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A COMMENT ON n-GROUPS

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Abstract

The main result of the article is Theorem 1, by which an n-group is described as an algebra with one n-ary and one $(n-1)$-ary operation, *for every* $N\backslash\{1\}$.

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1. Preliminaries

1.1. About the expression a_p^q

Let $p \in N$, $q \in N \cup \{0\}$ and let a be the maping of the set $\{i|i \in N \wedge i \geq p \wedge i \leq q\}$ into the set S ; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, ..., a_q \ ; & p < q \\ a_p \ ; & p = q \\ \text{empty sequence } (= \emptyset) \ ; & p > q. \end{cases}$$

Example.

$A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1})$, $j \in \{1, ..., n\}$, $n \in N \backslash\{1,2\}$, for $j = n$

stands for
$A(a_1, ..., a_{n-1}, A(a_n, ..., a_{2n-1}))$; and
$(\forall x_i \in Q)_1^q$ for $q > 1$ stands for
$(\forall x_1 \in Q)...(\forall x_q \in Q)$,
and for $q = 1$ it stands for
$(\forall x_1 \in Q)$.
For $q = 0$, say $(\forall x_i \in Q)_1^q(\forall y \in Q)$ stands for $(\forall y \in Q)$.
$(a_i \stackrel{def}{=} (\forall x_i \in Q).)$

In *some cases*, instead of a_p^q only, we write: sequence a_p^q (sequence a_p^q over a set S). For example: ... for every sequence a_p^q over a set S ... . And if $p \le q$, we usually write: $a_p^q \in S$.

1.2. About n-groups

Let $n \in N \setminus \{1\}$ and let A be the mapping of the set Q^n into the set Q . (Q, A) is said to be an n-semigroup iff for every $i \in \{2, ..., n\}$ and for all $x_1^{2n-1} \in Q$ the following equality holds:

$$A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}).$$

(Q, A) is an n-quasigroup iff for every $i \in \{1, ..., n\}$ and for all $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds. (Q, A) is said to be a Dörnte n-group (briefly: an n-group) iff (Q, A) is both, n-semigroup and n-quasigroup. For $n = 2$ it is a group. The notion of an n-group has been introduced in [1]. The following proposition holds:

Proposition 1. [2]*: An n-semigroup (Q, A) is an n-group iff for all $a_1^n \in Q$ there is exactly one $x \in Q$ and exactly one $y \in Q$, such that the equalities*

$$A(a_1^{n-1}, x) = a_n \quad \text{and} \quad A(y, a_1^{n-1}) = a_n$$

hold.

1.3. On a $\{1, n\}$-neutral operation in an n-groupoid

Let (Q, A) be an n-groupoid and $n \in N \setminus \{1\}$. Let also $\mathbf{e}$ be an $(n-2)$-operation in Q ; for $n = 2$ this is a nullary operation. We say that $\mathbf{e}$ is a $\{1, n\}$-neutral operation in the n-groupoid (Q, A) iff:

$$(\forall a_i \in Q)_1^{n-2}(\forall x \in Q)$$
$$(A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x \wedge A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x).$$

For $n = 2$, $\mathbf{e}(a_1^0)(= \mathbf{e}(\emptyset)) = e \in Q$ is a neutral element of the groupoid (Q, A). The notion of an $\{i,j\}$-neutral operation of an n-groupoid (: $n \in N \setminus \{1\}$, $(i,j) \in \{1,...,n\}^2, i < j$) has been introduced in [3]. The following propositions hold:

Proposition 2. [3]: *In an n-groupoid ($n \in N \setminus \{1\}$) there is at most one $\{1,n\}$-neutral operation.*

Proposition 3. [3]: *In every n-group there is a $\{1,n\}$-neutral operation.*

Proposition 4. [3]: *For $n \geq 3$, an n-semigroup (Q, A) is an n-group iff (Q, A) has a $\{1,n\}$-neutral operation*[1].

2. A characterization of n-groups, $n \in N \setminus \{1\}$

The following proposition about groups is well known:

Proposition 5. *A semigroup (Q, A) is a group iff there is a unary operation f in Q such that the following formula hold:*
(1) $(\forall a \in Q)(\forall x \in Q)\ A(f(a), A(a,x)) = x$
and
(2) $(\forall a \in Q)(\forall x \in Q)\ A(A(x,a), f(a)) = x.$

With regard to this, in a semigroup (Q, A) there is at most one operation f satisfying formulas (1) and (2).

The operation f is, in fact, the *inversing operation* in group (Q,A): $A(f(a),a) = A(a,f(a)) = e$, where e is the neutral element in a group.

Proposition 5 is a special case of the following one:

Theorem 1. *Let (Q, A) n-semigroup and $n \in N \setminus \{1\}$. Then:*

2.1.1: There is at most one $(n-1)$-ary operation f in Q such that the following formulas hold

[1] This result has been commented from the particular point of view in the paper [4].

(1) $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)$
$$A(f(a_1^{n-2},a),a_1^{n-2},A(a,a_1^{n-2},x)) = x$$
and
(2) $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)$
$$A(A(x,a_1^{n-2},a),a_1^{n-2},f(a_1^{n-2},a)) = x \;;$$

2.1.2 If there is an $(n-1)$-ary operation f in Q such that the formulas (1) *and* (2) *are satisfied, then (Q, A) is an n-group; and*

2.1.3: If (Q, A) is an n-group, then there is an $(n-1)$-ary operation f in Q such that the formulas (1) *and* (2) *hold.*

Proof.

1) Let (Q, A) be an n-semigroup, $n \in N\setminus\{1\}$. Let also f and $\bar{f}$ be $(n-1)$-ary operations in Q such that for every $x \in Q$, for every $a \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold:
$$A(f(a_1^{n-2},a),a_1^{n-2},A(a,a_1^{n-2},x)) = x;$$
$$A(A(x,a_1^{n-2},a),a_1^{n-2},f(a_1^{n-2},a)) = x;$$
$$A(\bar{f}(a_1^{n-2},a),a_1^{n-2},A(a,a_1^{n-2},x)) = x;$$
and
$$A(A(x,a_1^{n-2},a),a_1^{n-2},\bar{f}(a_1^{n-2},a)) = x;$$
Substituing x in the first equality by $\bar{f}(a_1^{n-2},a)$, and in the fourth by $f(a_1^{n-2},a)$, and using the fact that (Q, A) is an n-semigroup, we conclude that for every $a \in Q$ and for every sequence a_1^{n-2} over Q the equality
$$\bar{f}(a_1^{n-2},a) = f(a_1^{n-2},a)$$
holds, i.e. that $\bar{f} = f$.

2) Let (Q, A) be an n-semigroup, $n \in N\setminus\{1\}$, and let f satisfy the formulas (1) and (2). Let also x, y and a be arbitrary elements of Q and a_1^{n-2} an arbitrary sequence over Q so that the equality
$$A(a,a_1^{n-2},x) = A(a,a_1^{n-2},y)$$
holds. Then, by the monotonity of A and by the formula (1), we have the implication
$$A(a,a_1^{n-2},x) = A(a,a_1^{n-2},y) \Rightarrow x = y,$$
and thereby we can conclude that for all $x, y, a \in Q$ and for every sequence a_1^{n-2} over Q the equivalence
(3) $A(a,a_1^{n-2},x) = A(a,a_1^{n-2},y) \iff x = y,$
holds. Similary, using the formula (2), we conclude that for all $x, y, a \in Q$ and for every sequence a_1^{n-2} over Q the equivalence
(4) $A(a,a_1^{n-2},x) = A(a,a_1^{n-2},y) \iff x = y,$

holds. Further, considering propositions conected with the equivalences (3) and (4), we conclude that for all $x, y, a \in Q$ and for every sequence a_1^{n-2} over Q the following equivalences hold:

$$A(a, a_1^{n-2}, x) = b \iff x = A(f(a_1^{n-2}, a), a_1^{n-2}, b)$$

and

$$A(y, a_1^{n-2}, a) = b \iff y = A(b, a_1^{n-2}, f(a_1^{n-2}, a)),$$

therefore, we conclude that the equations

$$A(a, a_1^{n-2}, x) = b \text{ and } A(y, a_1^{n-2}, a) = b$$

with the unkowns x and y, for all $a, b \in Q$ and for every sequence a_1^{n-2} over Q, have unique solutions,

$$A(f(a_1^{n-2}, a), a_1^{n-2}, b) \quad \text{and} \quad A(b, a_1^{n-2}, f(a_1^{n-2}, a)),$$

respectively. Hence, by Proposition 5, we conclude that (Q, A) is an n-group.

3) Let (Q, A) be an n-group and $n \in N \setminus \{1\}$. Then, $(Q, \overset{2}{A})$, where

$$\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$$

is for all $x_1^{2n-1} \in Q$, a $(2n-1)$-group. Thereby, by Proposition 3, $(Q, \overset{2}{A})$ has a $\{1, 2n-1\}$-neutral operation. Thus, there is a $(2n-3)$-ary operation E, where $2n-3 \geq 1$. $(: n \geq 2)$, such that for all $x, a_1, ..., a_{2n-3} \in Q$ the following equalities hold

$$\overset{2}{A}(\mathsf{E}(a_1^{2n-3}), a_1^{2n-3}, x) = x \text{ and } \overset{2}{A}(x, a_1^{2n-3}, \mathsf{E}(a_1^{2n-3})) = x,$$

that is, the equalities

(5) $\quad A(\mathsf{E}(a_1^{2n-3}), a_1^{n-2}, A(a_{n-1}^{2n-3}, x)) = x$

and

(6) $\quad A(A(x, a_1^{n-2}, a_{n-1}), a_n^{2n-3}, \mathsf{E}(a_1^{2n-3})) = x.$

If we put in (5) and (6)

$$a_{n-1} = a, \; a_n^{2n-3} = a_1^{n-2}$$

and

$$f(a_1^{n-2}, a) \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2})$$

for every $a \in Q$ and for every sequence a_1^{n-2} over Q, we conclude that for all $x, a \in Q$ and for every sequence a_1^{n-2} over Q the equalities

$$A(f(a_1^{n-2}, a), a_1^{n-2}, A(a, a_1^{n-2}, x)) = x$$

and

$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, f(a_1^{n-2}, a)) = x$$

hold. The operation f thus satisfies the formulas (1) and (2).

As for the case $n = 2$ we shall say that the operation f is an *inversing operation* in the n-group (Q, A).

3. Two theorems generalizing propositions about inversing in a group

A consequence of Theorem 1 and Proposition 3 is the following

Theorem 2. *Let (Q, A) be an n-group, $\mathbf{e}$ its $\{1,n\}$-neutral operation, f its inversing operation and $n \in N \setminus \{1\}$. Then the following formula holds:*

$$(\forall a_i \in Q)_1^{n-2} (\forall a \in Q) \; (A(f(a_1^{n-2}, a), a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}) \wedge$$
$$\wedge\; A(a, a_1^{n-2}, f(a_1^{n-2}, a)) = \mathbf{e}(a_1^{n-2}))\,.$$

Theorem 3. *Let (Q, A) be an n-group, f its inversing operation and $n \in N \setminus \{1\}$. Then the formula*

$$(\forall a \in Q)\, (\forall b \in Q)\, (\forall c_i \in Q)_1^{n-2} \; f(c_1^{n-2}, A(a, c_1^{n-2}, b)) =$$
$$= A(f(c_1^{n-2}, b), c_1^{n-2}, f(c_1^{n-2}, a))$$

is satisfied.

Proof. Let a and b be arbitrary elements of the set Q, and c_1^{n-2} arbitrary sequence over Q. Then, by Theorem 1 and by the assumption that $\mathbf{e}$ is a $\{1,n\}$-neutral operation of an n-group (Q, A) (:1.3), we conclude that for every $x \in Q$ the equivalence

$$A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2}) \iff$$
$$x = f(c_1^{n-2}, A(a, c_1^{n-2}, b))$$

holds, and hence $f(c_1^{n-2}, A(a, c_1^{n-2}, b))$ is a solution of the equation

(1) $\quad A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2})$

(for the unknown x). With regard to this, for every $x \in Q$ the following sequence of equivalences hold

$$A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2}) \iff$$
$$A(a, c_1^{n-2}, A(b, c_1^{n-2}, x)) = \mathbf{e}(c_1^{n-2}) \iff$$
$$A(b, c_1^{n-2}, x) = A(f(c_1^{n-2}, a), c_1^{n-2}, \mathbf{e}(c_1^{n-2})) \iff$$
$$A(b, c_1^{n-2}, x) = f(c_1^{n-2}, a) \iff$$
$$x = A(f(c_1^{n-2}, b), c_1^{n-2}, f(c_1^{n-2}, a))$$

and hence we have

$$A(A(a, c_1^{n-2}, b), c_1^{n-2}, x) = \mathbf{e}(c_1^{n-2}) \iff$$
$$x = A(f(c_1^{n-2}, b), c_1^{n-2}, f(c_1^{n-2}, a))\,.$$

Thereby we conclude that $A(f(c_1^{n-2}, b), c_1^{n-2}, f(c_1^{n-2}, a))$ is also a solution of the equation (1). □

4. A proposition on inversing in an n-group which is a triviality for $n = 2$

Theorem 4. *Let* (Q, A) *be an* n*-group,* **e** *its* $\{1,n\}$*-neutral operation,* f *its inversing operation and*
$n \in N \setminus \{1\}$ *. Then, the formula*

(1) $(\forall x \in Q)\,(\forall y \in Q)\,(\forall a_i \in Q)_1^{n-2}\,(\forall b_i \in Q)_1^{n-2}$
$$A(x, b_1^{n-2}, y)) = A(A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(b_1^{n-2}))), a_1^{n-2}, y)$$
holds.

Proof.
1) For $n = 2$ the formula (1) reduces to the formula:
$$(\forall x \in Q)\,(\forall y \in Q)\ A(x, y) = A(y, x) .$$
2) Let $n \geq 3$. Let also $x, y, a_1^{n-2}, b_1^{n-2}$ be arbitrary elements of the set Q. Then, by the assumptions, we have the following equalities
$$A(z, a_1^{n-2}, y) = A(z, a_1^{n-2}, A(\mathbf{e}(b_1^{n-2}), b_1^{n-2}, y)) =$$
$$= A(A(z, a_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{n-2}, y),$$
i.e. the equality
$$A(A(z, a_1^{n-2}, \mathbf{e}(b_1^{n-2})), b_1^{n-2}, y) = A(z, a_1^{n-2}, y) .$$
Thereby, since for all $x, y, a_1^{n-2}, b_1^{n-2} \in Q$ the equivalence
$$A(z, a_1^{n-2}, \mathbf{e}(b_1^{n-2})) = x \iff z = A(x, a_1^{n-2}, f(a_1^{n-2}, \mathbf{e}(b_1^{n-2})))$$
holds, we conclude that the formula (1) is satisfied. □

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REZIME

JEDNO ZAPAŽANJE O n-GRUPAMA

Osnovni rezultat rada jeste Teorema 1, kojom se n-grupa opisuje kao algebra sa jednom n-arnom i jednom $(n-1)$-arnom operacijom *pri svakom* $n \in N \setminus \{1\}$.

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