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ON BALANCED LINEAR 4-PARTITIONS OF THE (n, n)-GRID

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Abstract

It is shown that the number of balanced linear 4-partitions of the $n \times n$-grid G can be expressed by means of the number of minimal central crosses of that grid. This relationship enables an easy derivation of the formulae for the number of balanced linear 4-partitions of G, in the form of two sums, which depend on parity of n. The corresponding two asymptotic formulae are of the form

$$\frac{A}{\pi^2}\, n^2 + O(n \log n),$$

where $A = 2$ for n even and $A = 6$ for n odd.

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1. Introduction

Digital pictures are usually given by digital gray values on grids of size $n \times n$. Different partitions of the $n \times n$ - grid by straight lines (linear partitions, [4]) may be the basis of certain routines in image analysis. The paper [5] shows that exhaustion procedures (which go through all the linear partitions) could be impractical even for small values of n, for the number of such partitions is asymptotically equal to

$$\frac{3}{\pi^2}\, n^4 + O(n^3 \log n)\ .$$

This paper is concerned with the balanced linear 4-partitions of the set of points of the $n \times n$-grid $G(n)$. These partitions correspond to the separations of G by two lines into four (almost) equicardinal sets. They may be viewed as the pairs of specially located linear partitions. A characterization and enumeration of these partitions is given. It is not surprising that the number of balanced linear 4-partitions of $G(n)$ is reduced by a factor of order n^2, in comparision with the number of general linear partitions.

The definition of balanced linear 4-partitions depends on parity of n, as well as the proofs and the derived results concerning the number of these partitions. Theorems 1 and 2 in Sections 3 and 4 establish the $1-1$ correspondence for n even, respectively the $1-4$ correspondence for n odd, between balanced linear 4-partitions and so-called minimal central crosses of $G(n)$.

Theorems 3 and 4 of Section 5 determine the number of balanced linear 4-partitions of $G(n)$, in the form of two sums (for n even and for n odd). The paper also includes the proofs of asymptotic estimations (Theorems 5 and 6) for the number of balanced linear 4-partitions. The auxiliary concept of minimal central cross of $G(n)$ plays an important role in deriving results in the form of sums.

2. Preliminaries

Let $G(n)$ denote the $n \times n$ grid, i.e., the set of n^2 points in the plane, which can be represented in a coordinate system as:
$\{\ (i,j) \mid (1-n)/2 \le i,\ j \le (n-1)/2\ \}$ for n odd and as:
$\{\ (i/2, j/2) \mid\ i \text{ odd}, j \text{ odd}, 1-n \le i,\ j \le n-1,\ \}$ for n even.

Given a finite point set S in the plane, the *balanced linear 4-partition* (Fig. 1) determined by lines p and q (shortly *BL 4-partition* (p,q)) of S is a partition of S into four point sets S_1, S_2, S_3, S_4, which satisfy the following two conditions:

a) Each one of the sets S_i, $1 \leq i \leq 4$, is located in one distinct open angle determined by the lines p and q

b) The cardinality difference of any two sets S_i, S_j $(1 \leq i,j \leq 4)$ is not greater than one.

Remark. It is obvious that all the four sets S_i are equicardinal for n even, while one of these sets has one element more than each one of the remaining three for n odd.

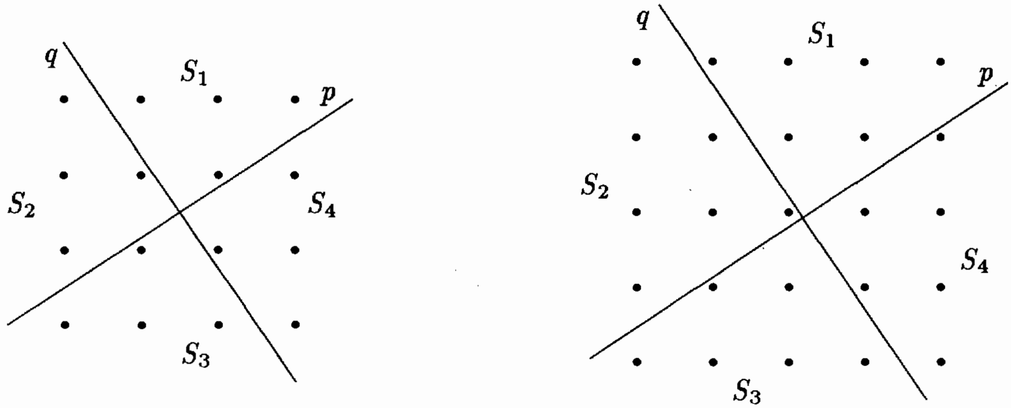

Figure 1. Two BL 4-partitions of $G(4)$ and $G(5)$

The *center* C of the grid $G(n)$ is the point $(0,0)$. It is obvious that the point C is the center of symmetry of $G(n)$ and that C belongs to $G(n)$ iff n is odd.

A *minimal central cross*, shortly *MC-cross* of $G(n)$ is a set $\{\{K,L\},\{M,N\}\}$, (Fig. 2) , which satisfies the following four conditions:

a) K, L, M and N are four distinct points of $G(n)$

b) The center C of $G(n)$ is the center of the both line segments KL and MN

c) The line segments KL and MN are mutually orthogonal

d) There are no points of $G(n)$ in the interior of the line segments CK and CM (due to symmetry, the same conclusion holds for the line segments CL and CN)

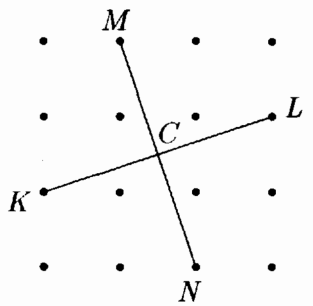

Figure 2. An MC-cross of $G(4)$

"$x \perp y$" will denote that the integers x and y are relatively prime.

3. Even case

Theorem 1. *The number of BL 4-partitions of $G(n)$ for n even is equal to the number of its MC-crosses.*

Proof. Given an MC-cross $\{\{K, L\}, \{M, N\}\}$, the unique BL 4-partition (p, q) is naturally associated to it according to the following rule:

The lines p and q are respectively obtained by a small rotation (the angle of the rotation belongs to the open interval $(0, \alpha(n))$, where $\alpha(n) = \text{arctg } (1/(2n^2))$) in the positive direction of the lines KL and MN around the center C (Fig. 3). Such a rotation guarantees that there are no points of G in the interior of the two smaller angles determined by the lines p and KL, respectively by the lines q and MN ($\alpha(n)$ is a lower bound for an angle determined by some three points of $G(n)$).

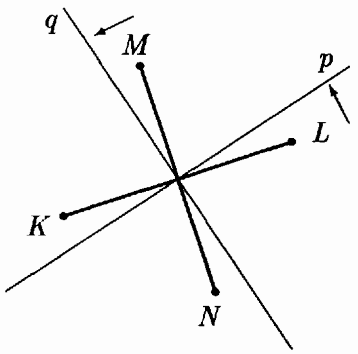

Figure 3. Associating the BL 4-partition to an MC-cross

It is easy to conclude that any point X_0 of $G(n)$ uniquely determines the associated cross $\{\{X_0, X_1\}, \{X_2, X_3\}\}$ satisfying the above conditions a), b), and c), as well as that the points X_0, X_1, X_2, X_3 belong to four distinct right angles determined by the lines p and q. This shows that there exists a BL 4-partition (p, q).

On the other hand, let be given a BL 4-partition (p, q) of $G(n)$. We may assume that the center C is the intersection point of the lines p and q:

Namely, if $C \notin p$ (similarly if $C \notin q$), then we may replace the line p by p', where $p' \parallel p$ and $C \in p'$. If p' has a non-empty intersection with $G(n)$, then we further replace p' by a line p" obtained from p' by a rotation around C for a small angle (smaller than $\alpha(n)$). No point of $G(n)$ may belong to the closed band determined by the lines p and p' (or p and p") (Fig. 4). This follows from the fact that exactly one half of points of $G(n)$ should belong to each one of the half-planes determined by p' or p" (due to the central symmetry w.r.t. C) and p (since p takes part in a BL 4-partition).

Figure 4. Closed bands (p, p') and $(p, p")$, without points of $G(n)$

Let us rotate the line p (passing through C) around C in the negative direction for an angle smaller than $\alpha(n)$ until it coincides with a line p_1, which has a non-empty intersection with $G(n)$. The line p_1 determines the unique MC-cross $\{\{K, L\}, \{M, N\}\}$, so that the grid points K and L belong to the line p_1. Let the BL 4-partition (p, r) be associated to this MC-cross as before (Fig. 5).

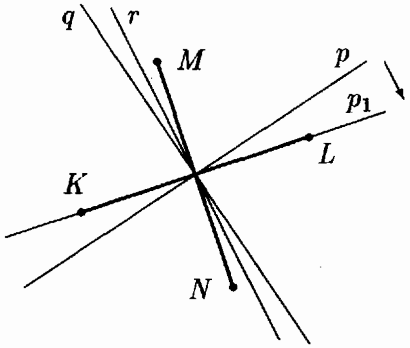

Figure 5. Associating the MC-cross to a BL 4-partition

If the BL 4-partitions (p, q) and (p, r) do not coincide, then there exist points of $G(n)$ in the interior of the angles determined by q and r, which do not contain p. Each such point belongs to the difference of some two angles, which are both supposed to contain exactly one quarter of points of $G(n)$, a contradiction. □

4. Odd case

Theorem 2. *The number of BL 4-partitions of $G(n)$ for n odd is exactly four times greater than the number of its MC-crosses.*

Proof. Given a grid $G(n)$, for some n odd, let $D(n)$ denote the "defect" configuration $G(n) - C$. A reasoning completely analogous to that which was applied in the proof of Theorem 1 may be applied in order to prove that the number of BL 4-partitions of $D(n)$ is equal to the number of MC-crosses of $G(n)$.

We primarily prove that each BL 4-partition of $D(n)$ determines four BL 4-partitions of $G(n)$. The pairs of underlying lines of these four partitions are obtained by small translations (Fig. 6) of the underlying lines p and q: $(p', q'), (p', q''), (p'', q'), (p'', q'')$; "small" means that no point of $G(n)$ is reached during a translation.

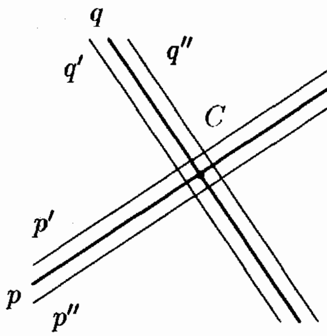

Figure 6. Four BL 4-partitions of $G(n)$ associated to a BL 4-partition (p, q) of $D(n)$

On the other hand, we are going to show that each BL 4-partition of $G(n)$ determines a unique BL 4-partition of $D(n)$, which is obtained by ignoring the center C.

Assume, on the contrary, that there exists a BL 4-partition (p, q) of $G(n)$ into sets S_1, S_2, S_3, S_4 of respective cardinalities $k + 1$, k, k, k (where $k = ((n-1)/2)^2 + (n-1)/2$), such that the center C belongs to some of the sets S_i, where $i \in \{2,3,4\}$, so that ignoring C does not result in a BL 4-partition of $D(n)$) (Fig. 7). Using the central symmetry of $G(n)$ w.r.t. C, as well as the lines p' and q' passing through C and parallel to the lines p and q respectively, it is easy to conclude that the set S_j located in the opposite angle determined by the lines p and q (w.r.t. the angle occupied by the points of S_i) possesses fewer than k points of $G(n)$, a contradiction. □

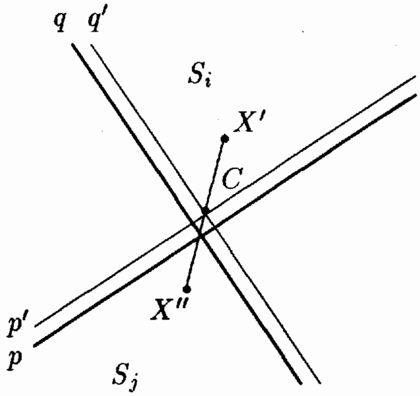

Figure 7. If $(|S_i| = k)$ and $(C \in S_i)$ then $|S_j| < k$

5. The number of BL 4-partitions of $G(n)$

Theorem 3. *The number of balanced linear 4-partitions of* $G(n)$ *for* n *even is equal to*

$$1 + 2 \sum_{\substack{x \perp y, \\ 0 < y < x \leq n-1, \\ x \text{ odd }, y \text{ odd}}} 1 .$$

Proof. Using Theorem 1, it suffices to count the MC-crosses of $G(n)$. On the other hand, each MC-cross (for n even) is uniquely determined by its point in the first quadrant, which has the form $(x/2, y/2)$, where $x \perp y$, and

$0 < x, y \leq n-1$ for some x, y odd. The above sum is obtained by extracting the point $(1/2, 1/2)$ and by using the symmetry w.r.t. the line $y = x$. □

Theorem 4. *The number of balanced linear 4-partitions of $G(n)$ for n odd is equal to*

$$8 + 8 \sum_{\substack{x \perp y, \\ 0 < y < x \leq (n-1)/2}} 1 \,.$$

Proof. Using Theorem 2, it suffices to count the MC-crosses of $G(n)$ and to multiply the obtained number by 4. Each MC-cross (for n odd) is again uniquely determined by its point which lies either in the open first quadrant or on the positive part of the x-axis. These points are of the form (x, y), where $x \perp y$ and $0 < x, y \leq (n-1)/2$, the special point $(1, 0)$ being added to them. The above sum is obtained by extracting the points $(1, 1)$ and $(1, 0)$, by using the symmetry w.r.t. the line $y = x$ and by multiplying the obtained sum by 4. □

Theorem 5. *The number of balanced linear 4-partitions of $G(n)$ for n odd is asymptotically equal to*

$$\frac{6}{\pi^2}\, n^2 + O(n \log n) \,.$$

Proof. The sum given in Theorem 4 is equal to

$$8 + 8 \sum_{k=1}^{(n-1)/2} \phi(k),$$

where $\phi(k)$ denotes the well-known Euler function. Theorem 330 of [2] gives that this sum is asymptotically equal to

$$8 \cdot 3\, \frac{(n-1)^2}{4\pi^2} + O(n \log n) \quad . \square$$

Theorem 6. *The number of balanced linear 4-partitions of $G(n)$ for n even is asymptotically equal to*

$$\frac{2}{\pi^2}\, n^2 + O(n \log n) \,.$$

The proof uses an exercise from [3], which implies that within the pairs (x, y) of natural numbers, which satisfy that $x \perp y$ and $1 < x, y \leq n$, the number of pairs with x odd and y even is asymptotically equal to the number of pairs with both x and y odd. Consequently, the number of pairs with both number odd is asymptotically equal to one third of the total nunber of pairs.

Example. The exact values of the number of balanced 4-partitions of the (n, n)-grid is for n between 1 and 100 inclusively are given in the following table:

0	1	8	3	16	7	32	13	48	19
80	29	96	41	144	49	176	65	224	83
256	95	336	117	368	137	464	155	512	183
576	213	640	233	768	257	816	293	960	317
1024	357	1120	399	1200	423	1376	469	1440	511
1600	543	1696	595	1840	635	1936	671	2160	729
2224	789	2464	825	2592	873	2752	939	2880	983
3072	1053	3168	1125	3456	1165	3600	1225	3792	1303
3920	1357	4240	1439	4336	1503	4672	1559	4832	1647
5024	1719	5200	1779	5568	1851	5696	1947	6032	2007

The exact numbers of balanced linear 4-partitions for n = 1000 and n = 1001 are 202661 and 608928, while the asymptotic formulas (Theorems 6 and 5) give the values 202642 and 609144 respectively. Thus the relative error done by the above approximation is less than 0.1 %.

6. Conclusion

The paper contains two closed formulas for the number of balanced linear 4-partitions of the (n, n)-grid, associated to the even and the odd case respectively, which are derived by using the auxiliary concept of minimal central cross. The enumeration of balanced linear 4-partitions can be made efficient by applying the asymptotically optimal algorithm for generating all the pairs (p, q) of mutually simple natural numbers p and q, which satisfy that $1 \leq p,\ q \leq m$ ([1]).

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REZIME

O URAVNOTEŽENIM LINEARNIM 4 - PARTICIJAMA $n \times n$ MREŽE

Pokazuje se da se broj uravnoteženih linearnih 4 - particija $n \times n$ mreže može izraziti preko broja minimalnih centralnih krstova te mreže. Ova veza se koristi za izvodjenje formula za broj uravnoteženih linearnih 4 - particija u obliku dve sume koje zavise od parnosti broja n. Dve odgovarajuće asimptotske formule su oblika

$$\frac{A}{\pi^2}\, n^2 + O(n \log n),$$

gde je $A = 2$ za n parno i $A = 6$ za n neparno.

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