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CONTRACTIONS IN PROBABILISTIC m - METRIC SPACES

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Abstract

In this paper we proved some contraction principles for selfmappings in a Gähler's complete $m-$ metric space.

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Introduction

In this paper we exhibit some contraction principles for selfmappings in a Gähler's complete $m-$ metric space. We formulate fixed - point theorems for contraction mappings in an $m-$ metric space and we prove two versions of Bessaga Converse Theorem of [3]. The m - metrization theorems of [34] give us a possibility to use these fixed - point results to prove some fixed - point theorems for contractions in $m-$ uniform spaces. In particular we have formulated some fixed - point theorems for contractions in probabilistic $m-$ metric spaces.

Our paper has not the capacity of a retrospective look on the whole fixed - point theory in $m-$ metric spaces and all the more in probabilistic $m-$ metric spaces, $m \geq 1$. We want to indicate near connections between the fixed - point theory in $m-$ metric spaces and fixed - points results in Gähler's $m-$ uniform spaces.

1. Definitions, lemmas and preliminary remarks

The purpose of this part of our paper is to cite some definitions and theorems which are formulated by S. Gähler (see [10] - [15]) and us (see [31] and [34]). The notation is the same as in original papers of Gähler, rather. In particular the $m-$ metrization theorems of Hicks and Sharma type are cited from [34].

Let m be a positive integer and let $M = \{0, ..., m\}$. In the sequel $\prod_M$ denotes the set of all permutations p of M.

Let X be a nonempty set. If $\alpha = (a_i)_{i\in M} \in X^M$, then p denotes the point $(\alpha_{p(i)})_{i\in M}$, where $p \in \prod_m$. For $\alpha \in X^M$, $\alpha_{a_i \to a} = (a_0, ..., a_{i-1}, a, a_{i+1}, ..., a_m)$, $\alpha_{a_i \to a, a_j \to a'} = (a_0, ..., a_{i-1}, a, a_{i+1}, ..., a_{j-1}, a, a_{j+1}, ..., a_m)$, $i \neq j$, $a, a' \in X, ...$

For $p \in \prod_M$ we denote

$$\Delta_p = \{\alpha \in X^M : p(\alpha) = \alpha\}.$$

The diagonal set $\Delta \subseteq X^M$ is defined in the following

$$\Delta = \bigcup_{p \in \prod_m \setminus \{id_M\}} \Delta_p,$$

where $id_M(j) = j$, $j \in M$.

The set $V \subseteq X^M$ is said to be symmetric iff $V = p(V)$ for each $p \in \prod_M$.

In the sequel V^{-1} denotes the following set

$$V^{-1} = \bigcup_{p \in \prod_M \setminus \{id_M\}} p(V).$$

For $V_0, ..., V_m \subseteq X^M$ we denote the set

$$\bigcirc_{k=0}^{m} V_k = V_0 \circ V_1 \circ ... \circ V_m$$
$$= \{\alpha \in X^M : \exists V \in X \alpha_{a_i \to V} V_{m-i}, \ i = 0, ..., m\}$$

and if $V_0 = V_1 = ... = V_m$, then we write $\circ V$ instead of $\bigcirc_{k=0}^{m} V_k$ (compare S.Gähler, [12]).

Let $\mathcal{U}$ be a family of subsets U of X^M. Following to Gähler [12] the family $\mathcal{U}$ is an m - uniform structure if $\mathcal{U}$ is a filter and in addition the conditions hold:

$\bigcup_1$ $\Delta \subseteq U$ for each $U \in \mathcal{U}$,

$\bigcup_2$ for each $U \in \mathcal{U}$ and $p \in \prod_M$, $p(U) \in \mathcal{U}$,

$\bigcup_3$ for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $\circ V \subseteq U$.

The ordered pair $(X,\mathcal{U})$ is then called an $m-$ uniform structure and members of $\mathcal{U}$ are called entourages. A subfamily $\mathcal{B}$ of $\mathcal{U}$ is said to be a basis for $\mathcal{U}$ iff every entourage contains some member of $\mathcal{B}$.

For $M = \{0, ..., m\}$ in the sequel $M_{k_1,...,k_l}$ denotes the set

$$M \setminus \{k_1, ..., k_l,\ 0 \leq k_1 < k_2 < ... < k_l \leq m,\ 0 < l < m.$$

Let $a, a' \in X$ and $\alpha = (a_i)_{i \in M_{kl}} \in X^{M_{kl}}$ for some $k, l,\ 0 \leq k < l \leq m$. Then in the sequel $[a, a, \alpha]$ denotes the point $\alpha' \in X^M$, such that $a'_k = a,\ a'_l = a'$ and $a'_i = a_i$ for $i \in M_{kl}$, and

$$U_\alpha[a] = \{a' \in X : [a, a', \alpha] \in U\}$$

for any $U \subseteq X^M$.

Theorem 1.1. *(see [34], Th. 2.1). Let $(X,\mathcal{U})$ be an $m-$ uniform structure for some $m \geq 1$. Then the family*

$$\mathcal{U}[a] = \{U[a] : U \in \mathcal{U}\},$$

where

$$U[a] = \bigcap_{s=1}^{n} U_{\alpha^s}[a] \quad \textit{for} \quad U \in \mathcal{U},$$

is a neighbourhood system on X.

The topology $\mathcal{T}_\mathcal{U}$ defined by the neighbourhood system is called a topology induced by $\mathcal{U}$ and an order triple $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is called then an $m-$ uniform space.

Let $(E, \leq)$ be a partially ordered set and the order relation have additional properties (compare S. Gähler [12]):

$(\leq .1)$ there exists $0 \in E$ that $(0, \epsilon) \in \leq$ for each $\epsilon \in E$,

$(\leq .2)$ for each $\epsilon, \epsilon' \in E_0 = E \setminus \{0\}$ there exists $\epsilon'' \in E_0$, such that (ϵ'', ϵ), $(\epsilon'', \epsilon') \in \leq$.

For $\epsilon, \epsilon' \in E$ we write $\epsilon \leq \epsilon'$ iff $(\epsilon, \epsilon') \in \leq$, $\epsilon \not\leq \epsilon'$ if $(\epsilon, \epsilon') \notin \leq$ and $\epsilon < \epsilon'$ if $\epsilon \leq \epsilon'$ and $\epsilon \neq \epsilon'$.

Theorem 1.2. *(see [34], Th. 3.1). Let $\mathcal{B}$ be a family of subsets of X^M of the form $\mathcal{B} = \{U_\epsilon \subseteq X^M : \epsilon \in E_0\}$, where E_0 is a partially ordered set with properties $(\leq .1) - (\leq .2)$. If the conditions hold:*

(1.1) for each $\epsilon \in E_0$, $\Delta \subseteq U$,

(1.2) $U_\sigma \subseteq U_\epsilon$ whenever $\sigma < \epsilon$ for each $\sigma, \epsilon \in E_0$,

(1.3) for each $\epsilon \in E_0$ there exists $\epsilon' \in E_0$, that $\circ U_{\epsilon'} \subseteq U_\epsilon$,

(1.4) for each $p \in \prod_M$ and $\epsilon \in E_0$, $p(U_\epsilon) \in \mathcal{B}$,

then $\mathcal{B}$ is the base of an $m-$ uniform structure $\mathcal{U}$ and $(X, \mathcal{U}, \mathcal{T}_\mathcal{U})$ is completely regular topological space. If in addition the property holds

$$\Delta = \bigcap_{\epsilon \in E_0} U_\epsilon, \tag{1.5}$$

then $(X, \mathcal{U}, \mathcal{T}_\mathcal{U})$ is a Hausdorff's topological space.

Let $(J, \leq)$ be a directed and partially ordered set. Let X be a nonempty set, m be a positive integer and let $\mathcal{B}$ be a family of nonempty subsets of X^M of the form

$$\mathcal{B} = \{U_{j,\epsilon} \subseteq X^M : j \in J,\ \epsilon \in (0, r)\},\ r \in R_+^\sharp,\ r > 0,$$

$R_+^\sharp = R_+ \cup \{+\infty\}$, such that

$(\mathcal{B}.1)_\mathcal{H}$ $\Delta = \bigcap_{(j,\epsilon) \in J \times (0,r)} U_{j,\epsilon}$

$(\mathcal{B}.2)_\mathcal{H}$ $U_{i,\sigma} \subseteq U_{j,\epsilon}$ whenever $(i, \sigma) \leq (j, \epsilon)$

$((j, \sigma) \leq (j, \epsilon)$ iff $i \leq j$ and $\sigma \leq \epsilon)$,

$(\mathcal{B}.3)_\mathcal{H}$ $U_{j,\epsilon} = U_{j,\epsilon}^{-1}$ for each $(j, \epsilon) \in J \times (0, r)$,

$(\mathcal{B}.4)_\mathcal{H}$ for each $j \in J$ and $\epsilon \in (0, r)$ there exists $\epsilon' \in (0, r)$, such that $\circ U_{j,\epsilon'} \subseteq U_{j,\epsilon}$,

$(\mathcal{B}.5)_\mathcal{H}$ for each $j \in J$, each $\epsilon \in (0, r)$ and each $\alpha \in X^M$, if $\alpha \in U_{j,\epsilon}$ then there exists $\epsilon' < \epsilon$ such that $\alpha \in U_{j,\epsilon'}$.

If $\mathcal{B}$ is a family as above, then $\mathcal{B}$ is the base of an $m-$ uniform structure fulfilling assumptions of Theorem 1.2. We say (see [34]) that an $m-$ uniform structure is an $m - \mathcal{H}-$ structure if it is generated by the base $\mathcal{B}$ fulfilling $(\mathcal{B}.1)_\mathcal{H} - (\mathcal{B}.5)_\mathcal{H}$.

If $(X,\mathcal{U})$ is an $m-\mathcal{H}-$ structure generated by $\mathcal{B}$, then the neighbourhood system $\{U_{j,\epsilon}[a] : j \in J,\ \epsilon \in (0,r),\ a \in X\}$ define the topology $\mathcal{T}_{\mathcal{U}}$ and the ordered triple $(X,\mathcal{U},\mathcal{T}_{\mathcal{U}})$ is called then an $m-\mathcal{H}-$ space.

A filter $\mathcal{F}$ in an $m-\mathcal{H}-$ space $(X,\mathcal{U},\mathcal{T}_{\mathcal{U}})$ is a Cauchy filter iff for each $(\alpha,j,\epsilon) \in X^{M_{01}} \times J \times (0,r)$, $F \times F \subseteq U_{j,\epsilon,\alpha}$ for some $F \in \mathcal{F}$. An $m-\mathcal{H}-$ space $(X,\mathcal{U},\mathcal{T}_{\mathcal{U}})$ is complete iff each Cauchy filter in X converges to a point of X.

Let X be a nonempty set and m be a positive integer. The function $\sigma : X^M \to E$, where E fulfils $(\leq .1)$ and $(\leq .2)$ is a generalized $m-$ metric over X and E (see S.Gähler [12], p. 177) if

M'_{1_a} $\sigma(\alpha) = 0$ for each $\alpha \in \Delta$,

M'_2 for each $\epsilon \in E_0$ and $p \in \prod_M$, there exists $\epsilon' \in E_0$ such that $\sigma(p(\alpha)) \not\geq \epsilon$ whenever $\sigma(\alpha) \not\geq \epsilon$,

M'_3 for each $\epsilon \in E_0$ there exists $\epsilon' \in E_0$ such that $\sigma(\alpha) \not\geq \epsilon$ whenever $\sigma(\alpha_{a_i \to a}) \geq \epsilon'$ for each $i \in M$, $a \in X$.

The ordered pair (X,σ) is said to be a generalized $m-$ metric space if σ has properties M'_{1_a}, M'_2 and M'_3.

If the generalized $m-$ metric space (X,σ) has the additional property (see [12], Th. 25):

M'_{1_b} for each two different points $x,y \in X$ there exists $\alpha \in X$ such that $\sigma[x,y,\alpha] \neq 0$,

then $(X,\mathcal{T}_\sigma)$ is a Hausdorff topological space and some additional properties on (X,σ) guarantee complete regularity of $\mathcal{T}_\sigma$.

Let $(J,\leq)$ be a directed and partially ordered set and X be a nonempty set. Let $\sigma = (\sigma_j)_{j\in J}$ be a family of functions $\sigma_j : X^M \to R_+$ such that the conditions hold

$(M_{1_a})_{\mathcal{H}}$ $\sigma_j(\alpha) = 0$ for each $\alpha \in \Delta$, $j \in J$,

$(M_{1_b})_{\mathcal{H}}$ for each different $a,a' \in X$ there exists $\alpha \in X^{M_{01}}$ and $j \in J$, such that $\sigma_j[a,a',\alpha] > 0$,

$(M_2)_{\mathcal{H}}$ $\sigma_j(\alpha) = \sigma_j(p(\alpha))$ for each $\alpha \in X^M$, $j \in J$ and $p \in \prod_M$,

$(M_3)_{\mathcal{H}}$ for each $j \in J$ and each $\epsilon \in (0,r)$ there exists $\delta > 0$ such that for each $\alpha \in X^M$ and $v \in X$ we have $\sigma_j(\alpha) < \epsilon$ whenever $\sigma_j(\alpha_{a_i \to v}) < \delta$, $i = 0,...,m$.

We say that the generalized $m-$ metric space (X,σ) fulfilling $(M_{1_a})_{\mathcal{H}} - (M_3)_{\mathcal{H}}$ (with the topology $\mathcal{T}_\sigma$ defined as usual) is a $\mathcal{H} - m-$ metric space.

A sequence (x_n) in an $\mathcal{H} - m-$ metric space (X,σ), $\sigma = (\sigma_j)_{j\in J}$ is a Cauchy sequence iff for every $\alpha \in X^{M_{01}}$ and $j \in J$, $\sigma_j(x_n, x_{n+p}, \alpha) \to 0$ as $n,p \to \infty$ and an $\mathcal{H} - m-$ metric space (X,σ) is sequentially complete iff each Cauchy sequence in X converges to a point of X.

Theorem 1.3. *(see Th. 6.1 of [34]).*
Let $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ be an $m - \mathcal{H}-$ metric space with the base $\mathcal{B} = \{U_{j,\epsilon} \subseteq X^M : j \in J,\ \epsilon \in (0,r)\}$, $r \in R_+^\sharp$, of its $m-$ uniformity $\mathcal{U}$. Then the family of functions

$$\sigma = (\sigma_j)_{j\in J}, \quad \sigma_j : X^M \to R_+,$$

$$\sigma_j(\alpha) = \begin{cases} \sup\{\epsilon \in (0,r) : \alpha \in U'_{j,\epsilon}\} & \text{if } \alpha \in \bigcup_{0<\epsilon<r} U'_{j,\epsilon} \\ 0; & \text{if } \alpha \in \bigcap_{0<\epsilon<r} U_{j,\epsilon} \end{cases}$$

$\alpha \in X^M$, where $U'_{j,\epsilon} = X^M \setminus U_{j,\epsilon}$, $\epsilon \in (0,r)$, has properties:

(1.6) the family $\sigma = (\sigma_j)_{j\in J}$ is an $-\mathcal{H} - m-$ metric on X,

(1.7) $\mathcal{T}_\sigma = \mathcal{T}_\mathcal{U}$,

(1.8) (X,σ) is sequentially complete iff $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is complete.

We say that the function $T : [0,r)^M \to [0,r)$, $r \in R_+^\sharp$, is a triangle function on $[0,r)$ if

(T.1) $T(a,...,a) \leq a$ for each $a \in [0,r)$,

(T.2) $T(\alpha) = T(p(\alpha))$ for each $\alpha \in [0,r)^M$ and $p \in \prod_M$,

(T.3) $T(\alpha) \leq T(\alpha^1)$ for each $\alpha, \alpha^1 \in [0,r)^M$, $\alpha \leq \alpha^1$ (i.e. $a_i \leq a_i^1$ for $i \in M$).

Let $(J,\leq)$ be a partially ordered and directed set. Let X be a non - void set, $\mathcal{B}$ be a family of nonempty subsets of X^M of the form $\mathcal{B} = \{U_{j,\epsilon} \subseteq X^M : j \in J,\ \epsilon \in (0,r)\}$, $r \in R_+$, such that $(\mathcal{B}.1)_\mathcal{H} - (\mathcal{B}.3)_\mathcal{H}$ and $(\mathcal{B}.5)_\mathcal{H}$ are fulfilled and in addition the following condition holds:

$(\mathcal{B}.4)_\mathcal{M}$ for each $j \in J$ and $\epsilon = (\epsilon_0, ..., \epsilon_m) \in (0,r)^M$ the relation holds $\bigcirc_{k=0}^m U_{j,\epsilon_k} \subseteq U_{j,T_j(\epsilon)}$, where T_j is a triangle function on $[0,r)$, $j \in J$.

We say that an $m-$ uniform structure $(X,\mathcal{U})$ is an $\mathcal{M} - m-$ structure if it is generated by $\mathcal{B}$ fulfilling $(\mathcal{B}.1)_\mathcal{H} - (\mathcal{B}.3)_\mathcal{H}$ and $(\mathcal{B}.5)_\mathcal{H}$ and $(\mathcal{B}.4)_\mathcal{M}$. The ordered triple $(X,\mathcal{U},\mathcal{T}_\mathcal{U})$ is called then an $m - \mathcal{M}-$ space.

Theorem 1.4. *(see [34], Th. 6.2).*
Let $(X, -, -_{-})$ *be an* $m - - -$ *space with the base* $- = -U_{j,\epsilon} : j - J,\ \epsilon -$ $(0, r)-$ *of its* $- \ - m-$ *structure* $-$ *and* $\sigma_j : X^M - R^{\sharp}_{+}$, $j - J$, *are defined as in Theorem 1.3. Then*

(1.9) (X, σ) *is an space generated by the family* $\sigma = (\sigma_j)_{j-J}$ *(i.e.* (X, σ) *is an* $m-$ *metric space),*

(1.10) $-_{-} = -_{\sigma}$

(1.11) as (1.8).

An $H - m-$ structure is the $- - m$ structure for which card $J = 1$ (see Example 4.2 of [34]) and the ordered triple $(X, -, -_{-})$ is called then an $m - H-$ space (for $m = 1$, see [31]).

An $M - m-$ structure is the $- \ - m-$ structure for which card $J = 1$ and $(X, -, -_{-})$ is called then an $m - M-$ space (for $m = 1$, see [31]).

If card $J = 1$, then the $- - m-$ metric space (X, σ) is called an $H - m-$ metric space and the $- \ - m-$ metric space is called then an $M - m-$ metric space, respectively (see [34]).

Let D be the set of all distribution functions (see B. Schweizer and A. Sklar [45]), i.e. D is the set of all non - decreasing, left continuous functions $f : R - [0, 1]$ with $\inf f = 0$ and $\sup f = 1$.

Let $F : X^M - R - D$, $F_\alpha(t) - F(\alpha, t)$, $t - R$, $\alpha - X^M$.

We say that F is a probabilistic $m-$ metric structure on X if the following conditions hold

(F.1) $F_\alpha(0) = 0$ for all $\alpha - X^M$,

(F.2) $F_\alpha(\epsilon) = 1$ for each $\epsilon > 0$ iff $\alpha - \Delta$,

(F.3) $F_\alpha = F_{p(\alpha)}$ for each $\alpha - X^M$ and $p - \prod_M$.

The pair (X, F) is called then a probabilistic $m-$ metric space ($PM - m-$ space).

An $H - m-$ space (for $m = 1$ compare T. Hicks and P.L. Sharma [20]) is a $PM - m-$ space that satisfies the following condition:

$(\text{F.4})_H$ for each $\epsilon > 0$ there exists $\delta > 0$ that $F_\alpha(\epsilon) > 1 - \epsilon$ whenever $F_{\alpha_{a_i \to v}}(\delta) > 1 - \delta$, for each $i - M$ and $v - X$.

We say that a $PM - m-$ space (X, F) is a Menger $m-$ space if it has properties (F.1) - (F.3) and

$(F.4)_M$ for $\epsilon = (\epsilon_0, ..., \epsilon_m) - (0,1)^M$,

$$F_\alpha(\sum_{i=0}^{m} \epsilon_i) > 1 - T(\epsilon), \quad \text{whenever} \quad F_{\alpha_{a_i \to v}}(\epsilon_i) > 1 - \epsilon_i$$

for each $i - M$, $\alpha - X^M$, $v - X$, where T is a triangle function fulfilling (T.1) - (T.3).

Lemma 1.1. *Let* $(\Omega, -, P)$ *be a probability space, and let* $A_1, ..., A_n - -$. *Then*

$$P(\bigcap_{i=1}^{n} A_i) - \sum_{i=1}^{n} P(A_i) - n + 1. \tag{1.12}$$

Proof. Obviously $P(A_1 - A_2) = P(A_1) + P(A_2) - P(A_1 - A_2)$ and therefore $P(A_1 - A_2) - P(A_1) + P(A_2) - 1$ for each $A_1, A_2 - -$. If (1.12) for some n holds, then

$$P(\bigcap_{i=1}^{n+1} A_i) = P(\bigcap_{i=1}^{n} A_i - A_{n+1}) - P(\bigcap_{i=1}^{n} A_i) + P(A_{n+1}) - 1$$

$$\sum_{i=1}^{n} P(A_i) - n + 1 + P(A_{n+1}) - 1 = \sum_{i=1}^{n+1} P(A_i) - (n+1) + 1. \square$$

Remark 1.2.

a) For $m = 1$, the $t-$ norms or triangle functions respectively, are considered in the large number of papers (see for example C.A. Drossos [5], T. Hicks and P.L. Sharma [19] and [20], K. Menger [29], B. Morrel and J.Nagata [35], E.Nishiura [38], B. Schweizer, A. Sklar and E. Thorp [45] - [46], A.N.Sherstnev [49], H. Sherwood [50] and [51] and R.R. Stevens [52]).

b) The following pair $(-, -)$ is well known as a model of a $PM - m-$ space: Let $(\Omega, -, P)$ be a probability space and let $-$ be the set of random variables $Y : \Omega - R$ (i.e. $Y^{-1}((- - , \epsilon)) - -$ for each $\epsilon > 0$). Then the set $A_\epsilon(Y_1, Y_2) = -\omega - \Omega : -Y_1(\omega) - Y_2(\omega) -\!\!< \epsilon-$belongs to $-$. Putting $F_{Y_1 Y_2}(\epsilon) = P(A_\epsilon(Y_1, Y_2))$ for $Y_1, Y_2 - -$ and $\epsilon > 0$, we get a $PM-$ space $(-, -)$, where

$- : - - - - -$ D, $-(Y_1(\omega), Y_2(\omega)) = F_{Y_1Y_2}(\omega)$, $\omega - \Omega$, $\epsilon > 0$. From Lemma 1.1 (compare Drossos [5]) $F_{Y_1Y_3}(\epsilon 1 + \epsilon_2) - T(F_{Y_1Y_2}(\epsilon_1),\ F_{Y_2Y_3}(\epsilon_2))$ for each Y_1, Y_2, Y_3 $- -$ and $\epsilon_1, \epsilon_2 > 0$, where $T(\epsilon_1, \epsilon_2) = \max{-}\epsilon_1 + \epsilon_2 - 1, 0-$ for $\epsilon_1, \epsilon_2 > 0$, is a triangle function (precisely, in this case T is a $t-$ norm of [45]).

2. Fixed points of contractions and remarks on converses of generalized Banach contraction principles in $m-$ metric spaces

Let (X, σ) be an $m-$ metric space, $m - 1$. We say that the mapping $f : X - X$ is $(m, k)-$ contraction, $1 - k - m$, if there exists $\lambda - [0, 1)$ such that for each $\alpha - X^M$ the inequality holds

$$\sigma(\alpha_{a_0 - f a_0, a_1 - f a_1, \ldots, a_k - f a_k}) - \lambda \sigma(\alpha). \tag{2.1.}$$

If in the above definition $k = 1$, then we say that f is an $m-$ contraction and if in addition $m = 1$, then f is a Banach contraction.

Remark 2.1. It is easy to verify that each $m-$ contraction f is continuous in the topology $-_\sigma$ of the $m-$ metric space (X, σ), $m - 1$. If $1 < k - m$ then in general $(m, k)-$ contraction is not continuous in (X, σ) for $m - 2$.

We have (compare for example the result of [33]) the following simple generalization of the Banach fixed point theorem for $m-$ contractions:

Theorem 2.1. *Let (X, σ) be a sequentially complete space generated by the family $\sigma = (\sigma_j)_{j-J}$ of pseudo - $m-$ metrics and $f : X - X$ be an $m-$ contraction, $m - 1$, i.e. the following inequality holds*

$$\sigma_j = (\alpha_{a_0 - f a_0, a_1 - f a_1}) - \lambda \sigma_j(\alpha) \tag{2.2}$$

for each $j - J$, $\alpha - X^M$ and some $\lambda - [0, 1)$.
Then there exists a unique fixed point $\bar{a}$ of f in X and $\lim_{n--} f^n a = \bar{a}$ for each $a - X$.

Proof. If card $J = 1$ and $m = 1$, then we get the well - known Banach fixed - point theorem in a complete metric space. Suppose that card $J - 1$ and

$m - 1$. For each $a - X$, $j - J$ and $\alpha - X^{M_{01}}$ we get the inequality

$$\sigma_j[a, f^n a, \alpha] - \sigma_j[a, fa, \alpha] + \sigma_j[fa, f^n a, \alpha]$$

$$+ \sum_{i=0}^{m-2} \sigma_j[a, fa, \alpha_{a_i} f^n a] - \sigma_j[a, fa, \alpha] + \lambda\sigma_j[a, f^{n-1}a, \alpha]$$

$$+\lambda \sum_{i=0}^{m-2} \sigma_j[a, a, \alpha_{a_i} f^{n-1} a] = \sigma_j[a, fa, \alpha] + \lambda\sigma_j[a, f^{n-1}a, \alpha]$$

$$- \sigma_j[a, fa, \alpha] + \lambda\text{-}\sigma_j[a, fa, \alpha] + \lambda\sigma_j[a, f^{n-1}a, \alpha]-- \dots$$

$$- (1 + \lambda + \dots + \lambda^n)\sigma_j[a, fa, \alpha] - (1 - \lambda)^{-1}\sigma_j[a, fa, \alpha].$$

Thus for any $j - J$, $n - N$ and $s - N$ we get the estimation

$$\sigma_j[f^n a, f^{n+s} a, \alpha] - \lambda^n \sigma_j[a, f^s a, \alpha]$$

$$\lambda^n (1 - \lambda)^{-1} \sigma_j[a, fa, \alpha].$$

Therefore $(f^n a)$ is a Cauchy sequence and $\lim_{n--} f^n a = \bar{a} - X$.

We have

$$\sigma_j[\bar{a}, fa, \alpha] - \sigma_j[\bar{a}, f^n a, \alpha] + \sigma_j[f^n a, f\bar{a}, \alpha]$$

$$+ \sum_{i=0}^{m-2} \sigma_j[\bar{a}, f^n a, \alpha_{a_i f\bar{a}}] - \sigma_j[\bar{a}, f^n a, \alpha]$$

$$+ \sum_{i=0}^{m-2} \sigma_j[\bar{a}, f^n a, \alpha_{a_i f a}] + \lambda\sigma_j[\bar{a}, f^{n-1}a, \alpha].$$

Taking $n - -$ we get $\sigma_j[\bar{a}, f\bar{a}, \alpha] = 0$ for any $\alpha - X^{M_{01}}$ and thus $\bar{a} = f\bar{a}$.

If for some $\bar{b} - X$, $\bar{b} = f\bar{b}$, then for each $\alpha - X^{M_{01}}$, $\sigma_j[\bar{a}, \bar{b}, \alpha] = \sigma_j[f\bar{a}, f\bar{b}, \alpha] - \lambda\sigma_j[\bar{a}, \bar{b}, \alpha]$ and $\bar{a} = \bar{b}$.□

Remark 2.2. By the assumptions of Theorem 2.1 the selfmapping f in X has the property

(2.3) $$Fixf = Perf = \neg\bar{a}\neg,$$

where $Fixf = \neg a - X : a = fa-$ and

$$Perf = \bigcup_{k-N} Fixf^k.$$

Therefore from the Bessaga's theorem [3] for each $\lambda - (0,1)$ there exists a complete metric (1 - metric) d_λ, that f is a contraction with respect to d_λ and (X, d_λ) is a complete metric space. On the other hand we get the simple generalization of Bessaga's theorem in the case of an $m-$ metric space.

Theorem 2.2. *Let X be a nonempty set and let $f : X - X$ be such that $Perf = -\bar{a}-$. Then for each $m - N$ and each $\lambda - (0,1)$ there exists an $m-$ metric σ_λ such that f is a $m-$ contraction in the sequential complete $m-$ metric space (X, σ_λ).*

Proof. C.Bessaga in [3] define the function $- : X - Z^\sharp$, $Z^\sharp = Z - -+- -$, such that $-(-(a)) - -(a) + 1$ for $a - X$ (this function $-$ exists by the Axiom of Choice). Now we define the function $\mu : X^m - Z^\sharp$ in the following way

$$\mu(\alpha) = \begin{cases} -, & \text{if } a_i = \bar{a} \text{ for each } i - M, \\ \sum_{a_i - a,\ i - M} -(a_i) & \text{if } a_i - \bar{a} \text{ for some } i - M. \end{cases}$$

Putting

$$\sigma_\lambda(\alpha_{a_0 - \bar{a}}) = 0 \quad \text{if} \quad \alpha_{a_0 - \bar{a}} - \Delta \quad \text{and}$$
$$\sigma_\lambda(\alpha_{a_0 - \bar{a}}) = \lambda^{\mu(\alpha_{a_0 \to \bar{a}})}, \quad \text{if} \quad \alpha_{a_0 - \bar{a}} - \Delta,$$

we define the required $m-$ metric in the following way

$$\sigma_\lambda(\alpha) = \begin{cases} 0, & \text{if } \alpha - \Delta \\ \sum_{i=0}^m \sigma_\lambda(\alpha_{a_i - \bar{a}}), & \text{if } \alpha - \Delta. \end{cases}$$

It is obvious, that if $a - b$, then $\sigma_\lambda[a, b, \alpha] > 0$ for some $\alpha - X^{M_{01}}$. From the definition, $\sigma_\lambda(\alpha) = \sigma_\lambda(p(\alpha))$ for each p from $\prod_M$.

We also have

$$\sum_{i=0}^m \sigma_\lambda(\alpha_{a_i - v}) = \sum_{i=0}^m \sum_{j=0}^m \sigma_\lambda(\alpha_{a_i - v, a_j - \bar{a}})$$

$$- \sum_{i=0}^m \sigma_\lambda(\alpha_{a_i - \bar{a}}) = \sigma_\lambda(\alpha)$$

and therefore (X, σ_λ) is an $m-$ metric space in the Gähler's sense.

For the proof of sequential completeness of (X, σ_λ) suppose that $(x_n) - X^N$ be such that $\lim_{n,p - -} \sigma_\lambda[x_n, x_{n+p}, \alpha] = 0$ for any $\alpha - X^{M_{01}}$. But then from the definition of σ_λ we get $x_n = \bar{a}$ for each $n - n_0$ and $\lim_{n--} x_n = \bar{a} - X$.

Remark 2.3.

a) It is well - known, that the Banach contraction theorem has the large number of generalizations and that Theorem 2.1 has generalizations in the case $m = 2$ (see for example K Iseki, P.L. Sharma, B.K. Sharma [23] and [48], M. Khan, B. Fisher [26] - [27], B.E. Rhoades [43], our papers [32] - [33] and others).

b) On the other hand for contraction mappings in a metric space ($m = 1$) are known converses (see for example A.A Ivanov [25], P. Meyers [30] and others). Furthermore, if $m = 1$, there are established converses for contraction mappings in uniform spaces (see for example V. Angelov [1]). Therefore there is a natural question: are some versions of Meyer's converses of Banach contraction principles for $m > 1$ true ? We have not an answer to this question.

c) In the fixed point - theory for $(m, k)-$ contractions $1 < k - m$ there are many questions. For example a (2, 2) - contraction in a sequentially complete 2 - metric space may not have a fixed point.

Example 2.1. Let card $X - 2$ and (X, σ) be a sequentially complete 2 - metric space. Let A be a nonempty subset of $X, A - X$. Given the different points $a, b - X$, such that $a - A$ and $b - A$ we define the non - continuous function $f : X - X$ of the form

$$f(x) = \begin{cases} a & \text{for } x - A \\ b & \text{for } x - A^{-} = X - A. \end{cases}$$

Then $(fx, fy, fz) = 0$ for each $(x, y, z) - X^3$. Thus for each $\lambda - [0, 1)$ and each $(x, y, z) - X^3$,

$$\sigma(fx, fy, fz) - \lambda\sigma(x, y, z).$$

The mapping f has not any fixed points i.e. Fix $f = -$. But $Per f - -$ because for example $f^2 a = a$ and $f^2 b = b$. It is easy to verify that f is not continuous in the topology $-_\sigma$ but f^2 is continuous in every point of X.

Our proposition on the fixed - point theorem for the $m, k-$ contraction mapping is the following theorem:

Theorem 2.3. *Let (X, σ), $\sigma = (\sigma_j)_{j-J}$ be a sequentially complete $- -$ $m-$ metric space with $m > 1$. Let $f : X - X$, $f(X) = X$, be $(m, k)-$*

contraction with $k - -2, ..., m-$ *Suppose, that there exists* $x - X$ *such that* $\sigma_j[x, f^n x, \alpha] - W_j < -$ *for any* $n - N$, $j - J$ *and* $\alpha - X^{M_{01}}$. *Then* $Fix f \neq - -$

Proof. For $\alpha - X^{M_{01}}$ there exists a sequense of points (z^s), $z^s - X^{M_{0,...,m-k-1}}$, $s -$ N, such that the inequality holds

$$\sigma_j[f^n x, f^{n+p} x, \alpha] =$$

$$\sigma_j(f^n x, f^{n+p} x, f^n z_0^n, ..., f^n z_{m-k-1}^n, a_{m-k}, ..., a_m) -$$

$$\lambda^n \sigma_j(x, f^p x, z_0^n, ..., z_{m-k-1}^n, a_{m-k}, ..., a_m) - \lambda^n -W_j,$$

$$n, p - N, \ \ j - J,$$

Thus $(f^n x)$ Is a Cauchy sequence and $\lim_{n--} f^n x = \bar{a} - X$. We have

$$\sigma_j[\bar{a}, f^n a, \alpha] - 0, \ \sigma_j[\bar{a}, f^n x, \alpha_{a_i} f a] - 0, \ i = 0, ..., m-k-1,$$

$$\sigma_j[fa, f^n x, \alpha] - \lambda \sigma_j[\bar{a}, f^{n-1} x, \alpha] - 0, \ \text{ as } n - - .$$

Thus $\sigma_j[\bar{a}, f\bar{a}, \alpha] < \epsilon$ for each $\epsilon > 0$ and each $j - J$. Consequently, we get $\bar{a} = f\bar{a}$.

Remark 2.4. If $f : X - X$ is the $(m,k)-$ contraction in an $- - m-$ metric space, then for each $\bar{a}_0, ..., \bar{a}_k - Perf$ we get the equality $\sigma_j(\bar{a}_0, ..., \bar{a}_k,$ $z_0, ..., z_{m-k-1}) = 0$ for each $(z_0, ..., z_{m-k-1}) - X^{M_{0,...,k}}$, $j - J$. Indeed, if $\bar{a}_i = f^{l_i} \bar{a}_i$, $i - -0, ..., k-$ then

$$\sigma_j(\bar{a}_0, ..., \bar{a}_k, z_0, ..., z_{m-k-1}) =$$

$$\sigma_j(f^1 a_0, ..., f^1 a_k, z_0, ..., z_{m-k-1}) -$$

$$\lambda^1 -\sigma_j(\bar{a}_0, ..., \bar{a}_k, z_0, ..., z_{m-k-1})$$

for $z = (z_0, ..., z_{m-k-1}) - X^{M_{0,...,k}}$, $j - J$, where l is a minimal common multiple of $l_0, ..., l_k$. This fact has the interesting interpretation in $m-$ normed spaces.

Remark 2.5. For $m - 1$, $(X, --)$ is called a linear $m-$ normed space (see S.Gähler [12] and A. White [56]) if X is a linear space and $--: X^{-0,...,m-1-} -$ R_+ is such that for each $\alpha = (a_0, a_1, ..., a_{m-1}) - X^{-0,...,m-1-}$ the conditions hold

(i) $-\alpha- = 0$ iff $a_0, ..., a_{m-1}$ are linearly dependent

(ii) $-\alpha-= -p(\alpha)-$for each $p - \prod_{-0,\dots,m-1\to}$

(iii) $-\alpha_{a_0}-\beta_{a_0}-= -\beta-\alpha-$for β real,

(iv) $-\alpha_{a_0}-a'_0+a''_0 -- -\alpha_{a_0}-a'-+ -\alpha_{a_0}-a''-$.

Obviously, $\sigma(\alpha) = -a_1 - a_0, a_2 - a_1, \dots, a_m - a_{m-1}-$for $\alpha = (a_0, \dots, a_{m-1}, a_m) - X^M$ is then a Gähler's $m-$ metric on X.

Example 2.2. Let $X = R^2$ and $--, --, ---$be defined on X in the following way:

$$-A-= \sqrt{x_A^2 + y_A^2 + z_A^2} \quad \text{(a usual Euclidean norm on } X\text{)},$$

$$-A, B-= abs \begin{vmatrix} i & j & k \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix} \quad \text{(a 2 - norm on } X\text{, comp. White [56])},$$

$$-A, B, C-= abs \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix} \quad \text{(a 3 - norm on } X\text{, respectively)},$$

where $A = (x_A, y_A, z_A)$, $B = (x_B, y_B, z_B)$, $C = (x_C, y_C, z_C)$. Then σ_1, σ_2 and σ_3, where $\sigma_1(A, B) = -B - A-$, $\sigma_2(A, B, C) = -B - A, C - B-$and $\sigma_3(A, B, C, D) = -B - A, C - B, D - C-$, are 1 - metric, 2 - metric and 3 - metric, respectively. It is easy to see that $(X, --)$, $(X, ---)$, $(X, ----)$ are Banach $k-$ normed spaces, $k = 1, 2, 3$ (compare A. White [56], Ex. 1.1).

Let $f : X - X$, $f(A) = (f_1 x_A, f_2 y_A, f_3 z_A)$.

If $\sigma_1(fA, fB) - \lambda\sigma_1(A, B)$ for $A, B - R^3$ and some $\lambda - [0, 1)$, then there exists a unique fixed point of f in R^3.

If $\sigma_2(fA, fB, fC) - \lambda\sigma_2(A, B, C)$, $A, B, C - R^3$, $\lambda - [0, 1)$, then in general f has not a fixed point in R^3. But, if $Perf \neq - -$, then for $A, B, C - Perf$, $A = B = C$ or A, B, C belong to the same line in R^3.

If $\sigma_3(fA, fB, fC) - \lambda\sigma_3(A, B, C, D)$, for $\lambda - [0, 1)$ and $A, B, C, D - R^3$, then in general f has no fixed points. But if $Perf \neq - -$, then for $A, B, C, D - Perf$, $A = B = C = D$ or A, B, C, D belong to the same line or A, B, C, D belong both to the same plane in R^3.

Remark 2.6. Let $X = R^n$, and $f : X - X$ be such that $Perf \neq - -$. Let us consider the sequence $-\sigma_1, \sigma_2, \dots, \sigma_n-$of $k-$ metrics, $k = 1, \dots, n$, which are generated by $k-$ norms, $k = 1, \dots, n$, $n - N$, respectively.

We define the sequence of relations $-_2, ..., -_{n+1}-$in the following way:

$$(A_0, A_1) - -_2 - X^2 \quad \text{iff} \quad A_0, A_1 - Perf \quad \text{and}$$

$$\sigma_1(fA_0, fA_1) - \lambda_1\sigma_1(A_0, A_1), ...,$$

$$(A_0, A_1, ..., A_n) - -_{n+1} - X^{n+1} \quad \text{iff} \quad A_0, ..., A_n - Perf \quad \text{and}$$

$$\sigma_n(fA_0, fA_1, ..., fA_n) - \lambda_n\sigma_n(A_0, A_1, ..., A_n),$$

$$\lambda_1, ..., \lambda_n - [0, 1).$$

It is easy to see that the finite sequence $-_2, ..., -_{n+1}-$is an $E-$ sequence (i.e. $n-$ ary equivalence relation of W. Szmielew [53]):

(i) $-_2$ is the identity relation on $Perf$,

(ii) (Reflexive law): $(A_0, ..., A_k) - -_{k+1} =- (A_0, ..., A_k, C) - -_{k+2}$ for any $C - Perf$, $k = 1, ..., n-1$,

(iii) (Transitive law): $(A_0, ..., A_k) - -_{k+1} - (A_0, ..., A_k, C_i) - -_{k+2}$, $k = 0, ..., k+1 =- (C_0, ..., C_{k+1}) - -_{k+2}$, $k = 1, ..., n-1$.

For $(A_0, A_1) - -_2$, $[A_0, A_1] := -C : (A_0, A_1, C) - -_3-$and for $(A_0, ..., A_k) - -_{k+1}$, $[A_0, ..., A_k] := -C : (A_0, ..., A_k, C) - -_{k+2}-$, $k = 2, ..., n-1$, and we say ([53], p. 9) that $[A_0, A_1]([A_0, ..., A_k]$, respectively) is then the equivalence class of $-_3$ (of $-_{k+2}$, respectively spanned by elements A_0, A_1 $(A_0, ..., A_k$, respectively), $k = 2, ..., n-1$.

Thus, if in Example 2.2, $Perf =- -$, then $[A_0]$ has only one element $A_0 - Perf$, $[A_0, A_1] = -C - Perf : C$ belongs to the line, which passes across $A_0, A_1 -, ..., [A_0, ..., A_{n-1}] = -C - Perf : C$ belongs to the hiperplane, which passes across $A_0, ..., A_{n-1}-$.

The following version of Bessaga's theorem [3] may be proved.

Theorem 2.4. *Let X be a nonempty set and let f be a surjective mapping of X in itself such that $Fix f =- -$ and card $Perf > 1$. Then for each $m - 2$ and each $\lambda - (0, 1)$ there exists an $m-$ metric σ_λ such that f is $(m, m)-$ contraction and (X, σ_λ) is sequentially complete.*

Proof. Let $-_2 - X^2$ be an equivalence relation defined by C. Bessaga in [3] in the following way:

$(a, a') - -_2$ iff there exist $p, q - N$, that $f^p a = f^q a'$.

Let $\phi : X/-_2 - X$ be a choice function such that $\phi([\bar{a}]) = \bar{a}$ if $\bar{a} - Perf$. If $a - [\bar{a}]$, where $\bar{a} - Perf$ and $a \neq - \bar{a}$, then there exist p, q, such that $f^p a = f^q \bar{a}$ and $f^1 a = \bar{a}$ for some 1. Thus there exists s, such that $f^{p+s} a = f^{q+s} \bar{a} = f^{nl} \bar{a} = \bar{a}$ for some $n - N$. Let s be a minimal number for which the above equality holds. Thus there exists p that $f^p a = \bar{a}$ and $f^{p-1} a \neq - \bar{a}$.

If $a - [a_0]$ and $a_0 - - Perf$ then there exist p, q that $f^p a = f^q a_0$. If there exist p^-, q^- that $f^{p'} a = f^{q'} a_0$, then $p - q = p^- - q^-$. Indeed, if $p - q = p^- - q^- + s$, and for example s is positive, then $f^q a_0 = f^p a = f^{p-p'} f^{p'} a = f^{p-p'} f^{q'} a_0 = f^{p-p'+q'} a_0$. Thus $f^q a_0 = f^{p-p'+q'-q+q} a_0 = f^{s+q} a_0$ and $f^s f^q a_0 = f^q a_0$, $f^p a = f^s f^p a$, i.e. $f^q a_0$ and $f^p a$ belong to $Perf$ and therefore $p - q = constans$.

Let $- : X - Z^\sharp$ be a function of the form

$$-(a) = \begin{cases} -, & \text{if } a - Perf \\ p, & \text{if } f^p a - Perf \text{ and } f^{p-1} a - Perf, \\ p - q, & \text{if } a - [a_0]_{-2} \text{ and } f^p a = f^q a_0, [a_0] - Perf = - . \end{cases}$$

Let $\nu : X^{M_0} - Z^\sharp$ have the form

$$\nu(a_0, \ldots, a_{m-1}) = \begin{cases} -, & \text{if } a_i - Perf \text{ for each } i - -0, \ldots, m-1-, \\ \min - (a_i) : i - -0, \ldots, m-1 -, & \text{if } a_i - Perf \text{ for some } i - -0, \ldots, m-1-. \end{cases}$$

Let $\bar{a}$ be a given fixed point of f in X.

Putting .

$$\sigma_\lambda(\alpha_{a_j - \bar{a}} = \begin{cases} 0, & \text{if } \alpha_{a_j - \bar{a}} - \Delta \\ \lambda^{\nu(a_0, \ldots, a_j, \ldots, a_{m-1})}, & \text{if } \alpha_{a_j - \bar{a}} - \Delta, \end{cases}$$

for $\alpha - X^M$, $j - M$, where $(a_0, \ldots, a_j, \ldots, a_{m-1}) = (a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{m-1})$. and taking

$$\sigma_\lambda(\alpha) = \begin{cases} 0, & \text{if } \alpha - \Delta \\ \sum_{i=0}^{m} \sigma_\lambda(\alpha_{a_i - \bar{a}}) & \text{if } a_i - a_j \text{ for each } i - j,\ 0 - i, j - m, \end{cases}$$

we get the desired $m-$ metric σ_λ compare the proof of Theorem 2.2. It is obvious, that $\sigma_\lambda(fa_0, ..., fa_m) - \lambda\sigma_\lambda(\alpha)$, and that (X, σ_λ) is a sequentially complete $m-$ metric space.

Remark 2.7.

a) From Theorem 2.4 we get the conclusion: each $(m,k)-$ contraction f (in $- - m-$ metric space (X,σ)) fulfilling assumptions of Theorem 2.3, $2 - k - m$, $m - 2$, is an $(m,m)-$ contraction in some sequentially complete $m-$ metric space (X, σ_λ) for any $\lambda - (0,1)$. If (X,σ) is an $m-$ metric space, then the following question remains without the answer: is there an $m-$ metric σ_λ, topologically equivalent to σ, that f is an $(m,m)-$ contraction in (X,σ_λ) for any $\lambda - (0,1)$? (Compare the papers [25], [30]).

b) It is easy to formulate some versions of Theorem 2.3, in which a fixed point of an $(m,k)-$ contraction, $2 - k - m$, is unique. But uniqueness of a fixed point is not a natural assertion for such $(m,k)-$ contractions, rather and it is true under very strong conditions. For example, if $\sigma_j(\alpha) - W_j < -$ for each $j - J$, then $(m,k)-$ contraction in a sequential complete $- - m-$ space $(X, \sigma_j)_{j-J}$ has a unique fixed point for $k - 2, ..., m-$, $m - 2$.

Remark 2.8. As a result of the "good properties" of an $m-$ metric there exist a large number of fixed - point theorems for contraction type mappings in a $m-$ metric space. Moreover, there exist the well - known models of such theorems for contraction mappings in a usual metric (see for example B. Rhoades [43]). On the other hand the proofs of fixed - point assertions in some $m-$ uniform spaces as for example in $PM - m-$ spaces slightly more complicated. The metrization theorems of -1 give us a possibility to translate the large number of contraction type conditions from $m-$ uniform spaces into $m-$ metric spaces, respectively.

3. Fixed points of m - contractions in uniform m - $\mathcal{H}-$ spaces and m - $\mathcal{M}-$ spaces

Theorems 1.3 - 1.4 give a possibility to formulate the large number of fixed - point theorems for $m-$ contraction mappings in $m- - -$ spaces and $m- - -$ spaces, respectively. The proofs of some fixed - point theorems are reduced to the proofs of some fixed - point theorems for $m-$ contraction mappings in $- - m-$ metric spaces and $- - m-$ metric spaces, respectively. We prove here some versions of Banach contraction principles in – and – – spaces.

Theorem 3.1. *Let $(X, -, -\_)$ be a sequentially complete – – space $(m - 1)$, with the base $- = -U_{j,\epsilon} - X^M : 0 < \epsilon < r,\ j - J-$ of its uniformity and let $f : X - X$ have the property*

(3.1) *for each* $t - (0, r)$, *if* $\alpha - U_{j,t}$ *then*

$$\alpha_{a_0 - fa_0, a_1 - fa_1} - U_{j,\lambda t},$$

for $\alpha - X^M$, $j - J$, where $\lambda - (0,1)$. Then there exists a unique fixed point $\bar{a}$ of f in X.

Proof. Let σ_j be a function of Theorem 1.3. The condition (3.1) implies the inequality (2.2) of Theorem 2.1 for $\sigma = (\sigma_j)_{j-J}$. Indeed, $\alpha - U_{j,\epsilon}$ iff $\sigma_j(\alpha) < \epsilon$ and $\sigma_j(\alpha_{a_0 - fa_0, a_1 - fa_1}) - \lambda\epsilon$ Since $\epsilon > \sigma_j(\alpha)$ was arbitrary, the inequality holds

$$\sigma_j(\alpha_{a_0 - fa_0, a_1 - fa_1}) - \lambda\sigma_j(\alpha), \quad \alpha - X^M.$$

From the definition of σ, $\sigma_j(\alpha) - r$ for any $\alpha - X^M$.

Let $x - X$. Then for each $n - N$ and $p - N$ the inequality holds

$$\sigma_j[f^n x, f^{n+p}x, \alpha] - \lambda^n \sigma_j[x, f^p x, \alpha] - \lambda^n r$$

for any $\alpha - X^{M_{01}}$. Thus $(f^n x)$ is a Cauchy sequence convergent to $\bar{a} - X$.

We also have

$$\lim_{n- -} \sigma_j[f^{n+1}x, \bar{a}, \alpha] = \lim_{n- -} \sigma_j[f^{n+1}x, f\bar{a}, \alpha] = 0$$

for each $\alpha - X^{M_{01}}$. Indeed, $\sigma_j[f^{n+1}x, f^n x, \alpha_{a_i - \bar{a}}] - 0$ as $n - -$, $i = 0, ..., m-2$, and from $(M.3)_-$, $\sigma_j[f^{n+1}x, \bar{a}, \alpha] - 0$. Now, from $\sigma_j[f^{n+1}x, f^n x, \alpha] - 0$, $\sigma_j[f^{n+1}x, f^n x, \alpha_{a_i - \bar{a}}] - 0$ and $\sigma_j[f^n x, f\bar{a}, \alpha] - \lambda\sigma_j[f^{n-1}x, \bar{a}, \alpha] - 0$, we get $\sigma_j[f^{n+1}x, f\bar{a}, \alpha] - 0$ as $n - -$ for each $i = 0, ..., m-2$.

Thus from $(M.3)_-$, for arbitrary $\epsilon > 0$, $\sigma_j[\bar{a}, f\bar{a}, \alpha] < \epsilon$ for any $\alpha - X^{M_{01}}$. Therefore $\bar{a} = f\bar{a}$ and if $\bar{b} = f\bar{b}$, then the inequality $\sigma_j[\bar{a}, \bar{b}, \alpha] - \lambda\sigma_j[\bar{a}, \bar{b}, \alpha]$, $j - J$, $\alpha - X^{M_{01}}$, implies $\bar{a} = \bar{b}$.

Remark 3.1. Let (X, σ) be a sequentially complete – – $m-$ metric space and $f : X - X$ be an $m-$ contraction mapping with a contraction constant

$\lambda - (0,1)$. Then there exists a unique fixed point $\bar{x}$ of f in X. Indeed, if $- = -U_{j,\epsilon} : j - J, \epsilon - (0,r) -$ is the base of the $m-$ uniformity $-$ of its $m - --$ space, then we get (3.1) for f. Obviously, $(X, -, -\_)$ is sequentially complete and all assumptions of Theorem 3.1 are fulfilled.

Remark 3.2. Let for $m - 1$, $(X, -, -\_)$ be a sequentially complete $m -- -$ space with the base $-$ defined as in Remark 3.1. of its uniformity $-$ and let $f : X - X$ fulfil the condition (3.1) of Theorem 3.1. Then there exists a unique fixed point $\bar{a}$ of f in X. Indeed, each $m - - -$ space is $m - --$ space, too.

Theorem 3.2. *Let $(X, -, -\_)$ be a sequentially complete $m - --$ space and $f : X - X$ be such that $f(X) = X$ and the following condition holds*

(3.2) *for each* $\epsilon - (0,r)$, *if* $\alpha - U_{j,\epsilon}$, *then*

$$\alpha_{a_0 - fa_0, \ldots, a_k - fa_k} - U_{j,\lambda\epsilon},$$

for each $\alpha - X^M$, $j - J$, some $k - -1, ..., m -$ and some $\lambda - (0,1)$. Then there exists a unique fixed point $\bar{x}$ of f in X.

Proof. Let σ be the family of σ_j, $j - J$, defined in Theorem 1.3. It is easy to verify, that the condition (3.2) obtains the form

$$\sigma_j(\alpha_{a_0 - fa_0, \ldots, a_k - fa_k}) - \lambda\sigma_j(\alpha), \tag{3.3}$$

for each $\alpha - X^M$ and $j - J$.

All assumptions of Theorem 2.3 are fulfilled and there exists $\bar{x} - X$, such that $\bar{x} = f\bar{x}$. But $\sigma_j - r$ for each $j - J$, and thus $\bar{x}$ is a unique fixed point of f in X (see Remark 2.7. b)).

Remark 3.3. We were mentioned in this paper that there are known the large number of fixed - point theorems for mappings fulfilling contraction type conditions in 2 - metric spaces. These conditions are sometimes very general and sometimes very complicated, too. There are also some nonlinear contractions in 2 - metric spaces. We can not translate some of them from a 2 - metric space to a 2 - uniform space. After all such fixed - point theorems in $m-$ uniform spaces give some profit as we mean.

Theorem 3.3. *Let $(X, -, -\_)$ be a sequentially complete $2 - --$ space and $f : X - X$ be such that the following condition holds*

(3.4) *for each* $t_1, t_2, t_3, t_4, t_5 - (0,r)$,

$$(fx, fy, z) - U_{j,\neg_j(t_1,t_2,t_3,t_4,t_5)} \text{ whenever } (x, y, z) - U_{j,t_1},$$
$$(x, fx, z) - U_{j,t_2}, (y, fy, z) - U_{j,t_3}, (x, fy, z) - U_{j,t_4}$$

and $(fx, y, z) - U_{j,t_5}$*, for each* $j - J$*,* $(x, y, z) - X^M$*,* $M = \neg 0, 1, 2\neg$*, where* $\sigma_j : R_+^5 - R_+$ *is non - decreasing and upper - semicontinuous function, such that* $t = 0$ *is a unique solution of the inequality* $t - \neg_j(t, t, t, t, t)$*,* $t - R_+$*,* $j - J$*. Then there exists a unique fixed point* $\bar{a}$ *of* f *in* X*.*

Proof. Let $\sigma = (\sigma_j)_{j-J}$ be a 2– metric defined as in Theorem 1.4. It is easy to verify that the condition (3.3) obtains the form:

$$\sigma_j(fx, fy, z) - \neg_j(\sigma_j(x, y, z), \sigma_j(x, fx, z), \sigma_j(y, fy, z), \sigma_j(x, fy, z), \sigma_j(fx, y, z)), \tag{3.5}$$

for each $(x, y, z) - X^M$, $j - J$, and as in the proof of Theorem 2.2 of the paper [32], the sequence $(f^n x)$ is convergent to $\bar{x} - X$ for each $x - X$. We have $\sigma_j(\bar{x}, f\bar{x}, z) = 0$ for each $j - J$ and each $z - X$ and thus $\bar{x} = f\bar{x}$. If $\bar{y} = f\bar{y}$, then

$$\sigma_j(\bar{x}, \bar{y}, z) = \sigma_j(f\bar{x}, f\bar{y}, z) -$$
$$\neg_j(\sigma_j(x, y, z), 0, 0, \sigma_j(x, y, z), \sigma_j(x, y, z)) -$$
$$\neg_j(\sigma_j(\bar{x}, \bar{y}, z), \sigma_j(\bar{x}, \bar{y}, z),, \sigma_j(\bar{x}, \bar{y}, z), \sigma_j(\bar{x}, \bar{y}, z), \sigma_j(\bar{x}, \bar{y}, z))$$

for each $z - X$ and each $j - J$. From the properties of $\neg_j$, $\sigma_j(\bar{x}, \bar{y}, z) = 0$ for each $z - X$ and for any $j - J$. Therefore $\bar{x} = \bar{y}$ and the proof is complete.

Remark 3.4. The cited in the proof of Theorem 3.2, the fixed - point result was proved in [32] by less restrictive conditions on the comparative function σ_j, $j - J$. Precisely, we assume in [32], that $- : R_+^5 - R_+$ is non - decreasing function such that $A : R_+ - R_+$, $A(t) = a(t, t, t, t, t)$, $t - R_+$, is upper - semi - continuous and $t = 0$ is a unique solution of the inequality $t - A(t)$, $t - R_+$.

4. Fixed points of contraction mappings in probabilistic m - spaces

The following result is a conclusion from the result of $\neg 3$.

Theorem 4.1. *Let $(X, F, -_F)$ be a sequentially complete probabilistic $m-H-$ space, $m - 1$. Let $f : X - X$ be such that the condition holds*

(4.1) *for each* $t - R_+, F_{\alpha_{a_0} - f a_0, a_1 - f a_1} \lambda t > 1 - \lambda t$

whenever $F_\alpha(t) > 1 - t$, *for all* $\alpha - X^M$, $(M = -0, ..., m-)$ *and some* $\lambda - (0, 1)$. *Then there exists a unique fixed point* $\bar{x}$ *of* f *in* X.

Proof. Taking $U_\epsilon = -\alpha - X^M : F_\alpha(\epsilon) > 1 - \epsilon -, \epsilon - R$, we get the base $- = -U_\epsilon : 0 < \epsilon < 1-$of the $H - m-$ structure $-$. Obviously, the condition (4.1) on f obtains the form (3.1) of Theorem 3.1 (with $r = 1$) in a sequentially complete $m - H-$ space $(X, -, -_-)$ and $-_- = -_F$. Thus from Theorem 3.1, there exists a unique fixed point $\bar{x}$ of f in X.

Example 4.1. Let $(\Omega, -, P)$ be a probability space and (V, σ) be a $h - m-$ metric space, $m - 1$. Let $-$ be a set of functions from Ω into V and $- : -^M - D$, where D is the set of distribution functions defined in -1. Let us suppose that the additional conditions hold:

(i) For all $\alpha - -^M$ and all $\epsilon - R_+$, the set $A_\epsilon(\alpha) = -\omega - \Omega : \sigma(\alpha(\omega)) < \epsilon-$ belongs to $-$, (i.e. the composite function $\sigma(\alpha)$ from Ω into R_+ is $P-$ measurable),

(ii) For all $\alpha - -^M$, $\epsilon - R$, $F_\alpha(\epsilon) = -(\alpha)(\epsilon) = P(A_\epsilon(\alpha))$

(iii) If for $a, b - -$, the equality holds $F_{[a,b,\alpha]}(\epsilon) = 1$ for each $\epsilon > 0$ and every $\alpha - -^{M_{01}}$, then $a = b$.

It is easy to see that conditions $(F.1) - (F.3)$ of $PM - m-$ space are fulfilled for the pair $(-, -)$.

If $\epsilon > 0$, then there exists $\delta > 0$, that $\sigma(\alpha(\omega)) < \epsilon$ whenever $\sigma(\alpha_{a_i - v}(\omega)) < \delta$ for some $v - -$ and each $\omega - \Omega$. Thus, from Lemma 1.1, if $\delta < \frac{\epsilon}{m+1}$, then

$$F_\alpha(\epsilon) = P(A_\epsilon(\alpha)) = P-\omega - \Omega : \sigma(\alpha(\omega)) < \epsilon--$$

$$P(A_\delta(\alpha_{a_0 - v}) - A_\delta(\alpha_{a_1 - v}) - ... - A_\delta(\alpha_{a_m - v})) -$$

$$P(A_\delta(\alpha_{a_0 - v})) + ... + P(A_\delta(a_{a_m - v})) - m > (m+1)(1-\delta) - m =$$

$$1 - (m+1)\delta > 1 - \epsilon$$

and $(F.4)_H$ holds, too. Therefore $(-, F)$ is a probabilistic $H - m-$ structure and $(-, F, -_F)$ is a probabilistic $m - H-$ space, respectively.

Let $g : - - \; -$ be such that the following "contraction condition" with a constant $\lambda - (0,1)$ holds

$$(4.2) \qquad P(A_{\lambda\epsilon}(\alpha_{a_0 - \, g a_0, a_1 - \, g a_1})) > 1 - \lambda\epsilon$$

whenever $P(A_\epsilon(\alpha)) > 1 - \epsilon$, for each $\alpha - -^M$ and $\epsilon > 0$.

If we assume, that $(-, F, -_F)$ is sequentially complete, then from Theorem 4.1, there exists a unique $\bar{a} - -$, such that $g\bar{a} = \bar{a}$ i.e. $(g\bar{a})(\omega) = \bar{a}(\omega)$ for any $\omega - \Omega$.

Remark 4.1.

a) If in Example 4.1, the function $\sigma : V^M - \; R_+$, is an $m-$ metric on V, then obviously σ is an $H - m-$ metric, too, and we get the same assertion in Example 4.1 putting an $m-$ metric σ_m instead of an $H - m-$ metric σ. On the other hand in this case the following "classical" form of condition $(F.4)_M$ holds:

$$(4.3) \qquad F_\alpha(\sum_{i=0}^{m} \epsilon_i) - T(F_{\alpha_{a_0 \to v}}(\epsilon_0), ..., F_{\alpha_{a_m \to v}}(\epsilon_m))$$

for each $\alpha - -^M$, $v - V$ and $\epsilon_0, ..., \epsilon_m > 0$, where $T(t_0, ..., t_m) = \max \neg t_0 + ... + t_m - m, 0 \neg$

Indeed,

$$F_\alpha(\epsilon_0 + ... + \epsilon_m) = P(A_{\epsilon_0 + ... + \epsilon_m}(\alpha)) =$$

$$P \neg \omega - \Omega : \sigma_m(\alpha(\omega)) < \epsilon_0 + ... + \epsilon_m - -$$

$$P(A_{\epsilon_0}(\alpha_{a_0 - \, v})) + ... + P(A_{\epsilon_m}(\alpha_{a_m - \, v})) - m$$

and thus (4.3) holds.

b) If the pair $(-, -)$ is as above, then we say that $(-, -)$ is a Sherwood's canonical $m - E-$ space (for $m = 1$, see [45], Df. 9.1.1) with base $(\Omega, -, P)$ and target (V, σ).

c) If $(-, -)$ is an $m - E-$ space with base $(\Omega, -, P)$ and target (V, σ), then for every ω in Ω the function $\sigma_\omega : -^M - \; R_+$, $\sigma_\omega(\alpha) = \sigma(\alpha(\omega))$ is an $m-$ pseudometric on $-$ for each $\omega - \Omega$. Obviously, for each distinct $p, q - -$, there is $\alpha - -^{M_{01}}$ and at least one $\omega - \Omega$, that $\sigma_\omega[p, q, \alpha] \neq - 0$. We say that the $m - E-$ space $(-, -)$ is pseudometrically generated, i.e. there is a probability space $(\Omega, -, P)$, called the base, such that

(i) for each $\omega - \Omega$ there is an $m-$ pseudometric σ_ω in $-$,

(ii) for any distinc $p, q - -$ there is an $\alpha - -^{M_{01}}$ and such that $\sigma_\omega[p, q, \alpha] > 0$,

(iii) for each $\alpha - -^M$, $\epsilon > 0$, the set $-\omega - \Omega : \sigma_\omega(\alpha) < \epsilon-$ belongs to $-$,

(iv) for all $\alpha - -^M$, $-(\alpha) = F_\alpha$, where $F_\alpha(\epsilon) = P-\omega - \Omega : \sigma_\omega(\alpha) < \epsilon-$.

If in addition each σ_ω is a $m-$ metric on $-$, then $(-, -)$ is $m-$ metrically generated.

d) For $m = 1$, H.Sherwood [1969] has established the fact that, up to isometry, the class of $E-$ spaces and the class of pseudometrically generated spaces coincide ([45], Th. 9.2.2).

Theorem 4.2. *Let $(X, F, -_F)$ be a sequentially complete probabilistic $H - m-$ space and $f : X - X$ be such that $f(X) = X$ and there exists $\lambda - (0,1)$ that the condition holds*

$$\text{(4.4)} \qquad \textit{for each} \quad t - R_+, \quad \textit{if} \quad F_\alpha(t) > 1 - t,$$

then $f_{\alpha_{a_0 \to fa_0, \ldots, a_k \to fa_k}}(\lambda t) > 1 - \lambda t$ for all $\alpha - X^M$, where $k - -1, \ldots, m-$. Then there exists a unique fixed point $\bar{x}$ of f in X.

Proof. Taking $U_\epsilon = -\alpha - X^M : F_\alpha(\epsilon) > 1 - \epsilon-$, $\epsilon - R_+$, we get the base $- = -U_\epsilon : 0 < \epsilon < 1-$of the $H - m-$ structure $-$. Next, the condition (4.4) on f has the form (3.2) of Theorem 3.2. Thus there exists a unique $\bar{x} - X$, that $\bar{x} = f\bar{x}$.

Example 4.2. Let $(\Omega, -, P)$ be a probability space and (V, σ) be an $H - m-$ metric space, $m - 1$. Suppose that $(-, -)$ is the same an in Example 4.1. Let $g : - - -$, $g(-) = -$, be such that

$$\text{(4.5)} \qquad P(A_{\lambda\epsilon}(\alpha_{a_0 - ga_0, \ldots, a_k - ga_k})) > 1 - \lambda\epsilon,$$

whenever $P(A_\epsilon(\alpha)) > 1 - \epsilon$,
for each $\alpha - -^M$, $\epsilon > 0$, some $k - -1, \ldots, m-$and $\lambda - (0,1)$. Let us suppose that $(-, -, -_-)$ is sequentially complete. Then all assumptions of Theorem 4.2 are fulfilled and therefore there exists a unique $\bar{a} = g\bar{a}$.

Theorem 4.3. *Let $(X, F, -_F)$ be a sequentially complete probabilistic $2-$ metric space and $f : X - X$ be such that the condition holds*

$$\text{(4.6)} \qquad \textit{for} \quad t_1, t_2, t_3, t_4, t_5 > 0,$$

$$F_{(fx,fy,z)}(-(t_2t_1,t_2,t_3,t_4,t_5)) > 1 - -(t_1,t_2,t_3,t_4,t_5)$$

whenever

$$F_{(x,y,z)}(t_1) > 1 - t_1,\ F_{(x,fx,z)}(t_2) > 1 - t_2,$$

$$F_{(y,fy,z)}(t_3) > 1 - t_3,\ F_{(x,fy,z)}(t_4) > 1 - t_4 \ \ and$$

$$F_{(fx,y,z)}(t_5) > 1 - t_5,$$

for each $(x,y,z) - X^M$, $(M = -0,1,2-)$ *where* $- : R_+^5 - R_+$ *is non - decreasing and uper - semicontinuous function such that* $t = 0$ *is a unique solution of the inequality* $t - -(t,t,t,t,t)$, $t - R_+$. *Then there exists a unique fixed point* $\bar{a}$ *of* f *in* X.

Proof. If $U_\epsilon = -(x,y,z) - X^M : F_{(x,y,z)}(\epsilon) > 1 - \epsilon -$, $\epsilon - (0,1)$, then $- = -U_\epsilon : 0 < \epsilon < 1-$ is the base of the $M - 2-$ uniformity $-$ and the assumptions of Theorem 3.2 are fulfilled.

Example 4.3. Let $(\Omega, -, P)$ be a probability space let $-$ be a space of functions from Ω into $V = R^n$, $n - 2$. Let $-,-$ be some $2-$ norm on V. Then $-,-$ generate the $2-$ metric σ on R^n. Let $A_\epsilon(\alpha) = -\omega - \Omega : \sigma(\alpha(\omega)) < \epsilon-$ belong to $-$ and let $F_\alpha(\epsilon) = P(A_\epsilon(\alpha))$ for each $\alpha - -^M$ and each $\epsilon > 0$. If (iii) of Example 4.1 holds (for $m = 2$), then $(-,-)$ is a probabilistic $M-2-$ structure, where $-(\alpha)(\omega) = F_\alpha(\epsilon)$, $\alpha - -^M$, $\epsilon > 0$. Suppose that $(-,-,-)$ is sequentially complete and $g : - - \ -$ fulfils the following condition:

(4.7) for all $t_1, t_2, t_3, t_4, t_5 > 0$,

$$P(A_{-(t_1,t_2,t_3,t_4,t_5)}(\alpha_{a_0 - ga_0, a_1 - ga_1})) > 1 - -(t_1,t_2,t_3,t_4,t_5)$$

whenever

$$P(A_{t_1}(\alpha)) > 1 - t_1,\ P(A_{t_2}(\alpha_{a_1 - ga_0})) > 1 - t_2,$$

$$P(A_{t_3}(\alpha)) > 1 - t_3,\ P(A_{t_4}(\alpha_{a_1 - ga_0})) > 1 - t_4,$$

and

$$P(A_{t_5}(\alpha_{a_0 - ga_0})) > 1 - t_5,$$

for each $\alpha - -^M$, where $- : R_+^5 - R_+$ is such as in Theorem 4.3. Then there exists a unique $\bar{a} - -$ such that $g\bar{a} = \bar{a}$.

Remark 4.2. If $(L, -, -)$ is a completely distributive lattice with order revessing involution, then an $L-$ fuzzy set (shortly a fuzzy set) on a set X is any map $A : X - L$ (see for ex. [6]). The fuzzy set theory has beginning with th classic paper [57] of L.Zadeh [1965] for the case $L = [0,1] - R$. When L is a lattice $-0,1-$ then the collection of fuzzy sets corresponds to the characteristic functions of ordinary sets. If in Example 4.1 in the sequel Λ denotes the family of all functions λ_ϵ, $\epsilon - R$, where $\lambda_\epsilon : X^M - [0,1]$, $\alpha - \lambda_\epsilon(\alpha) = F_\alpha(\epsilon)$, then Λ is the collection of fuzzy sets on X^M. Therefore the following observation is obvious: the $H - m-$ metric σ_F of Example 4.1 is de facto generating by Λ.

Remark 4.3. There are known two different classes of definitions of fuzzy metrics. The probabilistic function $F : X - X - R - [0,1]$ is the classic model of one of them (see I. Kramosil, J, Michalek [28]). In our paper we consider only this "fuzzy metric" it's a generalization, respectively, which is a distance function on X (some $m-$ distance function of Gähler on X, respectively). The second class of definitions is more sensible for the fuzzy set theory. In this case a fuzzy set metric is a distance function on the space of fuzzy sets defined on X. For such definition see for example [6]. The usual system of metric axioms is too restrictive for a fuzzy distance function, rather M.A. Erceg in [6] gives a definition of the pseudo - quasi fuzzy metric. For example, $p, q-$ metric does not fulfil the axiom of symmetry. M.Erceg provedin [6] that his $p.q.-$ metric generates some quasi - uniformity of B. Hutton [22] (for the case $L = -0,1-$see M.C. Murdeshwar, S.A. Naimpally [36]) and that this quasi - uniformity generate the same topology. The results of M.Erceg and other's (see for example I. Reilly [42]) give a possibility to formulate some fixed - point theorems for selfmappings on a space of fuzzy sets.

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REZIME

KONTRAKCIJE U VEROVATNOSNIM m - METRIČKIM PROSTORIMA

U ovom radu su dokazane neke teoreme o kontrakciji za samopreslikavanja u Gählerovom kompletnom metričkom prostoru

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