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SOME DIFFERENCE SCHEMES DERIVED VIA FINITE ELEMENT METHOD

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Abstract

The family of difference schemes via finite element method is derived. The El Mistikawy and Werle scheme is a member of the family. The scheme having the forth order of the classical convergence and second order of uniform convergence derived in [8] is a memeber of the family, also.

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1. Introduction

Let us consider the following singularly perturbed problem

$$(1)\qquad \begin{cases} Ly = -\varepsilon y'' + p(x)y' = f(x), & x \in I = [0,1], \\ y(0) = 0, \quad y(1) = 0, \end{cases}$$

where ε is a small positive parameter, p and f are sufficiently smooth functions and $p(x) \geq p > 0$. By using exponential spline $e(x)$ from [4], $e(x) \in C^1(I)$, as a collocation function, a family of difference schemes is derived in [4]. The well known Allan-Southwel- Il'in and El Mistikawy-Werle (EMW) schemes are members of this family. The new scheme colled Improved EMW scheme is analysed in [6]. The linear combination of the schemes IEMW and EMW is considered in [8]. In this paper the mentioned schemes are derived via finite element method by using suitable test functions.

2. The weak form

The weak form of the problem (1) is: Find $y \in H^1(0,1)$ so that

$$B(y,v) = -\varepsilon(y',v') + (py',v) = (f,v),$$

for all $v \in H^1(0,1)$, where $H^1[0,1]$ is Sobolev's space with norm $||u||_1 = ((u,u)+(u',u'))^{1/2}$, $(\cdot,\cdot)$ denotes inner product in $L_2(0,1)$. If we choose subspaces S^h (trial space) and T^h (test space) from $H^1(0,1)$ we can define the problem: Find $u^h \in S^h$ such that

$$B(y^h,v^h) = (f,v^h), \tag{2}$$

for all $v^h \in T^h$ such that $v^h(0) = v^h(1) = 0$. Let $\{\phi_j\}_1^{n-1}$ and $\{\psi_j\}_1^{n-1}$ be a set of basis functions for S^h and T^h, respectively. Let $xj = h * j, j = 0(1)n, h = 1/n$. Then

$$y^h = \sum_{j=1}^{n-1} y_j\phi_j, \quad v^h = \sum_{j=1}^{n-1} v_j\psi_j.$$

Since

$$supp(\phi_j) = [x_{j-1},x_{j+1}]; \phi(x_j) = 1; \sum_{j=1}^{n-1}\phi_j(x) = 1, x \in [x_1,x_{n-1}]$$

and the same relations are valid for function ψ, we have $u^h(x_j) = u_j$. With this, (2) reduces to

$$\sum_{k=j-1}^{j+1} (\phi_j', -\varepsilon\psi_j' + p(x)\psi_j)u_k = (f,\psi_j), j = 1(1)n-1, \tag{3}$$

$$u_0 = u_n = 0.$$

3. Choice of Trial and Test function

The test function we choose is of the form

$$\psi(x) = \lambda\tilde{\psi}(x) + (1-\lambda)\dot{\psi}(x),$$

where

$$\tilde{\psi}(x) = \begin{cases} \tilde{e}_j(x - x_{j-1}) & \text{for} \quad x \in I_j \\ 1 - \tilde{e}_{j+1}(x - x_j) & \text{for} \quad x \in I_{j+1}, \end{cases}$$

$\tilde{e}_j(x) = (\exp(\tilde{p}x/\varepsilon) - 1)/(\exp(\tilde{\rho}_j) - 1)$, $\tilde{\rho}_j = \tilde{p}_j h/\varepsilon$, and

$$\dot{\psi}(x) = \begin{cases} \dot{e}_j(x - x_{j-1}) & \text{for} \quad x \in I_j \\ 1 - \dot{e}_{j+1}(x - x_j) & \text{for} \quad x \in I_{j+1}, \end{cases}$$

and $\dot{e}_j(x) = (\exp(\dot{p}x/\varepsilon) - 1)/(\exp(\dot{\rho}_j) - 1)$ $\dot{\rho}_j = \dot{p}_j h/\varepsilon$, $\tilde{p}_j = (p_{j-1} + p_j)/2$, $\dot{p}_j = p(x_j - h/2)$, $I_j = [x_{j-1}, x_j]$ and λ is a real number. Note that the function $\tilde{e}_j(x)$ is the solution of the adjoint problem

$$-\varepsilon\tilde{e}_j'' + \tilde{p}_j\tilde{e}_j' = 0$$

$$\tilde{e}_j(0) = 0, \ \ \tilde{e}_j(h) = 1.$$

The similarly is for $\dot{e}_j$.

For same details about test functions see [3] or [5].

4. Discretization of the Problem

The quadrature rules in (3) we determine in the following way

$$(p(x)\phi, \psi)_j \simeq \lambda\tilde{p}_j(\phi', \tilde{\psi})_j + (1-\lambda)\dot{p}_j(\phi', \dot{\psi})_j$$

where index j denotes the integral on the interval $[x_{j-1}, x_j]$ and integrals $(\phi', \tilde{\psi})$ and $(\phi', \tilde{\psi})$ are evaluted exactly. Namely, we choose test and trial function such that these terms will be easy to evaluate. In this case, the choice of the test functions gives that $(-\varepsilon\tilde{\psi}_j' + \tilde{p}_j\tilde{\psi}_j) = const.$ and $(-\varepsilon\dot{\psi}_j' + \dot{p}_j\dot{\psi}_j) = const.$ Because of that, we obtain that independetly of the choice of trial function we can evaluate the integrals exactly. After some elementary calculations we obtain the family:

$$(4) \qquad \begin{cases} Ru_j = Qf_j, \quad j = 1(1)n \\ \\ u_0 = 0, \quad u_1 = 0, \end{cases}$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = \lambda q_j^- f_{j-1} + (1-\lambda) q_j^\mp f_{j-1/2} + \lambda q_j^c f_j + (1-\lambda) q_j^\pm f_{j+1/2} + (1-\lambda) q_j^+ f_{j+1}$$

$$r_j^- = \lambda r_1^- + (1-\lambda) r_2^-, \quad r_j^+ = \lambda r_1^+ + (1-\lambda) r_2^+,$$

$$r_j^c = \lambda r_1^c + (1-\lambda) r_2^c, \quad r_1^- = \varepsilon R^-(\mu_1^-)/h^2, \quad r_2^- = \varepsilon R^-(\mu_2^-)/h^2,$$

$$R^+(a) = a/(1-\exp(-a)), \quad R^-(a) = a\, exp(-a)/(1-exp(-a)),$$

$$r_1^c = -r_1^- - r_1^+, \quad r_2^c = -r_2^- - r_2^+$$

$$\mu_1^- = h\tilde{p}_j/\varepsilon, \quad \mu_1^+ = h\tilde{p_{j+1}}/\varepsilon, \mu_2^- = h\dot{p}_j/\varepsilon, \quad \mu_2^+ = h\dot{p_{j+1}}/\varepsilon,$$

$$q_j^- = \frac{1 - R^-(\mu_1^-)}{2\mu_1^-}, \quad q_j^+ = \frac{R^+(\mu_1^+) - 1}{2\mu_1^+}, \quad q_j^c = q_j^- + q_j^+$$

$$q_j^\mp = \frac{1 - R^-(\mu_2^-)}{\mu_2^-}, \quad q_j^\pm = \frac{R^+(\mu_2^+) - 1}{\mu_2^+},$$

Thus, when $\lambda = 1$ we obtain the EMW scheme. When $\lambda = 0$ we obtain IEMW scheme analysed in [6]. In order to obtain the scheme with greater accuracy , we analyse the truncation error $\tau_j(y)$.

$$\tau_j(y) = Ry_j - Q(Ly_j) = T_{j0}y_j + T_{j1}y_j' + T_{j2}y_j'' + O(h^3/\varepsilon)$$

where $T_{j0} = T_{j1} = 0$ and

$$T_{j2} = \frac{h^2}{2}(r_j^- + r_j^+) - \lambda(\varepsilon(q_j^- + q_j^c + q_j^+) - h(\tilde{p}_j q_j^- - \tilde{p_{j+1}} qj^+))$$

$$-\varepsilon(1-\lambda)(q_j^\mp + q_j^\pm) - h(q_j^\mp \dot{p}_j + q_j^\pm \dot{p_{j+1}})/2.$$

After some Taylor's expansions we obtain

$$T_{j2} = \lambda\frac{-h^2}{6\varepsilon}(p'(\beta_1) + p'(\beta_2)) + (1-\lambda)\frac{-h^2}{24\varepsilon}(p'(\beta_3) + p'(\beta_4)) + O(\frac{h^3}{\varepsilon})$$

where

$$x_{j-1} < \beta_1 < x_j < \beta_2 < x_{j+1}.$$

$$x_{j-1/2} < \beta_3 < x_j < \beta_4 < x_{j+1/2}.$$

Since, $p(x)$ is smooth function we can put $\lambda = 4/3$ and then we have

$$T_{j2} = O(\frac{h^3}{\varepsilon}).$$

In that case the scheme (4) becomes the scheme derived in [8]. The following theorems holds.

Theorem 1. *Let $y(x) \in C^4(I)$. Let u_j be approximation of $y(x_j)$ obtained using scheme (4) for $\lambda = 0$ or for $\lambda = 1$. Then*

$$|y(x_j) - u_j| \leq Mh^2$$

where M is constant independent of ε and h.

Proof. For $\lambda = 1$ we obtain the EMW scheme and the proof is given in [1]. For $\lambda = 0$ the scheme reduces to IEMW scheme from [6] and the proof is given there.

Theorem 2. *Let $y(x) \in C^6(I)$. Let u_j be approximation of $y(x_j)$ obtained using scheme (4) for $\lambda = 4/3$. Then*

$$|y(x_j) - u_j| \leq M\frac{h^4}{h^2 + \varepsilon^2}$$

where M is constant independent of ε and h.

Proof. For $\lambda = 4/3$ the scheme reduces to one derived in [8] and the proof is given there.

Numerical results presented in [6] and [8] suggest that for very smal ε should be use $\lambda = \varepsilon$. In that case some advantages of IEMW scheme become important (see [6]).

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REZIME

NEKE DIFERENCNE ŠEME IZVEDENE POMOĆU METODE KONAČNIH ELEMENATA

Primenom metode konačnih elemenata izvedena je familija diferencnih šema za singularno perturbovane probleme. Kao članovi familije pojavljuju se šeme izvedene u ranijim radovima autora kao i poznata El Mistikawy Werle šema.

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