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SOME PROPERTIES OF A MATRIX ARISING FROM BOUNDARY VALUE PROBLEM

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Abstract

In this paper we consider a matrix arising from the discretization of singularly perturbed boundary value problem on a nonequidistant mesh. For such a matrix an estimation for spectral radius of the corresponding Jacobi matrix is obtained.

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1. Introduction

When solving singularly perturbed two - point boundary value problem with a small perturbation parameter $\varepsilon \in (0, \varepsilon_0), \varepsilon_0 \ll 1$, which arise in practise (see [1], [2]), a system of equations is obtained. Such problems were treated numerically in various papers (see [2], [3] where a survey is given).

Under some standard assumptions the solution of singularly perturbed problem has in general two boundary layers. Because of that, a non - equidistant discretization mesh, which is dense in the layers, is used. The mesh is generated by a suitable mesh generating function which maps equidistant points into appropriate mesh points, and the fourth order Hermite scheme and the central difference scheme are used.

The mesh points will be given by

$$I_h = \{t_i = \lambda(ih),\ i = 0, 1, \ldots, n-1\},$$

where $n \in N$, $h = 1/n$ and the mesh generating function is

$$\lambda(t) = \begin{cases} \eta(t) = a\varepsilon t/(q-t), & t \in [0, \alpha] \\ \pi(t) = \eta(\alpha) + \eta'(\alpha)(t-\alpha), & t \in [\alpha, 0.5] \\ 1 - \lambda(1-t), & t \in [0.5, 1] \end{cases} .$$

Here, a and q are constants (independent of ε) such that

$$q \in (0, 0.5), \quad a\varepsilon_0 \le q.$$

α is a unique point from $(0, q)$ which is the abscissa of the contact point of the tangent line from $(0.5, 0.5)$ to λ and it can be found exactly

$$\alpha = \frac{q - \sqrt{aq\varepsilon(1 - 2q + 2a\varepsilon)}}{1 + 2a\varepsilon}.$$

Using the fourth order Hermite scheme and the central difference scheme we obtain a system of equations, for which the main role has the matrix

$$A = \begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & \cdots & \cdots & \cdots & \\ & & a_{n-2} & b_{n-2} & c_{n-2} \\ 0 & & & a_{n-1} & b_{n-1} \end{bmatrix}, \tag{1}$$

where

$$a_i = \frac{-2}{h_i(h_i + h_{i+1})}, \quad b_i = \frac{2}{h_i h_{i+1}}, \quad c_i = \frac{-2}{h_{i+1}(h_i + h_{i+1})}$$

and $h_i = t_i - t_{i-1}$, $i = 1, \ldots, n-1$.

Main results of this paper is an estimation for the spectral radius of Jacobi matrix for matrix A.

2. Estimation of the spectral radius

If $A = D(E - L - U)$ is the standard splitting of matrix A into it's diagonal matrix D, strictly lower triangular matrix $-L$ and strictly upper triangular matrix $-U$, the corresponding Jacobi matrix is

$$\tilde{J} = D^{-1}(L + U).$$

Matrix A is 2 - cyclic if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} D_1 & B_1 \\ B_2 & D_2 \end{bmatrix},$$

where D_1 and D_2 are diagonal matrices.

Matrix A is consistently ordered if the eigenvalues of $\delta D^{-1}L + \delta^{-1}D^{-1}U$ are independent of δ for all $\delta \neq 0$. The tridiagonal matrix A of (1) is consistently ordered and 2 - cyclic.

If we determine the value of $\rho(\tilde{J})$, using the well - known theorem 1, see [6], we can obtain the optimum relaxation parameter for the SOR method.

Theorem 1. *Let A be nonsingular, consistently ordered and 2 - cyclic matrix with all nonzero diagonal elements and let all the eigenvalues of the corresponding Jacobi matrix $\tilde{J} = D^{-1}(L + U)$ be real. Then:*

a) if $\rho(\tilde{J}) < 1$, then the SOR method converges for $0 < \omega < 2$,

b) there exists the optimum relaxation factor

$$\omega_b = \frac{2}{1 + \sqrt{1 - \rho(\tilde{J})^2}}, \quad \rho(\mathcal{L}_{\omega_b}) = \min_{0<\omega<2} \rho(\mathcal{L}_\omega).$$

Because of the definition of mesh generating function $\lambda(t)$ we shall suppose $n = 2m$. Then, Jacobi matrix has the following form

$$\tilde{J} = \begin{bmatrix} 0 & \tilde{c}_1 & & & 0 \\ \tilde{a}_2 & 0 & \tilde{c}_2 & & \\ & \cdots & \cdots & \cdots & \\ & & \tilde{a}_{n-2} & 0 & \tilde{c}_{n-2} \\ & & & \tilde{a}_{n-1} & 0 \end{bmatrix}, \tag{2}$$

with

$$\tilde{a}_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad \tilde{c}_i = \frac{h_i}{h_i + h_{i+1}}, \quad i = 1, \ldots, n-1.$$

Lemma 1. *[5] All the eigenvalues of $\tilde{J}$ are real.*

Proof. Let

$$S = diag\left(1, \sqrt{\frac{\tilde{c}_1}{\tilde{a}_2}}, \sqrt{\frac{\tilde{c}_1\tilde{c}_2}{\tilde{a}_2\tilde{a}_3}}, \ldots, \sqrt{\frac{\tilde{c}_1 \ldots \tilde{c}_{n-2}}{\tilde{a}_2 \ldots \tilde{a}_{n-1}}}\right).$$

Then

$$(3) \qquad J = S\tilde{J}S^{-1} = \begin{bmatrix} 0 & d_1 & & & 0 \\ d_1 & 0 & d_2 & & \\ & \cdots & \cdots & \cdots & \\ & & d_{n-3} & 0 & d_{n-2} \\ 0 & & & d_{n-2} & 0 \end{bmatrix}, \quad d_i = \sqrt{\tilde{a}_{i+1}\tilde{c}_i}.$$

So, matrix $\tilde{J}$ is similar to the symmetric matrix and the lemma is proved.
□

From now on, we are going to analyse the eigenvalues of J.

Lemma 2. *[5] If λ is an eigenvalue of the matrix J, then $-\lambda$ is an eigenvalue of J, too.*

Let J_0 be $(n-1) \times (n-1)$ tridiagonal matrix:

$$J_0 = \begin{bmatrix} 0 & \frac{1}{2} & & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & & & \frac{1}{2} & 0 \end{bmatrix}.$$

In the case $h_i = h$, $i = 1, \ldots, n-1$, i.e. when we use the equidistant mesh, matrix J_0 is Jacobi matrix for A.

Lemma 3. *[6] Matrix J_0 has eigenvalues*

$$\lambda_k = \cos\frac{k\pi}{n}, \ k = 1, \ldots, n-1,$$

and corresponding eigenvectors

$$z_k = [\sin\frac{k\pi}{n}, \sin\frac{2k\pi}{n}, \ldots, \sin\frac{(n-1)k\pi}{n}]^T, \ k = 1, \ldots, n-1.$$

Next theorem gives the upper bound for $\rho(J)$.

Theorem 2. *For $m > \frac{2}{1-2\alpha}$*

$$\rho(J) \leq \rho(J_0).$$

Proof. We have to prove

$$\sqrt{\tilde{a}_{i+1}\tilde{c}_i} \leq \frac{1}{2}, \ i = 1, \ldots, n-1, \tag{4}$$

what means

$$0 \leq J \leq J_0,$$

and that implies, see [6],

$$\rho(J) \leq \rho(J_0).$$

Because of the definition of the mesh generating function it's enought to consider the case $i = 1, \ldots, m$. By the mean value theorem we have

$$h_i = \lambda(ih) - \lambda((i-1)h) = \lambda'(\theta)h, \ \theta \in ((i-1)h, ih),$$

$$h_{i+1} = \lambda((i+1)h) - \lambda(ih) = \lambda'(\tau)h, \ \tau \in (ih, (i+1)h).$$

Also,

$$\lambda'(t) = \begin{cases} \eta'(t), & t \leq \alpha \\ \eta'(\alpha), & t \in [\alpha, 0.5] \end{cases},$$

and

$$\eta'(t) \geq 0.$$

So, we can conclude that

$$\eta'(\theta) \leq \eta'(\tau),$$

and that means

$$h_{i+1} \geq h_i, \ i = 1, \dots, m.$$

As $m > \frac{2}{1-2\alpha}$ there exist $k \leq m-3$ such that $kh \leq \alpha \leq (k+1)h$. Matrix J is block diagonal matrix

$$J = diag(D_1, D_2, D_3, D_2, D_1),$$

where D_1 is $(k-2) \times (k-2)$ matrix,

$$D_1 = \begin{bmatrix} 0 & \frac{1}{2} & & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & & & \frac{1}{2} & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & x & 0 & 0 \\ x & 0 & y & 0 \\ 0 & y & 0 & z \\ 0 & 0 & z & 0 \end{bmatrix},$$

and D_3 has the same structure as D_1 and dimension $(n-2k-4) \times (n-2k-4)$. Matrix D_1 arise from the mesh points $i \in \{1, 2, \dots, k-2\}$ which have the property $t_{i+2} \leq \alpha$.

Nonzero elements have the following properties

$$\tilde{a}_{i+1} = \frac{q - ih}{2(q - ih - h)}, \quad \tilde{c}_i = \frac{q - ih - h}{2(q - ih)},$$

so

$$\tilde{a}_{i+1}\tilde{c}_i = \frac{1}{4}.$$

For $i = k+3, \dots, m$ we have $t_{i-1} \geq \alpha$, $h_i = \eta(\alpha)h = h_{i+1} = h_{i+2}$, so

$$\tilde{a}_{i+1}\tilde{c}_i = \frac{1}{4}.$$

Matrix D_2 arise for $i = k-1, k, k+1, k+2$. Nonzero elements of that matrix are

$$x(k) = \sqrt{\tilde{a}_k \tilde{c}_{k-1}} = \frac{\sqrt{2}}{2}\sqrt{\frac{k - 2mq + (k - 2\alpha m)^2}{-1 + 4\alpha m - 4mq + (k - 2\alpha m)^2}},$$

$$y(k) = \sqrt{\tilde{a}_{k+1}\tilde{c}_k} =$$

$$= 2\sqrt{\frac{m^2(\alpha - q)^2}{(2k - 4mq + (k - 2\alpha m)^2)(-1 + 4\alpha m - 4mq + (k - 2\alpha m)^2)}},$$

$$z(k) = \sqrt{\tilde{a}_{k+2}\tilde{c}_{k+1}} = \frac{\sqrt{2}}{2}\sqrt{\frac{k-2mq+(k-2\alpha m)^2}{2k-4mq+(k-2\alpha m)^2}}.$$

Since

$$kh \le \alpha \le (k+1)h,$$

we have

$$h_i = \eta(ih) - \eta((i-1)h),\ i = k-1, k,$$

$$h_i = \pi(ih) - \pi((i-1)h),\ i = k+2, k+3$$

and

$$h_{k+1} = \pi((k+1)h) - \eta(kh).$$

It is easy to see that

$$sgn(x'(k)) = sgn(2m\alpha - k - 1)(4mq - 6m\alpha + k + 1).$$

As $2m\alpha - 1 \le k \le 2m\alpha$ and $q > \alpha$, we have $x'(k) \le 0$, i.e. $x(k)$ is decreasing function of k, so

$$G + \frac{1}{2} = x(2m\alpha) \le x(k) \le x(2m\alpha - 1) = \frac{1}{2}, \tag{5}$$

$$G = \frac{1}{2}\left(-1 + \sqrt{\frac{q-\alpha}{q-\alpha+\frac{1}{2n}}}\right).$$

Similarly,

$$sgn(z'(k)) = sgn(2m\alpha - k)(4mq - 2m\alpha - k),$$

and $z'(k) \ge 0$, for $2m\alpha - 1 \le k \le 2m\alpha$, $q > \alpha$ and

$$G + \frac{1}{2} = z(2m\alpha - 1) \le z(k) \le z(2m\alpha) = \frac{1}{2}. \tag{6}$$

Also,

$$sgn(y'(k)) = sgn(1 + 2k - 4\alpha m)(1 - k - (k - 2\alpha m)^2 - 2\alpha m + 4nq).$$

As

$$(1 - k - (k - 2\alpha m)^2 - 2\alpha m + 4nq) > 0,$$

for $2m\alpha - 1 \le k \le 2m\alpha$, and

$$1 + 2k - 4m\alpha = 0, \text{ for } k = 2m\alpha - 0.5,$$

function $y(k)$ is decreasing for $2m\alpha - 1 \le k \le 2m\alpha - 0.5$ and increasing for $2m\alpha - 0.5 \le k \le 2m\alpha$, and

$$D + \frac{1}{2} = y(2m\alpha - 0.5) \le y(k) \le y(2m\alpha) = x(2m\alpha), \tag{7}$$

$$D + \frac{1}{2} = y(2m\alpha - 0.5) \le y(k) \le y(2m\alpha - 1) = z(2m\alpha - 1). \tag{8}$$

$$D = \frac{3}{32(q - \alpha + \frac{3}{8n})}.\square$$

It is easy to see that $y(k) \le x(k)$, and $y(k) \le z(k)$, by inequalities (5), (6),(7),(8).

The lower bound for $\rho(J)$ will be estimated by the following theorem.

Theorem 3. *[4] Let A, B be Hermite matrices and let $\lambda_i(A), \lambda_i(B), \lambda_i(A+B)$ denote their eigenvalues respectively. If*

$$\lambda_1(A) \le \lambda_2(A) \le \ldots \le \lambda_n(A)$$

and

$$\lambda_1(B) \le \lambda_2(B) \le \ldots \le \lambda_n(B)$$

then

$$\lambda_k(A) + \lambda_1(B) \le \lambda_k(A+B) \le \lambda_k(A) + \lambda_n(B).$$

Theorem 4. *For $m > \frac{2}{1-2\alpha}$*

$$\rho(J) \ge \rho^- = \rho(J_0) - \frac{1}{\sqrt{2}}\sqrt{2G^2 + D^2 + |D|\sqrt{D^2 + 4G^2}}.$$

Proof. Matrix J can be written as

$$J = J_0 + diag(0, \tilde{D}_2, 0, \tilde{D}_2, 0),$$

with

$$\tilde{D}_2 = \begin{bmatrix} 0 & x-\frac{1}{2} & 0 & 0 \\ x-\frac{1}{2} & 0 & y-\frac{1}{2} & 0 \\ 0 & y-\frac{1}{2} & 0 & z-\frac{1}{2} \\ 0 & 0 & z-\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & p & 0 & 0 \\ p & 0 & r & 0 \\ 0 & r & 0 & s \\ 0 & 0 & s & 0 \end{bmatrix}.$$

Matrix $\tilde{D}_2$ has real eigenvalues $\pm\lambda_1, \pm\lambda_2$, which can be obtained as a solution of the equation

$$\lambda^4 - (p^2 + r^2 + s^2)\lambda^2 + p^2 s^2 = 0.$$

Let $\lambda_1 \geq \lambda_2 \geq 0$. Then $\rho(\tilde{D}_2) = \lambda_1$. It is easy to see that $\lambda_1 = \rho(p, r, s)$, where

$$\rho(p,r,s) = \frac{1}{\sqrt{2}}\sqrt{p^2 + r^2 + s^2 + \sqrt{(p^2 + r^2 - 2ps + s^2)(p^2 + r^2 + 2ps + s^2)}}.$$

We are going to determine $\max \rho(p, r, s)$, for $kh \leq \alpha \leq (k+1)h$.

By direct computation we obtain

$$sgn\left(\frac{\partial\rho}{\partial p}(p,r,s)\right) = sgn(p + p(p^2 + r^2 - s^2)) =$$

$$sgn(p + p(p^2 + (r-s)(r+s))) = sgn(p),$$

$$sgn\left(\frac{\partial\rho}{\partial r}(p,r,s)\right) = sgn(r + r(p^2 + r^2 + s^2)) = sgn(r),$$

$$sgn\left(\frac{\partial\rho}{\partial s}(p,r,s)\right) = sgn(s + s(-p^2 + r^2 + s^2)) =$$

$$sgn(s + s((r-p)(r+p) + s^2)) = sgn(s),$$

and $\rho(p, r, s)$ is a decreasing function of p, r, s. Using the inequalities (5), (6), (7) and

$$p \geq G,\ r \geq D, s \geq G,$$

we can conclude that

$$\rho(\tilde{D}_2) = \rho(G, D, G).$$

The statement of theorem is obtained by direct application of theorem 3. □

As an application of the previous theorem we have next corollary where the interval for the optimum relaxation parameter of SOR method is obtained.

Corollary 1. *If ω_b is the optimal SOR parameter for the system with matrix A, then*

$$\omega_b \in \left[\frac{2}{1+\sqrt{1-(cos\frac{\pi}{n}-\rho^{-})^2}}, \frac{2}{1+\sin\frac{\pi}{n}} \right].$$

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REZIME

O NEKIM OSOBINAMA MATRICE NASTALE DISKRETIZACIJOM KONTURNOG PROBLEMA

Posmatrana je matrica koja nastaje pri diskretizaciji singularno perturbovanog konturnog problema na neekvidistantnoj mreži i odredjen je spektralni radijus odgovarajuće Jakobijeve matrice.

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