 \in M$, then the function $d_{m_1,m_2} : S$ x $S \to R$ defined by

$$d_{m_1,m_2}(x, y) = \sup_{\varepsilon \geq 0} \{m_1(\varepsilon) \leq g \cdot F_{x,y}(m_2(\varepsilon))\}$$

is a metric on S which generates the (ε, λ)–uniformity. Also, the next equivalency holds

$$d_{m_1,m_2}(x, y) < \varepsilon \Leftrightarrow g \cdot F_{x,y}(m_2(\varepsilon)) < m_1(\varepsilon).$$

3. Fixed point in probabilistic metric spaces

Throughout this section we allways assume that $(X, \mathcal{F}, t)$ is a complete Menger space with T–norm t such that $\sup_{a<1} t(a, a) = 1$.

If the function $\varphi : R^+ \to R^+$ is a nondecreasing, semicontinuous from the right and $\varphi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$, then

$$\lim_{n \to \infty} \varphi^n(\varepsilon) = 0 \quad \text{for all} \quad \varepsilon > 0 \tag{1}$$

Let $f_i : X \to X$, $i \in N$ be the family of mappings, $\{n_i\}_{i \in N}$ the sequence of natural numbers and let the next implication holds

$$\max_{u,v \in \{x,y,f_i^{n_i}x, f_j^{n_j}y\}} g \cdot F_{u,v}(m_2(\varepsilon)) < m_1(\varepsilon) \Rightarrow$$

$$\Rightarrow g \cdot F_{f_i^{n_i}x, f_j^{n_j}y}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon)) \tag{2}$$

for all $x, y \in X$. If x_1 is any element of X we can form the sequence

$$x_{i+1} = f_i^{n_i} x_i \ , \ i \in N. \tag{3}$$

Lemma 1. *If the family of selfmappings $\{f_i\}_{i\in N}$ satisfies (2) then for every $x_1 \in X$ the sequence $\{x_i\}_{i\in N}$ (3) is a Cauchy sequence.*

Proof. Let x_1 be an element from X and $x_{i+1} = f_i^{n_i} x_i, i \in N$. In order to prove that $\{x_i\}_{i\in N}$ is a Cauchy sequence we proceed as follows. Since $g(0) < \infty$ and $\lim_{\varepsilon\to\infty} m_1(\varepsilon) = \infty$, there exists $\varepsilon > 0$ such that $g(0) < m_1(\varepsilon)$. Then, after identification of x and y from (2) with elements of sequence $\{x_i\}$, we get

$$\max_{u,v\in\{x_i,x_j,f_i^{n_i}x_i,f_j^{n_j}x_j\}} g\cdot F_{u,v}(m_2(\varepsilon)) \le g(0) < m_1(\varepsilon)$$

for all $i, j \in \{1, 2, \ldots\}$ and all $\varepsilon > 0$. Since the family $\{f_i\}_{i\in N}$ satisfies (1) we obtain that

$$g\cdot F_{f_i^{n_i}x_i,f_j^{n_j}x_j}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon))$$

i.e.

$$g\cdot F_{x_{i+1},x_{j+1}} m_2(\varphi(\varepsilon)) < m_1(\varphi(\varepsilon)) \quad \text{for all} \quad i, j \in \{1, 2, \ldots\}. \tag{4}$$

This means that

$$\max_{u,v\in\{x_i,x_j,x_{i+1},x_{j+1}\}} g\cdot F_{u,v}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon))$$

for all $i, j \in \{2, 3, \ldots\}$, which implies the inequality

$$g\cdot F_{x_{i+1},x_{j+1}}(m_2(\varphi^2(\varepsilon)) < m_1(\varphi^2(\varepsilon))$$

for all $i, j \in \{2, 3, \ldots\}$.

Continuing that procedure we get

$$g\cdot F_{x_i,x_j} m_2(\varphi^k(\varepsilon)) < m_1(\varphi^k(\varepsilon))$$

for all $i, j \in \{k+1, k+2, \ldots\}$.

Since the mapping φ satisfies the condition (1), for all $t > 0$ and $\lambda \in (0,1)$, there exists $k_0(t,\lambda)$ such that $m_2(\varphi^k(\varepsilon)) < t$ and $m_1(\varphi^k(\varepsilon)) < g(1-\lambda)$ for all $k > k_0$. Now, we obtain

$$g \cdot F_{x_i,x_j}(m_2(\varphi^k(\varepsilon))) < m_1(\varphi^k(\varepsilon)) < g(1-\lambda),$$

that is

$$F_{x_i,x_j}(m_2(\varphi^k(\varepsilon))) > 1-\lambda,$$

and

$$F_{x_i,x_j}(t) > F_{x_i,x_j}(m_2(\varphi^k(\varepsilon))) > 1-\lambda$$

for all $i,j \in \{k_0+1, k_0+2, \ldots\}$.

So, we have proved that $\{x_i\}_{i\in N}$ is a Cauchy sequence.

Theorem 1. *Let $(X,\mathcal{F},t)$ be a complete Menger space with T-norm t such that $\sup_{a<1} t(a,a) = 1$ and let the family $\{f_i\}_{i\in N}$ of selfmappings of X be such that the implication (2) holds. Then the family $\{f_i\}_{i\in N}$ has a unique common fixed point which is the limit of the sequence (3).*

Proof. From Lemma 1 we have that the sequence $\{x_i\}_{i\in N}$ formed by

$$x_{i+1} = f_i^{n_i} x_i \quad , \quad i \in N$$

is a Cauchy sequence and from the completness of X it follows $\lim_{i\to\infty} x_i = z \in X$. Now we shall prove that z is a common periodic point of $\{f_i\}_{i\in N}$, that is, that

$$f_i^{n_i} z = z \quad , \quad i \in N.$$

Let A_0 be the set of all discontinuity points of $F_{z,f_i^{n_i}z}(\varepsilon)$. Since φ^k and m_2 are strictly increasing, we know that $\varphi^{-k}(m_2^{-1}(A))$ is the set of all discontinuity points of $F_{z,f_i^{n_i}z}(m_2(\varphi^k(\varepsilon)))$. Moreover, $A_0, \varphi^{-k}(m_2^{-1}(A_0)), \quad k = 1,2,\ldots$ are all cauntable, therefore

$$A = A_0 \cup \left(\bigcup_{k=1}^{\infty} \varphi^{-k}(m_2^{-1}(A_0))\right)$$

is also cauntable. Let $\overline{R} = R \backslash A$. Since $g(0) < \infty$ and $\lim_{\varepsilon\to\infty} m_1(\varepsilon) = \infty$, by the density of real numbers there exists $\varepsilon > 0$ such that $\varepsilon \in \overline{R}$ and

$$\max_{u,v\in\{x_j,z,x_{j+1},f_i^{n_i}z\}} g \cdot F_{u,v}(m_2(\varepsilon)) \le g(0) < m_1(\varepsilon)$$

for all $j \in N$, and this implies that

$$g \cdot F_{x_{j+1}, f_i^{n_i} z}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon)).$$

From the last inequality and Lemma 1. we have

$$F_{x_{j+1}, f_i^{n_i} z}(m_2(\varphi(\varepsilon))) > g^{-1}(m_1(\varphi(\varepsilon)) \qquad \text{for all } j \in N$$

and when $j \to \infty$

$$F_{z, f_i^{n_i} z}(m_2\varphi(\varepsilon)) > g^{-1}(m_2(\varphi(\varepsilon))$$

which implies

$$g \cdot F_{z, f_i^{n_i} z}(m_2(\varphi(\varepsilon)) < m_2(\varphi(\varepsilon)).$$

Further, as it was shown in (4) where ε was choosen analogously, we get

$$g \cdot F_{x_j, x_{j+1}}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon))$$

for all $j \in \{2, 3 \ldots\}$. So, we can write

$$\max_{u,v \in \{x_j, z, x_{j+1}, f_i^{n_i} z\}} g \cdot F_{u,v}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon)) \quad , \quad j \in \{2, 3, \ldots\}$$

which implies that

$$g \cdot F_{x_j, f_i^{n_i} z}(m_2(\varphi^2(\varepsilon))) < m_1(\varphi^2(\varepsilon))) < m_1(\varphi^2(\varepsilon)) \quad , \quad j \in \{2, 3, \ldots\}$$

and when $j \to \infty$ we have

$$g \cdot F_{z, f_i^{n_i} z}(m_2(\varphi^2(\varepsilon))) < m_1(\varphi^2(\varepsilon)).$$

Continuing this procedure we get that

$$g \cdot F_{z, f_i^{n_i} z}(m_2(\varphi^k(\varepsilon)) < m_1(\varphi^k(\varepsilon)) \quad , \quad k \in N, \varepsilon \in \overline{R}.$$

Since $\lim_{k \to \infty} \varphi^k(\varepsilon) = 0$ for all $t > 0$ and $\lambda \in (0, 1)$ there exists $k_0(t, \lambda)$ such that $m_2(\varphi^k(\varepsilon)) < t$ and $m_1(\varphi^k(\varepsilon)) < g(1 - \lambda)$ for all $k > k_0$. Then we obtain

$$g \cdot F_{z, f_i^{n_i} z}(m_2(\varphi^k(\varepsilon))) < m_1(\varphi^k(\varepsilon)) < g(1 - \lambda) \;\Rightarrow$$

$$F_{z, f_i^{n_i} z}(m_2(\varphi^k(\varepsilon))) > 1 - \lambda \;\Rightarrow$$

$$F_{z,f_i^{n_i}z}(t) > F_{z,f_i^{n_i}z}(m_2\varphi^k(\varepsilon))) > 1-\lambda,$$

which means that $z = f_i^{n_i}z$.

So we have proved that z is a common periodic point for the family $\{f_i\}_{i\in N}$. To prove that z is the unique fixed point of $f_i^{n_i}$ we suppose that $y \in X$ is another fixed point of same $f_i^{n_i}, i \in N$, that is, $y = f_i^{n_i}y$. Then

$$\max_{u,v\in\{z,y,f_i^{n_i}z,f_i^{n_i}y\}} g\cdot F_{u,v}(m_2(\varepsilon)) < m_1(\varepsilon) \;\Rightarrow$$

$$g\cdot F_{z,y}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon)) \qquad \text{and since } z = f_i^{n_i}z, y = f_i^{n_i}y$$

we have

$$g\cdot F_{z,y}(m_2\varphi^k(\varepsilon)) < m_1(\varphi^k(\varepsilon))$$

and for $k > k_0(t,\lambda)$

$$F_{z,y}(t) > 1-\lambda,$$

that is, $z = y$.

Since $f_if_i^{n_i}z = f_i^{n_i}f_iz = f_iz$ and z is the unique fixed point of $f_i^{n_i}$, we get that $f_iz = z$ for all $i \in N$.

4. A connection with metric spaces

Theorem 2. *Let $(X,\mathcal{F},t)$ be a complete Menger space and $\{f_i\}_{i\in N}$ the sequence which satisfies (2). If $t \le t_g$, where g is aditive generator of t_g, then $\{f_i\}_{i\in N}$ has a unique fixed point which is the limit of the sequence (3) for every $x_1 \in X$.*

Proof. We know that the function $d : X \text{ x } X \to R$ defined by

$$d_{m_1m_2}(x,y) = \sup\{\varepsilon : \varepsilon \ge 0, m_1(\varepsilon) \le g\cdot F_{x,y}(m_2(\varepsilon))\}$$

is a metric on X which generates the (ε,λ)–uniformity. It is obvious that the next equivalency holds

$$d_{m_1m_2}(x,y) < \varepsilon \;\Leftrightarrow\; g\cdot F_{x,y}(m_2(\varepsilon)) < m_1(\varepsilon).$$

Further, from the inequality

$$(5) \qquad \max_{u,v\in\{x,y,f_i^{n_i}x,f_j^{n_j}y\}} d_{m_1m_2}(u,v) < \varepsilon,$$

we get that

$$\max_{u,v\in\{x,y,f_i^{n_i}x,f_j^{n_j}y\}} g\cdot F_{u,v}(m_2(\varepsilon)) < m_1(\varepsilon),$$

which implies that

$$g\cdot F_{f_i^{n_i}x,f_j^{n_j}y}(m_2(\varphi(\varepsilon))) < m_1(\varphi(\varepsilon)),$$

that is,

$$d_{m_1m_2}(f_i^{n_i}x, f_j^{n_j}y) < \varphi(\varepsilon).$$

Combining the last inequality with (5), we obtain

$$\max_{u,v\in\{x,y,f_i^{n_i}x,f_j^{n_j}y\}} d_{m_1m_2}(u,v) < \varphi d_{m_1m_2}(f_i^{n_i}x, f_j^{n_j}y)$$

and from (2), the sequence $\{f_i^{n_i}\}_{i\in N}$ has a unique common fixed point which is the limit of the sequence (3) for every $x_1 \in X$.

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REZIME

NEKE GENERALIZACIJE KONTRAKCIJE U VEROVATNOSNIM PROSTORIMA

Posmatran je problem egzistencije i jedinstvenosti zajedničke nepokretne tačke za familiju samopreslikavanja u Mengerovim prostorima. Za tu familiju se pretpostavlja da zadovoljava uopštenje uslova kontrakcije.

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