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THE LEAST UPPER BOUND OF THE ADDITIVE MEASURES AND INTEGRALS

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Abstract

In this paper m integral, i.e., monotone, positive, homogenous, subadditive functional defined on step functions, with respect to p - submeasure m, is characterized as least upper bound of a collection of additive integrals.

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1. Introduction

Let T be a ring of subsets of a set $X \neq \emptyset$ and m a submeasure on T. Any monotone positively homogenuous and subadditive functional J defined on $F^+(T) = \{\sum_{i=1}^n a_i\chi_{A_i};\ a_i > 0,\ A_i \in T,\ n \in \mathbf{N}\}$ satisfying $J(\chi_A) = m(A)$ for every $A \in T$, is said to be an m-integral.

Paper [3] shows that such an m-integral exists if and only if m is a p-submeasure.

The present paper characterizes m-integrals by means od additive integrals. It shows that each m-integral is the least upper bound of collection of additive integrals.

It also shows that in general a submeasure posseses more integrals while among them in the sense of maximal and minimal need not exist.

2.

Let T be a ring of subsets of a nonempty set X. Let m be a set function $m: T \to [0,\infty)$, $m(\emptyset) = 0$. Then m is called

a) submeasure, if for every $A, B, C \in T$, $A \cup B \supset C$ we have

$$m(A) + m(B) \geq m(C)$$

b) p-submeasure, if for every two positive integers n, k and the sets $A, A_i \in T$ such that $\sum_{i=1}^{n} \chi_{A_i} \geq k\chi_A$ there is

$$\sum_{i=1}^{n} m(A_i) \geq km(A)$$

It is evident that each p-submeasure is a submeasure and that they are both monotone (i.e. if $A, B \in T$, $A \subset B$ then $m(A) \leq m(B)$).

Let T be a ring of subsets of the set $X \neq \emptyset$. Denote

$$F^+(T) = \{\sum_{i=1}^{n} a_i\chi_{A_i};\ a_i > 0,\ A_i \in T,\ n \in \mathrm{N}\}$$

$$F(T) = \{\sum_{i=1}^{n} a_i\chi_{A_i};\ A_i \in T,\ n \in \mathrm{N},\ a_i \text{ real}\}.$$

A function $J: F^+(T) \to [0.\infty)$ is said to be an integral if

a) J is monotone, i.e. $J(f) \geq J(g)$ if $f, g \in F^+(T)$, $f \geq g$

b) J is positivelty homogenuous, i.e. $J(c{\cdot}f) = c{\cdot}J(f)$, if $c > 0$, $f \in F^+(T)$

c) J is subadditive, i.e. $J(f) + J(g) \geq J(f+g)$ if $f, g \in F^+(T)$.

Let m be a submeasure on T. An integral $J: F^+(T) \to [0.\infty)$ is said to be an integral with respect to a submeasure m (m-integral) if

$$J(\chi_A) = m(A) \text{ for } A \in T.$$

A submeasure m is said to be integrable provided that there exists an m-integral

The following theorem is proved in [3].

Theorem 1. *Let m be a submeasure on a ring T. Then the following assertions are equivalent*

a) m is p-measure

b) m is an integrable submeasure.

3.

Theorem 2. *Let m_i, $i \in I$ be a collection of additive measures on a ring T. Let $\int f dm_i$ be an additive integral with respect to m_i. The following holds.*

a) The function $m : T \to [0, \infty)$ defined by

$$m(A) = sup\ \{m_i(A);\ i \in I\}\ for\ A \in T$$

is a p-submeasure.

b) The function $J : F^+(T) \to [0, \infty)$ defined by

$$J(f) = sup\{\int f dm_i;\ i \in I\}\ for\ f \in F^+(T)$$

is an m-integral.

Proof.
a) Evidently $m(\emptyset) = 0$. Let $k, n \in \mathbb{N}$, $A, A_j \in T$ and $k \cdot \chi_A \leq \sum_{j=1}^{n} \chi_{A_j}$. Since m_i are additive measures, we have

$$k \cdot m_i(A) \leq \sum_{j=1}^{n} m_i(A_j) \leq \sum_{j=1}^{n} m(A_j) \text{ for each } i \in I.$$

By the definition of m we have

$$k \cdot m(A) = k \cdot \sup\{m_i(A);\ i \in I\} \leq \sum_{j=1}^{n} m(A_j).$$

Hence m is a p-submeasure.
b) It follows directly from the definition of J that J is monotone and that $J(0) = 0$. Let $c > 0$. $f, g \in F^+(T)$. Then

$$J(c \cdot f) = \sup\{\int c \cdot f dm_i;\ i \in I\} = \sup\{c \cdot \int f dm_i;\ i \in I\}$$

$$= c \cdot \sup\{\int f dm_i;\ i \in I\} = c \cdot J(f).$$

$$J(f) + J(g) = \sup\{\int f dm_i;\ i \in I\} + \sup\{\int g dm_i;\ i \in I\}$$

$$\geq \sup\{\int f dm_i + \int g dm_i;\ i \in I\}$$

$$= \sup\{\int (f+g) dm_i;\ i \in I\} = J(f+g).$$

So J is an integral.

Let $A \in T$. Then

$$J(\chi_A) = \sup\{\int \chi_A dm_i;\ i \in I\} = \sup\{m_i(A);\ i \in I\} = m(A).$$

So J is an m-integral. □

Let T be a ring and J an integral on $F^+(T)$. Let $f \in F(T)$. Then evidently there are $f^+, f^- \in F^+(T)$ such that $f = f^+ - f^-$. Define the function J^* on $F(T)$ as follows

$$J^*(f) = J(f^+) - J(f^-) \text{ if } f \in F(T).$$

Theorem 3. *Let T be a ring and J an integral on $F^+(T)$. Then*

a) J^* is positively homogenuous

b) J^* is monotone.

Proof.
a) Let $c > 0$, $f \in F(T)$. If $g = c \cdot f$, then $g^+ = c \cdot f^+$, $g^- = c \cdot f^-$. Consequently

$$J^*(c \cdot f) = J^*(g) = J^*(g^+) - J^*(g^-)$$

$$= J^*(c \cdot f^+) - J^*(c \cdot f^-) = c \cdot J^*(f^+) - c \cdot J^*(f^-) = c \cdot J^*(f).$$

So J^* is positively homogenuous.
b) Let $f, g \in F(T)$, $f \geq g$. Then $f^+ \geq g^+$ and $f^- \leq g^-$. Hence

$$J^*(f) = J^*(f^+) - J^*(f^-) \geq J^*(g^+) - J^*(g^-) = J^*(g).$$

So J^* is monotone. □

Let T be a ring of subsets of the set $X \neq \emptyset$. The function $J :\to [0, \infty)$ is said to be an integral if J is monotone, positively homogenuous and subadditive of $F^+(T)$.

The following theorem is proved in [12] in a more general form ([12], Theorem 5.)

Theorem 4. *Let T be a ring of subsets of the set $X \neq \emptyset$. Let E be a linear space, $E \subset F(T)$. Let J be an integral on $F(T)$ and J_0 be an additive integral on E. Then there exists an additive integral J_1 on $F(T)$ such that*

a) J_1 is an extension of J_0

b) $J_1 \leq J$ on $F^+(T)$.

Theorem 5. *Let m be a p-submeasure on a ring T. Let J be an m-integral on $F^+(T)$ anf $f \in F^+(T)$. Then there exists an additive measure w on T, such that*

a) $\int f dw = J(f)$

b) $\int g dw \leq J(g)$ for every $g \in F^+(T)$.

Proof. Let $E = \{c \cdot f;$ c is real$\}$. Theorem 3 implies that J^* is an integral on $F(T)$. Then J^* is the additive integral on E. It is obtained from Theorem 4 that there exists the additive integral J_1 on $F(T)$ such that

1) $J_1 = J^*$ on E

2) $J_1 \leq J$ on $F^+(T)$.

Define the function w on T as follows

$$w(A) = J_1(\chi_A) \text{ if } A \in T.$$

Since J_1 is the additive integral on $F(T)$, the function w is the additive measure on T. Evidently ig $g \in F(T)$ then $\int g dw = J_1(g)$. So we obtain

a) $\int f dw = J_1(f) = J^*(f) = J(f)$

b) If $g \in F^+(T)$ then

$$\int g dw = J_1(g) \leq J^*(g) = J(g).$$

The theorem is proved. □

Theorem 6. *Let m be a p-submeasure on a ring T and J be an m-integral. Then there exists a collection of additive measures m_i, $i \in I$ which are defined on T and the following is satisfied*

a) $m(A) = \max\{m_i(A);\ i \in I\}$ *for* $A \in T$

b) $J = \max\{\int f dm_i;\ i \in I\}$ *for* $f \in F^+(T)$.

Proof. Choosing the collection of additive measures w, which correspond to the functions $f \in F^+(T)$ according to Theorem 5, we obtain the proof. □

Corollary 1. *Let m be a p-submeasure on a ring T. Then*

a)

$$m(A) = \max\{w(A);\ w \leq m,\ w \text{ is an additive measure on } T\},\ A \in T$$

b) m-integral J defined as

$$J(f) = \max\{\int f dw;\ w \leq m,\ w \text{ is an additive measure on } T\},$$

$f \in F^+(T)$.

The following example shows that p-submeasure may have more than one m-integral and that in general a pointwise smallest m-integral does not exist.

Example 1. Let $T = \{\emptyset, \{1\}.\{2\}, \{1,2\}\}$. Let m, a, b, c, d be defined on T in the following way

	m	a	b	c	d
$\emptyset$	0	0	0	0	0
$\{1\}$	2	2	1	2	0
$\{2\}$	2	0	2	1	2
$\{1,2\}$	3	2	3	3	2

Evidently m is a p-submeasure and a, b, c, d are additive measures on T. Because

$$m(A) = \max\{a(A), b(A)\} = \max\{c(A), d(A)\} \text{ for } A \in T,$$

then the integrals I, J defined as

$$I(f) = \max\{\int f da, \int f db\},\ J(f) = \max\{\int f dc, \int f dd\} \text{ for } f \in F^{+}(T)$$

are m-integrals.

Put $f = \chi_{\{1\}} + 2 \cdot \chi_{\{2\}}$, $g = 2 \cdot \chi_{\{1\}} + \chi_{\{2\}}$. Then the following holds

	$\int da$	$\int db$	$\int dc$	$\int dd$	I	J
f	2	5	4	4	5	4
g	4	4	5	2	4	5

So two different integrals may exist.

If there exists a pointwise smallest m-integral K, then

$$K(f) \leq J(f) = 4,\ K(g) \leq I(g) = 4, \text{ and } K(f+g) = K(3 \cdot \chi_{\{1,2\}}) = 9.$$

So K would not be subadditive.

Thus in general there does not exist pointwise smallest integral with respect to m.

References

[1] Aleksiak, V. N., Bernosikov, F. D., An extension of continuous outer measure on a Boolean algebra (in Russian), Izv. VUZ, 4(119), 1972, 3-9.

[2] Černek, P., About product of submeasures, Acta math. Univ. Comem., to appear.

[3] Černek, P., Product of p-submeasures, Math. Slovaca, to appear.

[4] Dobrakov, I., On submeasures I, Dissertationes Mathematical, 112, 1973.

[5] Dobrakov, I., Farková, J., On submeasures II, Math. Slovaca 30, 1980, 65-82.

[6] Drewnowski, L., Topological rings of sets, continuous functions, integration, I - III, Bull. Acad. Pol. Sci., 20, 1972, 269-286.

[7] Kalas, J., A construction of a subadditive measure from a set function defined on a semiring (in Russian), Mat. časop. SAV, 24, 1974, 263-273.

[8] Kalas, J., Limit theorems concerning an integral with respect to a subadditive measure (in Russian), Acta Math. Univ. Comen., to appear.

[9] Neubrunn, T. - Riečan, B.: Measure and integral (in Slovak), Bratislava 1981.

[10] Riečan, B., An extension of Daniell integration scheme, Mat. časop. SAV, 25, 1975, 211-219.

[11] Šipoš, J., Integral with respect to a pre-measure, Math. Slovaca, 29, 1979, 141-156.

[12] Šipoš, J, A note on Hahn-Banach extension theorem, Czech. math. J.

REZIME

SUPREMUM ADITIVNIH MERA I INTEGRALA

U radu se karakteriše m - integral, kao monotona, pozitivno homogena i subaditivna funkcionela definisana na jednostavnim funkcijama, u odnosu na p - submeru, kao supremum familije aditivnih integrala.

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