Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 24, 1 (1994), 359-367

Review of Research
Faculty of Science
Mathematics Series

ON DOCHEV'S RELATION IN PARALLEL AND SERIAL MODE [1]

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Abstract

In this paper we present a new proof of Dochev's relation for the sum of approximations appearing in the classical Weierstrass' formula considered in the parallel mode. In addition, we give an estimate of the deviation of this sum in the case of the serial mode and its practical application.

AMS Mathematics Subject Classification (1991): 65H05.
Key words and phrases: Polynomial zeros, Weierstrass' formula, iterative methods.

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a given monic polynomial with simple complex zeros $\zeta_1, ..., \zeta_n$, and let $z_1, ..., z_n$ be distinct complex numbers. Let us introduce another polynomial $Q(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_0$ of degree n with simple zeros $z_1, ..., z_n$, that is, $Q(z) = (z - z_1)\cdots(z - z_n)$. Then obviously $Q'(z_i) = \prod_{j\neq i}(z_i - z_j)$ and $z_1 + \cdots + z_n = -c_{n-1}$ (by Vièta's rules). In his constructive proof of the fundamental theorem of algebra Weierstrass [9] used the following formula

$$\hat{z}_i = z_i - \frac{P(z_i)}{Q'(z_i)} = z_i - \frac{P(z_i)}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n}(z_i - z_j)} \quad (i = 1, ..., n). \tag{1}$$

[1] This work was partially supported by the Science Fund of Serbia under Grant No. 0401.

Much later, Durand [3] applied (1) for the simultaneous determination of all zeros of a polynomial P, where $z_1, ..., z_n$ are the former approximations to the zeros $\zeta_1, ..., \zeta_n$ of the polynomial P and $\hat{z}_1, ..., \hat{z}_n$ are the new approximations. Dochev [2] was the first who proved a quadratic convergence of the method (1) which is often called *Weierstrass-Dochev's method.*

Considering (1) Dochev [2] has pointed to an interesting property of the sum of numbers $\hat{z}_1, ..., \hat{z}_n$, referred to as *Dochev's relation* and given in

Theorem 1. *If $\hat{z}_1, ..., \hat{z}_n$ are complex numbers defined by (1), then*

$$\hat{z}_1 + \cdots + \hat{z}_n = -a_{n-1}. \tag{2}$$

Thus, the sum of the later approximations has a constant value $-a_{n-1}$. This means that the "center of gravity" of these approximations is equal to the "center of gravity" of the zeros $\zeta_1, ..., \zeta_n$ (given by $-a_{n-1}$) in each iteration. Later, Laurie [5] have posed the above problem but under considerably stronger conditions: he required additionally that $\sum_{k=1}^{n} z_k = -a_{n-1}$. Various solutions of Dochev's relation (2), which does not take into account Laurie's condition, have been presented by H. Reisel in [4], G. V. Milovanović in [7, pp. 347-348] and Terano [8]. In this note we present a new proof of (2). The proof will be derived by the following lemma.

Lemma 1. *If $z_1, ..., z_n$ are simple zeros of the polynomial Q, then*

$$z_1 + \cdots + z_n = \frac{z_1^n}{Q'(z_1)} + \cdots + \frac{z_n^n}{Q'(z_n)}. \tag{3}$$

Proof. Let $\Gamma = \{z : |z| = r\}$ be a circle with the radius $r > \max_i |z_i|$. Then, by calculus of residues,

$$\frac{1}{2\pi i} \oint_\Gamma \frac{z^n}{Q(z)} dz = \sum_{k=1}^{n} Res_{z=z_k} \frac{z^n}{Q(z)} = \sum_{k=1}^{n} \frac{z_k^n}{Q'(z_k)}.$$

On the other hand, with the substitution $w = 1/z$ and the new contour $\Gamma' = \{w : |w| = 1/r\}$ the same integral can be calculated as follows

$$\frac{1}{2\pi i} \oint_\Gamma \frac{z^n}{Q(z)} dz = \frac{1}{2\pi i} \oint_{\Gamma'} \frac{1}{w^2} \frac{1/w^n}{Q(1/w)} dw$$

$$= \lim_{w\to 0} \frac{d}{dw}\Big(\frac{1}{c_0 w^n + \cdots + c_{n-1}w + 1}\Big) = -c_{n-1} = z_1 + \cdots + z_n.$$

Hence, the assertion of Lemma 1 follows comparing the right hand sides. □

A new proof of Dochev's relation. Let $B(z) = b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_0$ be a polynomial of degree $n-1$ and let $z_1, ..., z_n$ be simple zeros of Q, as above. According to Lagrange's interpolation formula we have

$$B(z) = \sum_{k=1}^{n} \frac{B(z_k)\prod_{j\neq k}(z - z_j)}{Q'(z_k)}.$$

Hence

$$(4) \qquad b_{n-1} = \lim_{z\to\infty} \frac{B(z)}{z^{n-1}} = \lim_{z\to\infty} \sum_{k=1}^{n} \frac{B(z_k)\prod_{j\neq k}(z - z_j)}{Q'(z_k)z^{n-1}} = \sum_{k=1}^{n} \frac{B(z_k)}{Q'(z_k)}.$$

If we define the polynomial $B(z) := P(z) - z^n$, using (3) and (4) we find

$$a_{n-1} = \sum_{k=1}^{n} \frac{B(z_k)}{Q'(z_k)} = \sum_{k=1}^{n} \frac{P(z_k)}{Q'(z_k)} - \sum_{k=1}^{n} \frac{z_k^n}{Q'(z_k)}$$

$$= \sum_{k=1}^{n} \frac{P(z_k)}{Q'(z_k)} - \sum_{k=1}^{n} z_k = -\sum_{k=1}^{n} \hat{z}_k. \square$$

As mentioned above, formula (1) defines an iteration method for the simultaneous determination of all zeros of a polynomial P. This method is realized in *parallel mode* and it is often called the *total-step method*. The convergence can be accelerated using the so-called Gauss-Seidel approach; namely, in the calculation of the new approximation $\hat{z}_i$ ($i \geq 2$) the already found approximations $\hat{z}_1, ..., \hat{z}_{i-1}$ are used (see [1]). In this manner the following *single-step* iteration method (or *serial mode*) is obtained:

$$(5) \qquad \hat{z}_i = z_i - \frac{P(z_i)}{\prod_{j<i}(z_i - \hat{z}_j)\prod_{j>i}(z_i - z_j)} \quad (i = 1, ..., n).$$

The iteration formula (5) has a somewhat different structure compared to (1) so that we can expect a "shift" of the sum of approximations $\hat{z}_1, ..., \hat{z}_n$ from the center of gravity of the zeros of P given by the coefficient $-a_{n-1}$. In the following we are going to find this deviation.

In the sequel, for two real or complex numbers α and β having the moduli of the same order, we will write $\alpha = O_M(\beta)$. If α is less or equal (in magnitude) to β then we will use the denotation $\alpha = O_L(\beta)$ (meaning "α is *at least* as β"). Let us introduce the notations

$$w_i = P(z_i)/Q'(z_i), \quad w = \max_i |w_i|, \quad \epsilon_i = z_i - \zeta_i, \quad \epsilon = \max_i |\epsilon_i|.$$

Since

$$\begin{aligned} w_i &= (z_i - \zeta_i)\prod_{j\neq i}\frac{z_i-\zeta_j}{z_i-z_j} = \epsilon_i \prod_{j\neq i}\Big(1+\frac{\epsilon_j}{z_i-z_j}\Big) \\ &= \epsilon_i\Big(1+\sum_{j\neq i}\frac{\epsilon_j}{z_i-z_j}+O_M(\epsilon^2)\Big) = \epsilon_i + O_M(\epsilon^2), \end{aligned}$$

we have $\epsilon = O_M(w)$.

Theorem 2. *If the approximations $\hat{z}_1, \ldots, \hat{z}_n$ are defined by (5), then*

$$d := \sum_{i=1}^{n} \hat{z}_i - (-a_{n-1}) = O_L(w^2) \tag{6}$$

if w is small enough and $d = O_M\big(\sum_{i=2}^{n} w_i\big)$, otherwise.

Proof. Assume that w is sufficiently small. Then from (5) we obtain

$$z_i - \hat{z}_j = z_i - z_j + \frac{P(z_j)}{\prod_{k<j}(z_j-\hat{z}_k)\prod_{k>j}(z_j-z_k)} = z_i - z_j + O_L(w_j).$$

In regard to this estimate and taking into account (2) and (5) we find

$$\begin{aligned} \sum_{i=1}^{n}\hat{z}_i &= \sum_{i=1}^{n} z_i - w_1 - \sum_{i=2}^{n}\frac{P(z_i)}{\prod_{j<i}(z_i-z_j)\Big(1+\frac{O_L(w_j)}{z_i-z_j}\Big)\prod_{j>i}(z_i-z_j)} \\ &= \sum_{i=1}^{n} z_i - w_1 - \sum_{i=2}^{n} w_i\Big(1-\sum_{j=1}^{i-1}\frac{O_L(w_j)}{z_i-z_j}\Big) + O_L(w^3) \\ &= \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} w_i + \sum_{i=2}^{n}\Big(w_i\sum_{j=1}^{i-1}\frac{O_L(w_j)}{z_i-z_j}\Big) + O_L(w^3) \\ &= -a_{n-1} + O_L(w^2). \end{aligned}$$

In the case when w_i are not small in magnitude we cannot expect that the single-step method (5) is better than the total-step method (1) so that $z_i - \hat{z}_j = z_i - z_j + O_M(w_j)$. Let us put

$$\alpha_j^{(i)} = \frac{O_M(w_j)}{z_i - z_j}, \quad S_{\mu,i} = \sum_{j_1<j_2<\cdots<j_\mu} \alpha_{j_1}^{(i)}\alpha_{j_2}^{(i)}\cdots\alpha_{j_\mu}^{(i)}, \quad y_i = \sum_{\mu=1}^{i-1} S_{\mu,i},$$

where $S_{\mu,i}$ is the symmetric function relative to $\alpha_j^{(i)}$ $(j = 1,...,i-1)$. To simplify our estimate procedure we adopt that $y_i/(y_i+1) = O_M(1)$ (being w_j are not small in magnitude) and find

$$\begin{aligned}\sum_{i=1}^{n} \hat{z}_i &= \sum_{i=1}^{n} z_i - w_1 - \sum_{i=2}^{n} \frac{P(z_i)}{\prod_{j<i}(z_i - z_j)\Big(1 + \dfrac{O_M(w_j)}{z_i - z_j}\Big)\prod_{j>i}(z_i - z_j)} \\ &= \sum_{i=1}^{n} z_i - w_1 - \sum_{i=2}^{n} \frac{w_i}{1+y_i} = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} w_i + \sum_{i=2}^{n} w_i \frac{y_i}{1+y_i} \\ &= -a_{n-1} + O_M\Big(\sum_{i=2}^{n} w_i\Big). \square\end{aligned}$$

A number of numerical results confirms the assertion of Theorem 2 (for instance, see Table 1 for the presented example). In the case when the method (5) shows a divergent behaviour the deviation $d^{(m)}$ coincides with the sum $\sum_{i=2}^{n} w_i$ to a satisfactory extent.

Some authors have noted that the relation (2) can be used for an analysis of the convergence of the total-step method (1). But, a great number of numerical experiments shows that the relation (2) is also valid (in the worst case, due to rounding errors, the deviation $d = \sum_i \hat{z}_i + a_{n-1}$ is very close to 0) in the case of a divergence of the iteration method (1) or its divergent behaviour, [2] which coincides with the presented theoretical results. Therefore, applying the parallel mode of Weierstrass-Dochev's method, it is not possible to provide any information on the behaviour of this method according

[2] Many authors conjecture that Weierstrass-Dochev's method converges for almost any initial approximations. In practice, this method can demonstate a divergent behaviour in a great number of the first iteration steps and, after that, it begins to converge (so-called *quasi-divergence* or *false-divergence*). This convergent behaviour has not been explained in literature yet.

to (2). On the other hand, from (6) we observe that the deviation d tends to 0 only if the single-step method (5) is convergent and takes values which are not reasonably small in magnitude if the method (5) is (quasi)divergent. Thus, in contrast to the parallel mode (1), we are able to say much more about the convergence of the serial mode of Weierstrass-Dochev's method (5) considering the values of the sum $\sum_i \hat{z}_i + a_{n-1}$, which has a practical importance in detecting the convergence. The presented consideration is illustrated in the following example.

Example. Let us consider the polynomial

$$P(z) = z^7 + z^5 - 10z^4 - z^3 - z + 10$$

with the simple zeros $\zeta_1 = 2$, $\zeta_{2,3} = \pm 1$, $\zeta_{4,5} = \pm i$, $\zeta_{6,7} = -1 \pm 2i$ and the center of gravity $-a_6 = 0$. The initial approximations $z_1^{(0)}, \ldots, z_7^{(0)}$ to these zeros are chosen to be equally spaced on the boundary of the circle with the radius $r = 800$ and centered at the origin, that is

$$z_k^{(0)} = 800 \exp\Big(i\frac{2k\pi}{7}\Big) \quad (k = 1, \ldots, 7).$$

We have taken a very large radius to demonstrate the convergence of Weierstrass-Dochev's method even for extremely bad starting approximations (although, sometimes, a rather great number of iteration steps is needed, as in our example).

To find the zeros of P we have applied the iteration methods (1) and (5). Together with the new approximations $z_1^{(m)}, \ldots, z_7^{(m)}$, where m is the iteration index, we have calculated the absolute value of the deviation $|d^{(m)}| = |\sum_{k=1}^{7} z_k^{(m)} + a_6| = |\sum_{k=1}^{7} z_k^{(m)}|$ and $\sum_{k=1}^{7} |P(z_k^{(m)})|$ in the m-th iteration. The stoping criterion has been stated by the inequality

$$\sum_{k=1}^{7} |P(z_k^{(m)})| < 10^{-7}.$$

The entries are displayed in Table 1. The convergence of the both methods (1) and (5) has been followed by the values $\sum_{k=1}^{7} |P(z_k^{(m)})|$ in the course of the iteration processes.

Table 1. The deviation $d^{(m)}$ and the convergence of the iteration methods (1) and (5). $A(h)$ means $A \times 10^h$.

	Total-step method		Single-step method(5)		
m	$\|d^{(m)}\|$	$\sum_k \|P(z_k^{(m)})\|$	$\|d^{(m)}\|$	$\sum_k \|P(z_k^{(m)})\|$	$\|\sum_{k=2}^{7} w_k^{(m)}\|$
1	2.07(-13)	4.99(22)	1.21(2)	4.15(22)	1.59(2)
10	2.83(-14)	3.02(16)	2.91(1)	3.29(15)	2.78(1)
20	5.09(-15)	6.22(11)	4.27(0)	6.50(9)	4.26(0)
30	8.33(-17)	1.28(9)	6.17(-1)	1.27(6)	6.63(-1)
37	2.51(-17)	6.74(3)	2.22(-2)	1.27(0)	2.50(-1)
38	5.03(-16)	2.30(3)	9.57(-5)	3.82(-3)	6.88(-3)
39	5.11(-16)	7.92(4)	8.34(-10)	2.80(-8)	6.88(-3)
58	4.96(-16)	2.3(-7)			

From Table 1 we observe that the deviation $|d^{(m)}|$ remains very small, independent on the behaviour of the total-step method (1) in a convergence sense. But, in the case of the single-step method (5), $|d^{(m)}|$ is not small in the first iterations as long as the approximations are very bad (which is indicated by large $\sum_{k=1}^{7} |P(z_k^{(m)})|$), and decreases rapidly when the method begins to converge. Finaly, the presented example also illustrates the well-known fact that the single-step method (5) converges faster than the total-step method (1).

It is interesting to note that an assertion similar to that given in Theorem 2 holds for some other iteration methods including also a whole class of methods based on Schröder's development. As an illustration we present one of the most frequently used methods for the simultaneous approximating polynomial zeros having the form (see Maehly [6])

$$\hat{z}_i = z_i - \frac{P(z_i)/P'(z_i)}{1 - \frac{P(z_i)}{P'(z_i)} \sum\limits_{j \neq i} (z_i - z_j)^{-1}} \quad (i = 1, ..., n). \tag{7}$$

If w (and, consequently, ϵ) is small enough we find

$$\begin{aligned} \frac{P(z_i)}{P'(z_i)} &= w_i \cdot \frac{Q'(z_i)}{P'(z_i)} = \frac{w_i}{P'(z_i)/P(z_i)} \cdot \frac{Q'(z_i)}{P(z_i)} \\ &= \frac{w_i}{\sum\limits_{j=1}^{n} \frac{1}{z_i - \zeta_j}} \cdot \frac{1}{z_i - \zeta_i} \prod_{j \neq i} \frac{z_i - z_j}{z_i - \zeta_j} \end{aligned}$$

$$
\begin{aligned}
&= \frac{w_i}{1+\epsilon_i \sum_{j\neq i} \frac{1}{z_i-\zeta_j}} \prod_{j\neq i}\Big(1-\frac{\epsilon_j}{z_i-\zeta_j}\Big) \\
&= w_i\big(1-\epsilon_i\sigma_i+(\epsilon_i\sigma_i)^2-\ldots\big)\Big(1-\sum_{j\neq i}\frac{\epsilon_j}{z_i-\zeta_j}+O_M(\epsilon^2)\Big) \\
&= w_i\big(1+O_M(\epsilon)\big),
\end{aligned}
$$

where $\sigma_i = \sum_{j\neq i}(z_i-\zeta_j)^{-1}$. Returning back to (7) we find

$$
\begin{aligned}
\hat{z}_i &= z_i - \frac{w_i\big(1+O_M(\epsilon)\big)}{1-w_i\big(1+O_M(\epsilon)\big)\sigma_i} \\
&= z_i - w_i\big(1+O_M(\epsilon)\big)\Big(1+w_i\big(1+O_M(\epsilon)\big)\sigma_i+O_M(w^2)\Big) \\
&= z_i - w_i + O_M(w^2).
\end{aligned}
$$

Hence, by summing, we obtain $d=\sum_{i=1}^{n}\hat{z}_i + a_{n-1} = O_M(w^2)$.

In the case when w is not small enough, from the expression $P(z_i)/P'(z_i) = w_iQ'(z_i)/P'(z_i)$ we immediately see that the deviation d cannot be small.

References

[1] Alefeld, G., Herzberger, J., On the convergence speed of some algorithms for the simultaneous approximation of polynomial zeros, SIAM J. Numer. Math., 11(1974), 237-243.

[2] Dochev, K., Modified Newton method for the simultaneous approximate calculation of all roots of a given algebraic equation (in Bulgarian), Mat. Spis. B"ulgar. Akad. Nauk 5 (1962), 136-139.

[3] Durand, E., Solution numériques des équations algébraiques, Tom. I: Équations du Type F(x)=0; Racines d'un Polynôme. Masson, Paris 1960.

[4] Kjellberg, G., Two observations on Durand-Kerner root-finding method, BIT, 24(1984), 554-559.

[5] Laurie, D.P., Problem H-315, The Fibonacci Quarterly, 18(1980) p. 190, 19(1981), p. 476.

[6] Maehly, V.H., Zur iterativen Auflösung algebraischer Gleichungen, Z. Angew. Math. Phys. 5(1954), 260-263.

[7] Mitrinović, D.S., Kečkić, J.D., The Cauchy method of residues. Theory and applications, Dordrecht-Boston-Lancaster 1984.

[8] Terano, T., On a global algorithm for algebraic equations, Ph. D. Theses, University of Tokyo, Tokyo 1978.

[9] Weierstrass, K., Neuer Beweis des Satzes, dass jede ganze rationale Funktion einer Veränderlichen dargestellt werden kann als ein Product aus linearen Funktionen derstelben Verändelichen, Ges. Werke 3(1903), 251-269 (Johnson Reprint Corp., New York 1967).

REZIME

O RELACIJI DOCHEVA U PARALELNOM I SERIJSKOM MODU

U ovom radu dat je nov dokaz relacije Docheva za sumu aproksimacija koje se javljaju u klasičnoj Weierstrassovoj formuli posmatranoj u paralelnom modu. Osim toga, data je ocena odstupanja ove sume u slučaju serijskog moda i njena praktična primena.

Received by the editors September 13, 1993